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Liouvillian and Darboux First Integrals of the Self-Assembling Micelle System

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Abstract

In this paper we prove that the planar self-assembling micelle system

$$\frac{dx}{dt} = \mu - xy^2 - x(r+\alpha), \qquad \frac{dy}{dt} = rx + xy^2 - \eta y\,,$$

has no Liouvillian, polynomial and Darboux first integrals. Moreover, we show that the system
has only one irreducible Darboux polynomial $\alpha x + \alpha y - \mu$ with the cofactor being $-\alpha$ if and only if $\eta = \alpha$ via the weight homogeneous polynomials and only two irreducible exponential factors e^{x+y} and $e^{(x+y)^2}$ with cofactors $\mu - \alpha x - \eta y$ and $2(x+y)(\mu - \alpha x - \eta y)$ respectively with $\left(x + y - \frac{\mu}{\alpha}\right)^{\lambda} e^{-\alpha\lambda t}$, be the unique Darbox invariant of system.

Keywords: Self-assembling micelle system, Invariant algebraic curves, Darboux first integrals , Darboux polynomials, Exponential factors, Weight homogeneous polynomials.

التكاملات الاولية ليوفيليان و داربوكس للتجميع الذاتى لميسيل

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الخلاصة

فى هذا البحث تطرقنا الى برهنة التجميع الذاتي لمستوى منظونة ميسيل

$$\frac{dx}{dt} = \mu - xy^2 - x(r+\alpha), \qquad \frac{dy}{dt} = rx + xy^2 - \eta y\,,$$

والتي ليست لها ولاتملك ليوفلين متعدده الحدود مع داربوكس التكاملي الأولي. وعلاوة على ذلك برهنا آن للنظام لها فقط باستخدام الوزن المتجانس المتعددي $\eta = \alpha$ اذا وفقط اذا $-\alpha$ مع معامل تشاركي $\alpha x +$ $\alpha y - \mu$ عامل محتزل داربوكس متعددي هو $2(x+y)(\mu - \alpha x - \eta y)$ و $\mu - \alpha x - \eta y$ مع

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العاملين التشاركين $e^{(x+y)^2}$ و e^{x+y}الحدود مع عاملين أسيين غير قابلين للأختزال هما .
$e^{-\alpha\lambda t}\left(x+y-\frac{\mu}{\alpha}\right)^{\lambda}$ على التوالي ولها متغير داربوكس وحيد وهو

1. Introduction.

Nonlinear system of ordinary differential equations is appeared in many branches of physics, mechanics, biology, and economics. Exact answers to those equations are important to better understand the key features of a wide range of phenomena and processes in natural science. But, even if there exists a solution, only for a few nonlinear system of ordinary differential equations it is possible to determine this exact solution. There is no another method for finding analytical solutions, see for instance [1]. The integrability theory of dynamical systems plays a quite important role in studying dynamics of many differential systems. Because, differential systems cannot be solved explicitly in general, the qualitative information provided by the theory of dynamical systems is the best that one can expect to obtain in general. For a 2-dimensional polynomial system the existence of a first integral determines completely its phase portrait. But in general, for a given differential system depending on parameters it is very hard problem to characterization the existence or nonexistence of first integrals. We recall that the study of invariant algebraic curves and first integrals are important aspects for studying of dynamical systems.

In this work we deal with the cubic planar differential systems suggested by Ball and Haymet [2], which is the self-assembling micelle system with chemical sinks is modeled by the following dynamical system:

$$\begin{aligned} \frac{dx}{dt} &= \mu - xy^2 - x(r+\alpha) = P(x,y), \\ \frac{dy}{dt} &= rx + xy^2 - \eta y = Q(x,y), \end{aligned} \tag{1}$$

here x and y are dimensionless concetrations of active free amphiphile and micelles respectively.The rate coefficients α and η represent combined quantities that include a common flow-rate component as well as separate chemical sink-rates for each species, and r and μ are intrinsic parameters. It is important to note that all of the parameters are real positive constants. The behaviors of system (1) may change dramatically as the parameters vary. The authors of [2], investigated limit cycles of system (1) using Hopf bifurcation and qualitative behaviors they have analyzed by means of pure mathematical methods with numeric simulation and visualization. In [3] the stability, limit cycles, and bifurcations for system (1) using algebraic methods. Moreover, bifurcation analysis such as Hopf bifurcation, saddle-node bifurcation, and Bogdanov–Takens bifurcation are also analyzed. In [4] the stability conditions on the parameters are obtained for system (1) for some special cases.

In this paper, we want to understand complex dynamics of system (1) by studying its integrability. The system is defined only for real values of the dependent variables we will consider it in the real plane and study some types of first integrals.

2. Background.

This part provides a short explation of the Darboux method, the existence of first integral, and the auxiliary results which are used in this study [5,6,7]. So as to confirm the main results some essential definitions and theorems are given.

Associated to the polynomial differential system (1) we have the vector field X given by

$$X = \left(\mu - xy^2 - x(r+\alpha)\right)\frac{\partial}{\partial x} + (rx + xy^2 - \eta y)\frac{\partial}{\partial y}.$$

The zero set of a real polynomial $f(x,y)$ is defined as an invariant algebraic curve for system (1) if

$$\left(\mu - xy^2 - x(r+\alpha)\right)\frac{\partial f(x,y)}{\partial x} + (rx + xy^2 - \eta y)\frac{\partial f(x,y)}{\partial y} = k(x,y)f(x,y), \tag{2}$$

in order for a real polynomial $k(x,y)$ which is a cofactor of $f(x,y)$ with a degree of at most 2. According, the cofactor form can be deduced as given below:

$$k(x,y) = \delta_{2,2}x^2 + \delta_{1,2}xy + \delta_{0,2}y^2 + \delta_{1,1}x + \delta_{0,1}y + \delta_{0,0}\,, \tag{3}$$

where $\delta_{j,i}$ are constants, $i = 0,1,2$, and $j = 0,1,2$. We also say that $f(x,y)$ is Darboux polynomial of system (1).
An invariant algebraic curve $f = 0$ is irreducible if it is irreducible in $\mathbb{C}[x,y]$. If $f(x,y)$ is Darboux polynomial of system (1), then the invariant algebraic curve in $\mathbb{R}^2$ is $f(x,y) = 0$. Note that it is invariant by the dynamics in the sense that if a trajectory starts on the curve it does not leave it and that the invariant algebraic curves are important because a sufficient number of then forces the existence of a first integral.

An exponential factor is a function of the form $E = \exp\left(\frac{g(x,y)}{f(x,y)}\right) \notin \mathbb{C}$ with $g(x,y), f(x,y) \in \mathbb{C}[x,y]$. Let $g(x,y)$ and $f(x,y)$ be the co-primes in the ring $\mathbb{C}[x,y]$, thus satisfying Eq.(4) below:

$$\left(\mu - xy^2 - x(r+\alpha)\right)\frac{\partial E}{\partial x} + (rx + xy^2 - \eta y)\frac{\partial E}{\partial y} = LE, \tag{4}$$

for the polynomial $L = L(x,y)$ with degree at most 2, which is called cofactor of E.

If system (1) has a first integral or an integrating factor of the form

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q}\,,$$

where f_i and F_j are the invariant algebraic curve and exponential factor of system (1) respectively and $\lambda_i\,, \mu_j \in \mathbb{C}$, then system (1) is said to be Darboux integrable. This kind of function is said to be a Darbouxian function.

Exponential factors of the form $E = \exp\left(\frac{g}{f}\right)$ with $f \neq 1$ appear when the multiplicity of the invariant algebraic curve $f = 0$ is a multiple of one, and with $f = 0$ appear when the multiplicity of the invariant straight line at infinity is a multiple of one, see [10] for more information exponential factors.

