DOI: 10.24996/ijs.2019.60.4.19

ISSN: 0067-2904

Investigating Particular Representations for Matrix Lie Groups SO(3) and SL(2,ℂ)

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Abstract
A complexified adjoint representations of the complexification Lie algebras associated with the special orthogonal group SO(3) and special linear group SL(2,ℂ) have been obtained. A new representation of their tensor product is naturally arisen and computed in details.
Keywords: Lie, groups, Complexification of Lie algebras, Tensor of representations.

تدارس تمثيلات خاصة لزمر لي المصفوفيه SO(3) و SL(2,ℂ)

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الخلاصة
تم ايجاد التمثيلات المساعدة المعقدة لجبور لي المرتبطة المعقدة للزمرة المتعامدة الخاصة SO(3) والزمرة الخطية الخاصة SL(2,ℂ). تمثيل جديد لضربهما المكثف نشأ بصورة طبيعية وتم حسابه بالتفصيل.

1. Introduction

Élie Catan introduced new sights on the theory of Lie algebras by showing that complexification of the Lie algebra of a compact group is a complex Lie algebra, which leads to classifying symmetric spaces [1, 2]. Many physical phenomena studied through analyzing their symmetry, one of the pioneer results in this direction is the discovery of Higgs boson. [3].Jonathan and Mich~ le V designed an orbital method to determine a special variety of unitary representation [4]. Moreover; Martin provides three different approaches to exhibit their close relationship to the pointwise tensor product [5].

Let G be any matrix lie group, g *its* associated Lie algebra and *Ad* is the adjoint representation of *G*, then, the adjoint representation *ad* of *g* related with *Ad* through the formula: $Ad(e^x) = e^{ad(x)}$ for each $x \in g$, which provides a tool to transfer information between Lie groups and Lie algebras[6]. An attempt has been made to compute the adjoint representations for the complexification of the associated Lie algebras $so(3)_{\mathbb{C}}$, and $sl(2,\mathbb{C})_{\mathbb{C}}$ of the matrix Lie groups SO(3) and SL(2,ℂ) respectively.

Seeking for new irreducible representations, tensor product representation of the tensor product Lie algebras $so(3)_{\mathbb{C}} \otimes sl(2,\mathbb{C})_{\mathbb{C}}$, has been computed in details.

2. Notations and preliminaries

Throughout, we adopt the standard notations and definitions of matrix lie group, matrix lie algebra and their representation's. For example, see [6].

Consider the basis $\{F_i\}_{i=1}^{3}$ for the special orthogonal matrix Lie algebra *so(3)* where;

$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ and $F_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ With commutation

Relations: $$[F_1, F_2] = F_3, [F_2, F_3] = F_1 \text{ and } [F_3, F_1] = F_2 \tag{2.1}$$

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Also, $\{X_i\}_{i=1}^{3}$ form a basis for the special linear matrix Lie algebra $sl(2,₵)$ where;
$X_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ With commutation relations:

$$[X_1, X_2] = 2X_2, [X_1, X_3] = -2X_3 \ and \ [X_2, X_3] = X_1 \quad (2.2)$$

Definition 2.1 For any Lie algebra $\boldsymbol{g}$, we define its complexification by: $\boldsymbol{g}_{₵} = \{x + iy | x \in \boldsymbol{g}, y \in \boldsymbol{g}\}$.
Moreover, every finite dimensional complex representation Φ of $\boldsymbol{g}$ can be extended to $\boldsymbol{g}_{₵}$ by:

$$\Phi_{₵}(x + iY) = \Phi(x) + i\Phi(y)$$

Definition 2.2 Let $\boldsymbol{g}$ be an arbitrary lie algebra, the adjoint representation $\boldsymbol{ad}$ of $\boldsymbol{g}$ is a lie algebra homomorphism $\boldsymbol{ad}: \boldsymbol{g} \to \mathbf{gl}(\boldsymbol{g}),\ for\ x \in \boldsymbol{g}$ defined by:

$$\boldsymbol{ad_x}(y) = [x, y]\ \forall\ y \in \boldsymbol{g}$$

Our aim is to find the tensor adjoint representation of the complexification matrix Lie algebras associated to *so(3)* and sl(2,ℂ).