Let $\mathbb{U}$ is an open subset $\mathbb{R}^2$. Here a non-constant analytic function $H\colon \mathbb{U} \to \mathbb{R}$ is said to be a first integral (F.I.) of system (1) on $\mathbb{U}$ if it is constant on all solutions curves $(x(t), y(t))$ of the vector field X associated to system (1) on $\mathbb{U}$; i.e. $H\big(x(t), y(t)\big) =$ constant for every values of t for which the solution $(x(t), y(t))$ is defined in $\mathbb{U}$. Clearly H is a F.I. of the vector field X on $\mathbb{U}$ if and only if

$$X(H) = P(x,y)\frac{\partial H}{\partial x} + Q(x,y)\frac{\partial H}{\partial y} \equiv 0, \tag{5}$$

on $\mathbb{U}$.

A polynomial first integral is a F.I. H which is a polynomial. Liouvillian integrable of system (1) is a F.I. given by integrals of elementary functions or by elementary functions. Here, a function is elementary if it is the expressed of trigonometric, logarithmic, exponential and polynomial functions, for more information of a Liouvillean F.I. see [8]. We say that system (1) is Liouvillian integrable if it has a Liouville F.I. .

A non-constant function $V: U \longrightarrow \mathbb{R}$ is an integrating factor of the polynomial vector field X on U if the condition hold;

$$\frac{\partial(VP)}{\partial x} = -\frac{\partial(VQ)}{\partial y},$$

on U.

If V is an integrating factor of system (1), then $H = \int V(x,y)P(x,y)dy + h(x)$ is a first integral of system (1), where the function $h(x)$ can be determined by $\frac{\partial H}{\partial x} = -V(x,y)Q(x,y)$. A polynomial $f(x,y)$ is called a weight-homogeneous polynomial if there exist $\lambda = (\lambda_1, \lambda_2) \in \mathbb{N}^2$ and $m \in \mathbb{N}$ such that for all $\in \mathbb{R}\backslash\{0\}$,

$$f(\lambda^{l_1}x, \lambda^{l_2}y) = \lambda^m f(x,y),$$

where $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}$ the set of positive integers. We shall specifically refer to $l = (l_1, l_2)$ as the weight of f, m as the weight degree, and $(x,y) \mapsto (\lambda^{l_1}x, \lambda^{l_2}y)$ as the weight change of variables.

Proposition 1. [9] System (1) has a rational F.I. if it has two Darboux polynomials with the same cofactor.

Theorem 2. [10] The two statements below are applicable.

1) If $E = e^{\frac{g}{f}}$ for the polynomial differential system (1), is an exponential factor and f is not a polynomial with a constant value, then $f = 0$ is an algebraic curve that is invariant.
2) $E = e^g$ can eventually be obtained from the multiplicity of the infinity invariant plane as an exponential factor.

The results found in [11] characterization under suitable assumptions of the algebraic multiplicity of an invariant algebraic curve using the number of exponential factors of system (1) like with the invariant algebraic curve.

Theorem 3. Given Darboux polynomial $f = 0$ of degree n of system (1), has algebraic multiplicity m if and only if the system (1) has $m - 1$ exponential factors of the kind exp $(\frac{g_i}{f})$, where g_i is a polynomial of degree at most i, where g_i and f are relatively primes for $i = 1, \dots, m-1$.

Proposition 4. [12] Assume that $f \in \mathbb{C}[x,y]$ and let $f = f^{n_1}{}_1 \dots f^{n_r}{}_r$ the factoring of its into irreducible factors over $\mathbb{C}[x,y]$. Then there's the case of a polynomial system (1), $f(x,y) = 0$ is an invariant algebraic curve with cofactor k_f if and only $f_i = 0$ is an invariant algebraic curve for each $i = 1, \dots, r$ with cofactor k_{f_i}. Moreover, $k_f = n_1 k_{f_1} + \cdots + n_r k_{f_r}$.

Proposition 5. [12] Assume that a polynomial system (1) admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors k_i for $i = 1, \dots, p$ and q exponential factors exp (g_j/h_j) with cofactors L_j for $j = 1, \dots, q$. Then,

1- There exist λ_i *and* $\mu_j \in \mathbb{C}$ not all zero in the sense that

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = 0 ,$$

if and only if the (multivalued) function

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} (\exp(g_1/h_1))^{\mu_1} \dots \exp(g_p/h_p))^{\mu_p} \tag{6}$$

is a first integral of system (1).

2- There exist λ_i and μ_j $\in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = -div(P, Q) ,$$

if and only if function (6) is an integrating factor of system (1).

3- If there exist λ_i and μ_j $\in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = -s ,$$

for some $s \in \mathbb{C}\backslash\{0\}$, then the (multivalued) function

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \dots \exp(g_p/h_p))^{\mu_p} \exp(st) ,$$

is an invariant of system (1).

Note that the function of the form (6) is said to be a Darboux function.

We use the following result from [8] to establish the findings relating to the Liouville first integral.

Theorem 6. The polynomial system (1) has a Liouville F.I. if and only if it has on integrating factor which is a Darboux function.

A considerable impact in the study of Liouvillian and Darboux integrability of polynomial dynamical systems has been made by Llibre and Valls, see for example [13,14].

3. Main results and their Proving.

In this section, the existence of Darboux F.I. (see Theorem 8) which is the primary outcome of the system (1) is described. Moreover, some other results relative to this topic is studied during this work such as a polynomial first integrals, invariant algebraic curves and exponential factors of system (1).

Theorem 7. System (1) does not admit polynomial F.I. .

Proof. Let $H(x, y)$ be a polynomial F.I. of degree n of the system (1). We can write $H(x, y) = H_0(x, y) + H_1(x, y) + \cdots + H_n(x, y)$, where each H_i is a homogeneous polynomial in its variables of degree i and we assume that $H_n \neq 0$. Thus, equation writes as

$$\left(\mu - xy^2 - x(r + \alpha)\right) \frac{\partial}{\partial x} \sum_{i=0}^{n} H_i(x, y) + (rx + xy^2 - \eta y) \frac{\partial}{\partial y} \sum_{i=0}^{n} H_i(x, y) = 0. \tag{7}$$

Since $H_n(x, y)$ is a homogeneous polynomial of variables x, y of degree n, so calculating the terms of degree $n + 2$ in Eq. (7), we can obtain

$$-xy^2\frac{\partial H_n(x,y)}{\partial x}+xy^2\frac{\partial H_n(x,y)}{\partial y}=0,$$

as a solution of the above partial differential equation for $H_n(x,y)$, we gain that

$$H_n(x,y)=G_n(x+y),$$

where $G_n$is an arbitrary polynomial.
Since $H_n(x,y)$ is a homogeneous polynomial of degree n, then it must be in the following form

$$H_n(x,y)=C_n(x+y)^n,$$

where C_n is an arbitrary constant.
Computing the terms of degree $n+1$ in Eq. (7), we obtain

$$-xy^2\frac{\partial H_{n-1}(x,y)}{\partial x}+xy^2\frac{\partial H_{n-1}(x,y)}{\partial y}=0,$$

as a solution of the above partial differential equation for $H_{n-1}(x,y)$, we can obtain

$$H_{n-1}(x,y)=G_{n-1}(x+y),$$

where G_{n-1} is an arbitrary polynomial function of $x+y$.

Since $H_{n-1}(x,y)$ is a homogeneous polynomial of degree $n-1$, then it must be in the following form

$$H_{n-1}(x,y)=C_{n-1}(x+y)^{n-1},$$

where C_{n-1} is an arbitrary constant.

Since $H_i(x,y)$ is a homogeneous polynomial of variables x,y of degree i, then computing the terms of degree n in Eq. (7), we obtain

$$-xy^2\frac{\partial H_{n-2}(x,y)}{\partial x}-(r+\alpha)x\frac{\partial H_n(x,y)}{\partial x}+rx\frac{\partial H_n(x,y)}{\partial y}+xy^2\frac{\partial H_{n-2}(x,y)}{\partial y}-\eta y\frac{\partial H_n(x,y)}{\partial y}=0,$$

by using Maple as a solution of the above partial differential equation for $H_{n-2}(x,y)$, we can obtain

$$H_{n-2}(x,y)=-n\eta C_n(x+y)^{n-2}(ln(x)-ln(-y))-\text{n}\alpha C_n\frac{(x+y)^{n-1}}{y}+G_{n-2}(x+y),$$

where G_{n-2} is a polynomial function of variables $x+y$.

Since $H_{n-2}(x,y)$ is a homogeneous polynomial of degree $n-2$, $H_n(x,y)\neq 0$ and $\alpha,\eta>0$ must be $nC_n=0$.

If $nC_n=0$, then we obtain that $H_n(x,y)$ becomes constant function which is a trivial first integral. Then there is no a polynomial first integral of system (1). □

Theorem 8. System (1) has only one irreducible Darboux polynomial expressed as $\alpha x+\alpha y-\mu$ with the cofactor being $-\alpha$ if and only if $\eta=\alpha$.

First, we begin with the following three lemmas to prove Theorem 8.

Lemma 9: The cofactor $k=\delta_{2,2}x^2+\delta_{1,2}xy+\delta_{0,2}y^2+\delta_{1,1}x+\delta_{0,1}y+\delta_{0,0}$ of the invariant is algebraic curve of system (1) satisfy the $\delta_{2,2}=\delta_{1,1}=\delta_{0,1}=0$, $\delta_{0,0}=r\delta_{0,2}-\alpha n-\frac{1}{2}\delta_{1,2}$ and $\delta_{1,2}\in \mathrm{N}\cup\{0\}$, with $\delta_{1,2}$ even number, where $\delta_{i,j}$ are constants, $i=0,1,2$ and $j=0,1,2$.

Proof. We write the Darboux polynomial of the form $f = \sum_{i=0}^{n} f_i(y)x^i$, where each f_i is a polynomial in the variable y. Then must be satisfy the equation,

$$\left(\mu - xy^2 - x(r+\alpha)\right)\sum_{i=0}^{n}\frac{df_i(y)x^i}{dx} + (rx + xy^2 - \eta y)\sum_{i=0}^{n}\frac{df_i(y)x^i}{dy} = \left(\delta_{2,2}x^2 + \delta_{1,2}xy + \delta_{0,2}y^2 + \delta_{1,1}x + \delta_{0,1}y + \delta_{0,0}\right)\sum_{i=0}^{n} f_i(y)x^i. \quad (8)$$

Compute the terms of x^{n+2} in Eq. (8), we have

$$\delta_{2,2}f_n(y) = 0 \text{, then } f_n(y) = 0 \text{ or } \delta_{2,2} = 0.$$

Case1**:** If $f_n(y) = 0$; hence, $f = f(y)$.
Then from Eq. (8), we can derive

$$(rx + xy^2 - \eta y)\frac{df(y)}{dy} = \left(\delta_{2,2}x^2 + \delta_{1,2}xy + \delta_{0,2}y^2 + \delta_{1,1}x + \delta_{0,1}y + \delta_{0,0}\right)f(y), \quad (9)$$

as a solution of the above differential equation, we gain that

$$f(y) = C_0((y^2+r)x - \eta y)^{\frac{\delta_{1,2}}{2}+\frac{\eta\delta_{0,2}}{2x^2}+\frac{\delta_{0,1}}{2x}}\, e^{\frac{-2(\left(-x^4\delta_{2,2}-x^3\delta_{1,1}+\left(r\delta_{0,2}-\frac{\eta\delta_{1,2}}{2}-\delta_{0,0}\right)x^2-\frac{\eta x\delta_{0,1}}{2}-\frac{\eta^2\delta_{0,2}}{2}\right)tan^{-1}\left(\frac{2xy-\eta}{\sqrt{4rx^2-\eta^2x^2}}\right)-\frac{\delta_{0,2}yx\sqrt{4rx^2-\eta^2}}{2})}{\sqrt{4rx^2-\eta^2}x^2}},$$

where C_0 is a constant.
Since $f(y)$ is a polynomial of y must be $\delta_{2,2} = \delta_{0,2} = \delta_{1,1} = \delta_{0,1} = 0$ and $\delta_{1,2} = \frac{-2}{\eta}\delta_{0,0}$, hence;

$$f(y) = C_0((y^2+r)x - \eta y)^{\frac{-\delta_{0,0}}{\eta}}.$$

Since $f(y)$ is a polynomial of y must be $\delta_{0,0} = 0$, hence $f(y) = C_0$ and its cofactor is $k = 0$, where C_0 is a constant, this is a trivial first integral.
Case2**:** If $\delta_{2,2} = 0$ and $f_n(y) \neq 0$.
Compute the terms of x^{n+1} in Eq. (8) we have,

$$(r + y^2)\frac{df_n(y)}{dy} = \left(\delta_{1,2}y + \delta_{1,1}\right)f_n(y). \quad (10)$$

This gives

$$f_n(y) = C_n(y^2+r)^{\frac{\delta_{1,2}}{2}}\, e^{\frac{\delta_{1,1}tan^{-1}(\frac{y}{\sqrt{r}})}{\sqrt{r}}}, \quad (11)$$

where C_n is a constant.

Taking into account that $f_n(y)$ be a polynomial, this implies that $\delta_{1,1} = 0$ and $\delta_{1,2} \geq 0$ and $\delta_{1,2}$ is even number, then equation (11) becomes,

$$f_n(y) = C_n(y^2+r)^{\frac{\delta_{1,2}}{2}}. \quad (12)$$

Compute the terms of x^n in Eq. (8), we have

$$n\left(-y^2 - (r+\alpha)\right)f_n(y) + (r+y^2)\frac{df_{n-1}(y)}{dy} - \eta y\frac{df_n(y)}{dy} = \delta_{1,2}yf_{n-1}(y) + \left(y^2\delta_{0,2} + y\delta_{0,1} + \delta_{0,0}\right)f_n(y),$$

as a solution of the above differential equation, we gain that

$$f_{n-1}(y)=\Big(C_n ny+C_n\delta_{0,2}y+\frac{1}{2}C_n\delta_{0,1}\,ln(y^2+r)+(2\alpha n+\delta_{1,2}\eta-2r\delta_{0,2}+2\delta_{0,0})\frac{1}{2\sqrt{r}}C_n tan^{-1}\left(\frac{y}{\sqrt{r}}\right)-\frac{C_n\eta\delta_{1,2}\,y}{2(y^2+r)}+C_{n-1}\Big)(y^2+r)^{\frac{\delta_{1,2}}{2}},$$

where C_{n-1} is a constants.
Since $r>0$ and $f_n(y)\neq 0$, then $f_{n-1}(y)$ must be polynomial, hence, $\delta_{0,1}=0$ and $2\alpha n+\delta_{1,2}\eta-2r\delta_{0,2}+2\delta_{0,0}=0$, hence; $\delta_{0,0}=r\delta_{0,2}-\alpha n-\frac{1}{2}\delta_{1,2}\eta$. □

Lemma 10. The coefficients $\delta_{0,2}=0$ and $\delta_{0,0}=-\alpha n-\frac{1}{2}\delta_{1,2}\eta$ in cofactor k.
Proof. From Lemma (9), the coefficients of cofactor $\delta_{2,2}=\delta_{1,1}=\delta_{0,1}=0,\delta_{0,0}=r\delta_{0,2}-\alpha n-\frac{1}{2}\delta_{1,2}\eta$ and $\delta_{1,2}\in \mathrm{N}\cup\{0\}$, with $\delta_{1,2}$ even number, then we can derive the cofactor k is

$$k=\delta_{1,2}xy+\delta_{0,2}y^2+r\delta_{0,2}-\alpha n-\frac{1}{2}\delta_{1,2}\eta,\ \delta_{1,2}\in \mathrm{N}\cup\{0\}. \quad (13)$$

To show that $\delta_{0,2}=0$, put simplicity, the variables weight change can be converted to this computation $x=X$, $y=\lambda^{-1}Y$, $t=\lambda^2T$, with $\lambda\in\mathbb{R}\setminus\{0\}$. After that, system (1) is changed to

$$X'=\lambda^2\mu-XY^2-\lambda^2X(r+\alpha),$$
$$Y'=\lambda^3rX+\lambda XY^2-\lambda^2\eta Y\ , \quad (14)$$

the derivatives of the variables connected to T are denoted by the primes.
Taking in to account that f is a Darboux polynomial of system (1) with cofactor k specified in Eq. (13), by the transformation (14) and setting, $F(X,Y)=\lambda^n f(X,\lambda^{-1}Y)$, where n defines the extent to which f and $K=\lambda^2k(X,\lambda^{-1}Y)=\delta_{1,2}XY\lambda+\delta_{0,2}Y^2+\lambda^2\left(r\delta_{0,2}-\alpha n-\frac{1}{2}\delta_{1,2}\eta\right)$.

Hence $F=\sum_{i=0}^{n}\lambda^iF_i$, where F_i is a weight homogeneous polynomial in the variables X and Y with weight degree $n-i$ for $i=0,1,\dots n$, we get

$$f=F|_{\lambda=1}.$$

From the definition of invariant algebraic curve we have

$$(\lambda^2\mu-XY^2-\lambda^2X(r+\alpha))\sum_{i=0}^{n}\lambda^i\frac{\partial F_i}{\partial X}+(\lambda^3rX+\lambda XY^2-\lambda^2\eta Y)\sum_{i=0}^{n}\lambda^i\frac{\partial F_i}{\partial Y}=(\delta_{1,2}XY\lambda+\delta_{0,2}Y^2+\lambda^2\left(r\delta_{0,2}-\alpha n-\frac{1}{2}\delta_{1,2}\eta\right))\sum_{i=0}^{n}\lambda^iF_i. \quad (15)$$

The calculation of the coefficients of λ^0 in Eq. (15) can generate

$$-XY^2\frac{\partial F_0}{\partial X}=\delta_{0,2}Y^2F_0. \quad (16)$$

The equation (16) has the general solution

$$F_0(X,Y)=G_n(Y)X^{-\delta_{0,2}}, \quad (17)$$

where G_n is an arbitrary function of Y.
Since $F_0(X,Y)$ must be a homogenous of degree n, we get that

$$F_0(X,Y)=C_nY^{n+\delta_{0,2}}X^{-\delta_{0,2}},$$

where C_n is an arbitrary constant.
By the weight change of since $F_0(X,Y)$ must be weight homogenous of degree n, we must be $\delta_{0,2}=0$. □

Lemma 11. The coefficient $\delta_{1,2}=0$ in cofactor k and $\eta=\alpha$.

Proof . From Lemmas (9) and (10), the coefficients of cofactor $\delta_{2,2}=\delta_{1,1}=\delta_{0,1}=\delta_{0,2}=0, \delta_{0,0}=-\alpha n-\frac{1}{2}\delta_{1,2}\eta$ and $\delta_{1,2}\in \mathrm{N}\cup\{0\}$, with $\delta_{1,2}$ even number, then we can derive the cofactor k is

$$k=\delta_{1,2}xy-\alpha n-\frac{1}{2}\delta_{1,2}\eta, \text{ where } \delta_{1,2}\in \mathrm{N}\cup\{0\}.$$

We write the Darboux polynomial of the form $f=\sum_{i=0}^{n} f_i(y)x^i$, where each f_i is a polynomial in the variable y. Then must be satisfy the equation,

$$\left(\mu-xy^2-x(r+\alpha)\right)\sum_{i=0}^{n}\frac{df_i(y)x^i}{dx}+(rx+xy^2-\eta y)\sum_{i=0}^{n}\frac{df_i(y)x^i}{dy}$$
$$=(\delta_{1,2}xy-\alpha n-\frac{1}{2}\delta_{1,2}\eta)\sum_{i=0}^{n} f_i(y)x^i. \tag{18}$$

Compute the terms of x^{n+1} in Eq. (18) we have,

$$(r+y^2)\frac{df_n(y)}{dy}=\left(\delta_{1,2}y\right)f_n(y). \tag{19}$$

To solve this differential equation, we obtain

$$f_n(y)=C_n(y^2+r)^{\frac{\delta_{1,2}}{2}}, \tag{20}$$

where C_n is a constant. Compute the terms of x^n in Eq. (18), we have

$$n\left(-y^2-(r+\alpha)\right)f_n(y)+(r+y^2)\frac{df_{n-1}(y)}{dy}-\eta y\frac{df_n(y)}{dy}=\delta_{1,2}yf_{n-1}(y)+\left(-\alpha n-\frac{1}{2}\delta_{1,2}\eta\right)f_n(y),$$

by solving the differential equation above, we deduce that

$$f_{n-1}(y)=\left(C_n ny-\frac{1}{2}(C_n\eta\delta_{1,2}\, y)(y^2+r)^{-1}+C_{n-1}\right)(y^2+r)^{\frac{\delta_{1,2}}{2}},$$

where C_{n-1} is a constant. Compute the terms of x^{n-1} in Eq. (18), we have

$$n\mu f_n(y)+(n-1)\left(-y^2-(r+\alpha)\right)f_{n-1}(y)+(r+y^2)\frac{df_{n-2}(y)}{dy}-\eta y\frac{df_{n-1}(y)}{dy}=\delta_{1,2}yf_{n-2}(y)+\left(-\alpha n-\frac{1}{2}\delta_{1,2}\eta\right)f_{n-1}(y),$$

by solving the differential equation above, we deduce that

$$f_{n-2}(y)=\frac{1}{2\sqrt{r}(y^2+r)^2}\left((y^2+r)^{\frac{\delta_{1,2}}{2}}(-2(y^2+r)^2(n\mu C_n+\alpha C_{n-1})tan^{-1}\left(\frac{y}{\sqrt{r}}\right)+(C_n(y^2+r)^2\left((-\alpha+\eta)n+\frac{1}{2}\eta\delta_{1,2}\right)\ln(y^2+r)+\left(nC_n(n-1)y^2+2C_{n-1}(n-1)y+n\eta C_n\delta_{1,2}+2C_{n-2}\right)r^2+\left(2nC_n(n-1)y^4+4C_{n-1}(n-1)y^3+\left(n\eta C_n\delta_{1,2}+4C_{n-2}\right)y^2-\eta C_{n-1}\delta_{1,2}y-\frac{1}{2}\alpha\eta C_n\delta_{1,2}\right)r+\left(nC_n(n-1)y^4+2C_{n-1}(n-1)y^3+2y^2C_{n-2}-\eta C_{n-1}\delta_{1,2}y+\frac{1}{4}C_n\left(\left((\delta_{1,2}-2)\eta-2\alpha\right)\eta\delta_{1,2}\right)y^2)\sqrt{r}\right)),$$

where C_{n-2} is a constant.

Since $f_{n-2}(y)$ is a polynomial must be $n\mu C_n+\alpha C_{n-1}=0$ and $(-\alpha+\eta)n+\frac{1}{2}\eta\delta_{1,2}=0$, hence $C_{n-1}=\frac{-n\mu C_n}{\alpha}$, $\delta_{1,2}=\frac{2n(\alpha-\eta)}{\eta}$ and

$$f_{n-2}(y)=\frac{1}{2\alpha(y^2+r)^2}\left((y^2+r)^{\frac{n(\alpha-\eta)}{\eta}}\left(((y^6\alpha-2y^5\mu+2ry^4\alpha-4(r+\frac{1}{2}\eta-\frac{1}{2}\alpha\right)\mu y^3+\alpha(r-\eta+\alpha)^2y^2-2r\mu(r-\eta+\alpha)y-2r^2\alpha(-\alpha+\eta)\right)n-y^6\alpha+2y^5\mu-2ry^4\alpha+4ry^3\mu-\alpha(\alpha^2-\eta^2+r^2)y^2+2r^2y\mu+r\alpha^2(-\alpha+\eta))nC_n+2\alpha C_{n-2}(y^2+r)^2)).$$

Compute the terms of x^{n-2} in Eq. (18), we have

$(n-1)\mu f_{n-1}(y)+(n-2)\big(-y^2-(r+\alpha)\big)f_{n-2}(y)+(r+y^2)\frac{df_{n-3}(y)}{dy}-\eta y\frac{df_{n-2}(y)}{dy}=$ $(\frac{2n(\alpha-\eta)}{\eta})yf_{n-3}(y)+\big(-\alpha n-n(f_{n-2}(y))\big)f_{n-2}(y),$

as the solution of this differential equation, we get the following

$$f_{n-3}(y)=\frac{1}{6r^{\frac{5}{2}}\alpha(y^2+r)^3}\Bigg(\Big(6r(y^2+r)^3\Big(n\alpha C_n(\eta-\alpha)r^2+(C_n(-2\alpha^3+2\alpha^2\eta+\mu^2)n^2-C_n\left(\tfrac{1}{2}\eta^2\alpha+\tfrac{1}{2}\eta\alpha^2-\alpha^3+\mu^2\right)n-2\alpha^2(C_{n-2}+\tfrac{1}{2}C_n))r-\tfrac{1}{4}C_n n\alpha^2(\eta+2\alpha)(\eta-\alpha)\Big)tan^{-1}\left(\frac{y}{\sqrt{r}}\right)+\Bigg(3nr^{\frac{5}{2}}\mu C_n(y^2+r)^3(\eta-\alpha)\ln(y^2+r)-\tfrac{3}{2}ny^5\alpha^2C_n(\eta+2\alpha)(\eta-\alpha)r^{\frac{3}{2}}+r^{\frac{5}{2}}\Big(\big(y^3\alpha-3\mu y^2-6\alpha(\eta-\alpha)y+6\mu(\eta-\alpha)\big)C_nn^3-3C_n(y^3\alpha-3\mu y^2-4\alpha(\eta-\alpha)y+2\mu(\eta-\alpha)\Big)n^2+2\left(\left(y^2\alpha-3y\mu-3\alpha\left(\eta-\alpha-\tfrac{1}{2}\right)\right)C_n+3C_{n-2}\alpha\right)yn+6\alpha(-2yC_{n-2}-yC_n+C_{n-3})\Bigg)r^3+(3(y^4\alpha-3\mu y^3-5\alpha(\eta-\alpha)y^2+4\mu\left(\eta-\alpha)y-2\alpha(\eta-\alpha)^2\right)C_nn^3y-9(y^5\alpha-3y^4\mu-\tfrac{11}{3}\alpha(\eta-\alpha)y^3+\tfrac{4}{3}\mu(\eta-\alpha)y^2-\tfrac{1}{3}\alpha^2(\eta-\alpha)y+\tfrac{1}{3}\mu\alpha(\eta-\alpha))C_nn^2+((6y^2\alpha-18y^4\mu-18\alpha(\eta-\alpha-\tfrac{1}{2})y^3+3(\eta+1)\alpha(\eta-\alpha)y-3\mu\alpha(\eta-\alpha))C_n+18\alpha\, C_{n-2}\left(y^2+\tfrac{1}{3}\eta-\tfrac{1}{3}\alpha)y\right)n+18y^2\alpha(-2yC_{n-2}-yC_n+C_{n-3}))r^3+3((y^5\alpha-3y^4\mu-4\alpha(\eta-\alpha\big)y^3+2\mu(\eta-\alpha\big)y^2-3\alpha(\eta-\alpha)^2y\Big)-\mu(\eta-\alpha)^2)C_nyn^3-3(y^6\alpha-3y^5\mu-\tfrac{10}{3}\alpha(\eta-\alpha)y^4+\tfrac{2}{3}\mu(\eta-\alpha)y^3-\tfrac{1}{3}\alpha(\eta+2\alpha)(\eta-\alpha)y^2+\tfrac{1}{3}\mu(\eta+2\alpha)(\eta-\alpha)y+\tfrac{1}{3}\alpha^2(\eta-\alpha)^2)C_nn^2+(\Big(2y^6\alpha-6y^5\mu-6\alpha\left(\eta-\alpha-\tfrac{1}{2}\right)y^4+2(\eta+1)\alpha(\eta-\alpha)y^2-2\mu\alpha(\eta-\alpha)y+\tfrac{1}{2}\alpha^2(\eta-\alpha)(\eta-2\alpha)\Big)C_n+6\alpha(y^2+\tfrac{2}{3}\eta-\tfrac{2}{3}\alpha)y^2)n+6y^3\alpha(-2yC_{n-2}-yC_n+C_{n-3}))r+y^3(C_n(y^6\alpha-3y^5\mu-3\alpha(\eta-\alpha)y^4-3\alpha(\eta-\alpha)^2y^2-3\mu(\eta-\alpha)^2y+\alpha(\eta-\alpha)^3)n^3-3(y^6\alpha-3y^5\mu-3\alpha(\eta-\alpha)y^4+(\alpha^3-\alpha\eta^2)y^2+(-\alpha^2\mu+\eta^2\mu)y-\alpha(\eta+\alpha)(\eta-\alpha)^2)C_nn^2+(\Big(2y^6\alpha-6y^5\mu-6\alpha\left(\eta-\alpha-\tfrac{1}{2}\right)y^4+3(\eta+1)\alpha(\eta-\alpha)y^2-3\mu\alpha(\eta-\alpha)y+2\alpha(\eta-\alpha)(-3\alpha^2+\eta^2)\Big)C_n+6C_{n-2}y^2\alpha(y^2-\alpha+\eta))n+6y^3\alpha(-2yC_{n-2}-yC_n+C_{n-3}))))(y^2+r)^{n\frac{\alpha}{\eta}-n}).$$

Since $f_{n-3}(y)$ is a polynomial, must be the coefficient of logarithm removed, we have $nr^{\frac{5}{2}}\mu C_n(\eta-\alpha)=0,$ but the parameters α, r, μ and η are positive real numbers and $nC_n\neq 0$ ($C_n\neq 0$ by Lemma 9, $f_n(y)\neq 0$ and $n\neq 0$ otherwise $f_n(y)$ be constant) then must be $\eta-\alpha=0,$ but $\delta_{1,2}=\frac{2n(\alpha-\eta)}{\eta},$ hence $\delta_{1,2}=0.$ □

Proof of Theorem 8. From Lemmas 9, 10 and 11, the cofactor of system (1) is $k=-\alpha n$, where $\eta=\alpha$. Let $F=\sum_{i=0}^{n}f_i\,(x,y)$, where f_i is a homogeneous polynomial in the variables x and y with degree i for $i=0,1,\dots n$ and $k=-n\alpha$ is a cofactor. Without missing the generality, it is supposed that $f_n(x,y)\neq 0$ and $n>0$ in each variables x and y .
From the definition of Darboux polynomial, we can have

$$\left(\mu - xy^2 - x(r+\alpha)\right)\sum_{i=0}^{n}\frac{\partial f_i(x,y)}{\partial x} + (rx + xy^2 - \alpha y)\sum_{i=0}^{n}\frac{\partial f_i(x,y)}{\partial y} = (-n\alpha)\sum_{i=0}^{n} f_i(x,y). \quad (21)$$

Using the terms of degree $n+2$ in Eq. (21), we get

$$-xy^2\frac{\partial f_n(x,y)}{\partial x} + xy^2\frac{\partial f_n(x,y)}{\partial y} = 0,$$

by solving the partial differential equation above, we deduce that

$$f_n(x,y) = G_n(x+y),$$

where G_n is an arbitrary polynomial function of $x+y$.

Since $f_n(x,y)$ is the homogeneous polynomial of degree n it should be in this form

$$f_n(x,y) = C_n(x+y)^n,$$

where C_n is an arbitrary constant.

Similarly, by using the terms of degree $n+1$ in Eq. (21), we get

$$f_{n-1}(x,y) = G_{n-1}(x+y),$$

where G_{n-1} is an arbitrary polynomial of $x+y$.

Since, $f_{n-1}(x,y)$ is the homogeneous polynomial of degree $n-1$. Then it must be in the following form

$$f_{n-1}(x,y) = C_{n-1}(x+y)^{n-1},$$

where C_{n-1} is an arbitrary constant.

Using the terms of degree n in Eq. (21), we get

$$-x(r+\alpha)\frac{\partial f_n(x,y)}{\partial x} - xy^2\frac{\partial f_{n-2}(x,y)}{\partial x} + (rx - \alpha y)\frac{\partial f_n(x,y)}{\partial y} + xy^2\frac{\partial f_{n-2}(x,y)}{\partial y} = (-n\alpha)f_n(x,y).$$

To solve this partial differential equation, we can obtain

$$f_{n-2}(x,y) = G_{n-2}(x+y),$$

where G_{n-2} is an arbitrary polynomial of $x+y$.

Since, $f_{n-2}(x,y)$ is the homogeneous polynomial of degree $n-2$. Then it must be in the following form

$$f_{n-2}(x,y) = C_{n-2}(x+y)^{n-2},$$

where C_{n-2} is an arbitrary constant.

The terms of degree $n-1$ in Eq. (21), we obtain that

$$\mu\frac{\partial f_n(x,y)}{\partial x} - x(r+\alpha)\frac{\partial f_{n-1}(x,y)}{\partial x} - xy^2\frac{\partial f_{n-3}(x,y)}{\partial x} + (rx - \alpha y)\frac{\partial f_{n-1}(x,y)}{\partial y} + xy^2\frac{\partial f_{n-3}(x,y)}{\partial y} = -n\alpha f_{n-1}(x,y).$$

To solve this differential equation, we can obtain

$$f_{n-3}(x,y) = (ln(x) - ln(-y))(\mu n C_n + \alpha C_{n-1})(x+y)^{n-3} + (\mu n C_n + \alpha C_{n-1})\frac{(x+y)^{n-2}}{y} + G_{n-3}(x+y)$$

where G_{n-3} is an arbitrary polynomial of $x+y$.

Since $f_{n-3}(x,y)$ is a homogeneous polynomial of degree $n-3$, should be

$$nC_n\mu + \alpha C_{n-1} = 0 \text{ then } C_{n-1} = \frac{-n\mu}{\alpha}C_n, \text{ hence}$$
$$f_{n-3} = C_{n-3}(x+y)^{n-3} \text{ and } f_{n-1} = \frac{-n\mu}{\alpha}C_n(x+y)^{n-1},$$

where C_{n-3} is an arbitrary constant.
Using the terms of degree $n-2$ in Eq. (21), we get

$$\mu\frac{\partial f_{n-1}(x,y)}{\partial x} - x(r+\alpha)\frac{\partial f_{n-2}(x,y)}{\partial x} - xy^2\frac{\partial f_{n-4}(x,y)}{\partial x} + (rx-\alpha y)\frac{\partial f_{n-2}(x,y)}{\partial y}xy^2\frac{\partial f_{n-4}(x,y)}{\partial y} = -n\alpha f_{n-2}(x,y).$$

To solve this differential equation, we can obtain
$f_{n-4}(x,y) = (2\alpha C_{n-2} + \mu(n-1)C_{n-1})\big(ln(x) - ln(-y)\big)(x+y)^{n-4} + (2\alpha C_{n-2} + \mu(n-1)C_{n-1})\frac{(x+y)^{n-3}}{y} + G_{n-4}(x+y)$,
where G_{n-4} is an arbitrary polynomial of $x+y$.
Since $f_{n-4}(x,y)$ is a homogeneous polynomial of degree $n-4$, must be
$2\alpha C_{n-2} + \mu(n-1)C_{n-1} = 0$, hence $C_{n-2} = \frac{-\mu(n-1)}{2\alpha}C_{n-1}$,
since $C_{n-1} = \frac{-n\mu}{\alpha}C_n$, we get $C_{n-2} = \frac{n(n-1)\mu^2}{2\alpha^2}C_n$, then

$$f_{n-4}(x,y) = C_{n-4}(x+y)^{n-4} \text{ and } f_{n-2}(x,y) = \frac{n(n-1)\mu^2}{2\alpha^2}C_n(x+y)^{n-2},$$

where C_{n-4} is an arbitrary constant.
Using the terms of degree $n-3$ in Eq. (21), we get

$$\mu\frac{\partial f_{n-2}(x,y)}{\partial x} - x(r+\alpha)\frac{\partial f_{n-3}(x,y)}{\partial x} - xy^2\frac{\partial f_{n-5}(x,y)}{\partial x} + (rx-\alpha y)\frac{\partial f_{n-3}(x,y)}{\partial y} + xy^2\frac{\partial f_{n-5}(x,y)}{\partial y} = -n\alpha f_{n-3}(x,y).$$

To solve this differential equation, we can obtain
$f_{n-5}(x,y) = (3\alpha C_{n-3} + \mu(n-2)C_{n-2})\big(ln(x) - ln(-y)\big)(x+y)^{n-5} + (3\alpha C_{n-3} + \mu(n-2)C_{n-2})\frac{(x+y)^{n-4}}{y} + G_{n-5}(x+y))$,

where G_{n-5} is an arbitrary polynomial of $x+y$.
Since $f_{n-5}(x,y)$ is a homogeneous polynomial of degree $n-5$, must be

$3\alpha C_{n-3} + \mu(n-2)C_{n-2} = 0$, hence; $C_{n-3} = \frac{\mu(n-2)}{3\alpha}C_{n-2}$, since $C_{n-2} = \frac{n(n-1)\mu^2}{2\alpha^2}$ we get $C_{n-3} = \frac{-n(n-1)(n-2)\mu^3}{6\alpha^3}C_n$, then

$$f_{n-5}(x,y) = C_{n-5}(x+y)^{n-5} \text{ and } f_{n-3}(x,y) = \frac{-n(n-1)(n-2)\mu^3}{6\alpha^3}C_n(x+y)^{n-3},$$

where C_{n-5} is an arbitrary constant.
Similarly, by computing $f_1(x,y)$ and $f_0(x,y)$, we can obtains

$f_1(x,y) = C_1(x+y)$,
where C_1 is an arbitrary constant.

$$C_1 = \frac{n(n-1)(n-2)\ldots 2\mu^{n-1}}{(n-1)!\,\alpha^{n-1}} C_n,$$

and

$$f_0(x,y) = \frac{n(n-1)(n-2)\ldots\mu^n}{n!\,\alpha^n} C_n,$$

Thus, $F = f_n(x,y) + f_{n-1}(x,y) + \cdots + f_0(x,y)$.
Hence, $F = C_n(x+y)^n - \frac{n\mu}{\alpha} C_n(x+y)^{n-1} + \frac{\mu^2 n(n-1)}{2!\,\alpha^2} C_n(x+y)^{n-2} - \frac{\mu^3 n(n-1)(n-2)}{3!\,\alpha^3} C_n(x+y)^{n-3} + \cdots + \frac{\mu^{n-1} n(n-1)\ldots 2}{(n-1)!\,\alpha^{n-1}} C_n(x+y) + \frac{\mu^n n(n-1)\ldots 1}{n!\,\alpha^n} C_n$.

By Binomial Theorem, then $F = C_n(x+y-\frac{\mu}{\alpha})^n$, where C_n is an arbitrary constant, with the cofactor $-n\alpha$. Then system (1) has a unique irreducible Darboux polynomial $\alpha x + \alpha y - \mu$ with the cofactor $-\alpha$ if and only if $\eta = \alpha$. □

Proposition 12. System (1) has only two irreducible exponential factors e^{x+y} and $e^{(x+y)^2}$ with cofactors $\mu - \alpha x - \eta y$ and $2(x+y)(\mu - \alpha x - \eta y)$ respectively.
Proof. By Theorem 8, system (1) has the irreducible Darboux polynomial $\alpha x + \alpha y - \mu$ when $\eta = \alpha$. Then in view of Theorem 2, system (1) can have an exponential factor of the form: either $E = \exp(g)$ with $g \in \mathbb{C}[x,y]$, or only when $\eta = \alpha$, $E = e^{\frac{g(x,y)}{(x+y-\frac{\mu}{\alpha})^s}}$ with $s \geq 1$ and such that $g \in \mathbb{C}[x,y]$ is co-prime with $x + y - \frac{\mu}{\alpha}$ and the degree of g no more than s.

We first prove that system(1) with $\eta = \alpha$ has no exponential factors of the form $E = e^{\frac{g(x,y)}{(x+y-\frac{\mu}{\alpha})^s}}$.

Suppose that system (1) with $\eta = \alpha$ has an exponential factor of the form $E = e^{\frac{g(x,y)}{(x+y-\frac{\mu}{\alpha})^s}}$ with $s \geq 1$ such that $x + y - \frac{\mu}{\alpha}$ is co-prime with $g \in \mathbb{C}[x,y]$. In view of Theorem 3, we can assume that $s = 1$ and that g has degree at most one (note that here $g_1 = x + y - \frac{\mu}{\alpha}$ has degree one). We write g has a polynomial of degree one in the variables x and y as follows:

$$g(x,y) = a_1 x + a_2 y + a_3,$$

where a_1, a_2 and a_3 are constant.
Clearly g satisfies

$$X(E) = (\mu - xy^2 - x(r+\alpha))\frac{\partial E}{\partial x} + (rx + xy^2 - \alpha y)\frac{\partial E}{\partial y} = LE,$$

then

$$\left(\mu - xy^2 - x(r+\alpha)\right)\frac{\partial}{\partial x}\left(\frac{g(x,y)}{\left(x+y-\frac{\mu}{\alpha}\right)}\right) + (rx + xy^2 - \alpha y)\frac{\partial}{\partial y}\left(\frac{g(x,y)}{\left(x+y-\frac{\mu}{\alpha}\right)}\right) = L,$$

where L is a cofactor of exponential factor E.
As a simply version of the above equation, it can be converted to;

$$\left(\mu - xy^2 - x(r+\alpha)\right)\frac{\partial g(x,y)}{\partial x} + (rx + xy^2 - \alpha y)\frac{\partial g(x,y)}{\partial y} - \alpha g(x,y) = L(x + y - \frac{\mu}{\alpha}). \quad (22)$$

The cofactor L must be a polynomial of degree 2 and so

$$L = b_0 x^2 + b_1 xy + b_2 y^2 + b_3 x + b_4 y + b_5\,, \quad (23)$$

where b_0, b_1, b_2, b_3, b_4 and b_5 are constant.

From Eq. (22)and using an algebraic manipulator ,

$$\left(\mu - xy^2 - x(r+\alpha)\right)a_1 + (rx + xy^2 - \alpha y)a_2 + \alpha(a_1x + a_2y + a_3) = (b_0x^2 + b_1xy + b_2y^2 + b_3x + b_4y + b_5)(x + y - \frac{\mu}{\alpha}). \quad (24)$$

Comparing the coefficients of Eq. (24), we obtain
$b_0 = b_1 = b_2 = b_3 = b_4 = b_5 = 0$ and $a_1 = a_2$, $a_3 = \frac{-\mu a_1}{\alpha}$ then $g = a_1(x + y - \frac{\mu}{\alpha})$ and $L = 0$. However this is not possible since g is co-prime with $x + y - \frac{\mu}{\alpha}$. Hence this case is not possible.

In summary, system (1) has an exponential factor it should be of the form $E = \exp(g)$ with $g \in \mathbb{C}[x,y]\backslash\mathbb{C}$. In this case, g satisfies

$$\left(\mu - xy^2 - x(r+\alpha)\right)\frac{\partial g(x,y)}{\partial x} + (rx + xy^2 - \alpha y)\frac{\partial g(x,y)}{\partial y} = L, \quad (25)$$

where $L = L(x,y)$ is polynomial of degree two in the variable x and y and that we can take as in Eq. (23).

Let g can be written in the homogeneous form $g(x,y) = \sum_{i=0}^{n} g_i(x,y)$, where each g_i is a homogeneous polynomial in the variables x and y of degree i and $n > 0$.
Using the terms of degree $n+2$ with $n \geq 3$ in Eq. (25), we get

$$-xy^2\frac{\partial g_n(x,y)}{\partial x} + xy^2\frac{\partial g_n(x,y)}{\partial y} = 0 .$$

We may obtain by solving the partial differential equation given above
$g_n(x,y) = G_n(x+y)$,
where G_n is an arbitrary polynomial of $x+y$.

Since $g_n(x,y)$ is a homogeneous polynomial of degree n, then it should be in the following form

$$g_n(x,y) = C_n(x+y)^n,$$

where C_n is an arbitrary constant.
Calculating the terms of degree $n+1$ with $n \geq 3$ in Eq. (25) we obtain

$$-xy^2\frac{\partial g_{n-1}(x,y)}{\partial x} + xy^2\frac{\partial g_{n-1}(x,y)}{\partial y} = 0 .$$

By solving the above-mentioned partial differential equation, we can obtain
$g_{n-1}(x,y) = G_{n-1}(x+y)$,
where G_{n-1} is an arbitrary polynomial of $x+y$.
Since $g_{n-1}(x,y)$ is a homogeneous polynomial of degree $n-1$, then it should be in the following form

$$g_{n-1}(x,y) = C_{n-1}(x+y)^{n-1},$$

where C_{n-1} is an arbitrary constant.

Calculating the terms of degree n with $n \geq 3$ in Eq. (25) we obtain

$$-xy^2\frac{\partial g_{n-2}(x,y)}{\partial x} - x(r+\alpha)\frac{\partial g_n(x,y)}{\partial x} + (rx - \alpha y)\frac{\partial g_n(x,y)}{\partial y} + xy^2\frac{\partial g_{n-2}(x,y)}{\partial y} = 0.$$

We may obtain by solving the partial differential equation given above

$$g_{n-2}(x,y) = -n\alpha C_n(ln(x) - ln(-y))(x+y)^{n-2} - n\alpha C_n \frac{(x+y)^{n-1}}{y} + G_{n-2}(x+y),$$

where G_{n-2} is an arbitrary polynomial of $x+y$.

Since $g_{n-2}(x,y)$ is a polynomial of degree $n-2$, $n>1$ and $\alpha>0$ then $C_n=0$, hence $g_n(x,y)=0$.

In contrast to the fact that $n \geq 3$, this results in g being a constant. Then $n \leq 2$ is required. In this situation, g is a polynomial of degree two in variables x and y that we write it as

$$g(x,y) = A_1x^2 + A_2xy + A_3y^2 + A_4x + A_5y + A_6,$$

where A_1, A_2, A_3, A_4, A_5 and A_6 are constant. Then, by equation (25), we obtain that $\big(\mu - xy^2 - x(r+\alpha)\big)(2A_1x + A_2y + A_4) + (rx + xy^2 - \alpha y)(A_2x + 2A_3y + A_5) = b_0x^2 + b_1xy + b_2y^2 + b_3x + b_4y + b_5$.

Comparing the coefficients, we obtain
$A_1 = A_3$, $A_2 = 2A_3$,$A_4 = A_5$, $b_0 = b_1 = -2A_2\alpha$, $b_2 = -A_2\alpha$, $b_3 = b_4 = A_2\mu - A_4\alpha$ and $b_5 = A_4\mu$.

That is $g(x,y) = A_6 + A_5(x+y) + A_3(x+y)^2$.

This implies that $e^{A_3(x+y)^2 + A_5(x+y) + A_6}$ is the exponential factor with cofactor $A_2\big(-2x(x+y) - \alpha + \mu(x+y)\big) - A_4(\alpha(x+y) - \mu)$. Then $exp(x+y)$ and $exp(x+y)^2$ are only exponential factors of system (1) with cofactors $\mu - \alpha x - \eta y$ and $2(x+y)(\mu - \alpha x - \eta y)$ respectively. □

We now prove the result related with Darboux F.I. of system (1).

Theorem 13. System (1) has not Darboux F.I. .
Proof. By Theorem 8 and Proposition 12, if $\eta = \alpha$ then $x + y - \frac{\mu}{\alpha}$ is the unique invariant algebraic curve of system (1) with the cofactor $k = \alpha$ with $e^{x+y}$and $e^{(x+y)^2}$ are the unique exponential factors of system (1) with cofactors $L_1 = \mu - \alpha x - \alpha y$ and $L_2 = 2(x+y)(\mu - \alpha x - \alpha y)$ respectively. From Proposition 5, system (1) has Darboux first integral if and only if

$$\lambda k + \mu_1 L_1 + \mu_2 L_2 = 0,\ \lambda, \mu_1, \mu_2 \in \mathbb{C}.$$

Then

$$\lambda\alpha + \mu_1(\mu - \alpha x - \alpha y) + \mu_2(2(x+y)(\mu - \alpha x - \alpha y)) = 0.$$

Comparing the coefficients, we obtain the following equations,

$$\lambda\alpha + \mu_1\mu = 0,$$
$$-\alpha\mu_1 + 2\mu_2\mu = 0,$$
$$-\mu_1\alpha + 2\mu_2\mu = 0,$$
$$-2\alpha\mu_2 = 0,$$
$$-2\alpha\mu_2 - 2\alpha\mu_2 = 0,$$
$$-2\alpha\mu_2 = 0,$$

since the parameter α is positive real constant, then $\lambda = \mu_1 = \mu_2 = 0$, by Proposition 5, the system (1) has no Darboux F.I. . □

Let us now prove that system (1) does not have an integrating factor of Darboux type as a consequence of Theorems 6, 8 and Proposition 5.

Corollary 14. System (1) has no integrating factors of Darboux type.
Proof. By Theorem 8, system (1) has only one Darboux polynomial if and only if $\eta = \alpha$, in which case it is $x + y - \frac{\mu}{\alpha}$ with the cofactor $k = -\alpha$, and by Proposition 12, System (1) has only two exponential factors e^{x+y} with cofactor $L_1 = \mu - \alpha x - \eta y$ and $e^{(x+y)^2}$ with cofactor $L_2 = 2(x+y)(\mu - \alpha x - \eta y)$ respectively. In order that system (1) has a Liouvillian F.I., by Theorem 6, system (1) must have an integrating factor of Darboux type. From Proposition 5, system (1) has an integrating factor of Darboux-type if and only if

$$\lambda k + \mu_1 L_1 + \mu_2 L_2 = -div(P,Q),$$
$$-\alpha\lambda + \mu_1(\mu - \alpha x - \eta y) + \mu_2\big(2(x+y)(\mu - \alpha x - \eta y)\big) = 2xy - y^2 - r - 2\alpha,$$

where $\lambda, \mu_1, \mu_2 \in \mathbb{C}$. Since $\alpha, \eta, r > 0$, then above equation has no solution. Hence system (1) has no integrating factor of Darboux type, then the result follows directly by Proposition 5. □

As a consequence of Corollary 14 and Theorem 6, we obtain directly the main result.
Theorem 15. System (1) has no Liovillian F.I. .

Proposition 16. System (1) has a unique irreducible Darbox invariant.
Proof. By Theorem 8, if $\eta = \alpha$ the $x + y - \frac{\mu}{\alpha}$ is the unique invariant algebraic curve of system (1), with the cofactor $k = \alpha$ and by Proposition 12, e^{x+y} and $e^{(x+y)^2}$ are the unique exponential factors of system (1) with cofactors $L_1 = \mu - \alpha x - \alpha y$ and $L_2 = 2(x+y)(\mu - \alpha x - \alpha y)$ respectively. From Proposition 5, system (1) has Darboux invariant if and only if

$$\lambda k + \mu_1 L_1 + \mu_2 L_2 = -s, \;\; \lambda, \mu_1, \mu_2 \in \mathbb{C} \; s \in \mathbb{C}\backslash\{0\},$$

then

$$\lambda\alpha + u_1(\mu - \alpha x - \alpha y) + u_2(2(x+y)(\mu - \alpha x - \alpha y)) = -s\,.$$

Comparing the coefficients, we obtain

$$\lambda\alpha + \mu_1\mu = -s,$$
$$-\alpha\mu_1 + 2\mu_2\mu = 0,$$
$$-\mu_1\alpha + 2\mu_2\mu = 0,$$
$$-2\alpha\mu_2 = 0,$$
$$-2\alpha\mu_2 - 2\alpha\mu_2 = 0,$$
$$-2\alpha\mu_2 = 0,$$

since the parameter α is positive real constant, then $\lambda = \frac{-s}{\alpha}$ and $\mu_1 = \mu_2 = 0$, by Proposition 5, then $\left(x + y - \frac{\mu}{\alpha}\right)^{\lambda} e^{-\alpha\lambda t}$, be the unique Darboux invariant of system (1). □

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