3. Complexification representation of $\boldsymbol{so(3)_{₵}}$

Definition 3.1 The special orthogonal Lie algebra $so(3)$ associated to the special orthogonal Lie group SO(3) is the set of all 3×3 real traceless matrices x such that $x^{tr} = -x$ and its adjoint representation is;

$$\boldsymbol{ad_x}: \boldsymbol{so(3)} \to \mathbf{gl}(\boldsymbol{so(3)})$$

Let $x \in \boldsymbol{so(3)}$ then $x = \alpha F_1 + \beta F_2 + \gamma F_3$ for some $\alpha, \beta, \gamma \in R$ **(*R* is the set of all real numbers)**
For any $y \in \boldsymbol{so(3)}$ $\exists c_i \in R$, $i = 1,2,3$ such that $y = c_1F_1 + c_2F_2 + c_3F_3$, by definition 2.2 we have;
$\boldsymbol{ad_x}(y) = [x, y] = [\alpha F_1 + \beta F_2 + \gamma F_3, c_1F_1 + c_2F_2 + c_3F_3]$, from the properties of Lie bracket [,] the right expression take the form;
$\boldsymbol{ad_x}(y) = (\alpha c_2 - \beta c_1)F_3 + (\gamma c_1 - \alpha c_3)F_2 + (\beta c_3 - \gamma c_2)F_3$ Which can be simplify to get:

$$\boldsymbol{ad_x}(y) = \begin{pmatrix} 0 & \beta c_1 - \alpha c_2 & \gamma c_1 - \alpha c_3 \\ \alpha c_2 - \beta c_1 & 0 & \gamma c_2 - \beta c_3 \\ \alpha c_3 - \gamma c_1 & \beta c_3 - \gamma c_2 & 0 \end{pmatrix} \quad (3.1)$$

Now, $\boldsymbol{ad}_{₵}: \boldsymbol{so(3)}_{₵} \to \boldsymbol{gl}(\boldsymbol{so(3)}_{₵})$ can be written by definition 2.1 as follows:
Let $\psi = x + iy \in \boldsymbol{so(3)}_{₵}$, for each $z \in so(3)_{₵}$

$$\boldsymbol{ad_{₵z}(\psi) = ad_z(x) + i\, ad_z(y) = [z, x] + i[z, y]}$$

Since the basis of a Lie algebra $\boldsymbol{g}$ can be considered as a basis for its complexification $\boldsymbol{g}_{₵}$ the element z can be written as: $z = r_1F_1 + r_2F_2 + r_3F_3$ for some $r_i \in R, i = 1,2,3$. Therefore, using (3.1) we have;

$$\boldsymbol{ad_{₵z}(\psi)} = \begin{pmatrix} 0 & \alpha r_2 - \beta r_1 & \alpha r_3 - \gamma r_1 \\ \beta r_1 - \alpha r_2 & 0 & \beta r_3 - \gamma r_2 \\ \gamma r_1 - \alpha r_3 & \gamma r_2 - \beta r_3 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & r_2c_1 - r_1c_2 & r_3c_1 - r_1c_3 \\ r_1c_2 - r_2c_1 & 0 & r_3c_2 - r_2c_32 \\ r_1c_3 - r_3c_1 & r_2c_3 - r_3c_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \omega_1 & \omega_2 \\ \omega_3 & 0 & \omega_4 \\ \omega_5 & \omega_6 & 0 \end{pmatrix} \text{ where}$$

$$\left.\begin{aligned} &\omega_1 = (\alpha r_2 - \beta r_1) + i(r_2c_1 - r_1c_2), \omega_2 = (\alpha r_3 - \gamma r_1) + i(r_3c_1 - r_1c_3), \\ &\omega_3 = (\beta r_1 - \alpha r_2) + i(r_1c_2 - r_2c_1), \omega_4 = (\beta r_3 - \gamma r_2) + i(r_3c_2 - r_2c_3), \\ &\omega_5 = (\gamma r_1 - \alpha r_3) + i(r_1c_3 - r_3c_1), \omega_6 = (\gamma r_2 - \beta r_3) + i(r_2c_3 - r_3c_2). \end{aligned}\right\} \quad (3.2)$$

4. Complexification representation of $\boldsymbol{sl(2,₵)_{₵}}$

Definition 4.1 The special linear Lie algebra sl(2,ℂ) associated to the special linear group SL(2,ℂ) is the set of all 3×3 complex traceless matrices, for $A \in \mathrm{sl}(2,₵)$ its adjoint representation is given by;

$$\boldsymbol{ad_A}: \mathbf{sl(2,₵)} \to \mathbf{gl}(\mathbf{sl(2,₵)})$$

The element A can be written as a linear combination of basis elements, that is;
$A = s_1X_1 + s_2X_2 + s_3X_3$ and for each $B \in \mathrm{sl}(2,₵)$, $B = k_1X_1 + k_2X_2 + k_3X_3$ for some $s_i, k_i \in ₵$, $i = 1,2,3$

$$ad_A(B) = [A, B] = [s_1X_1 + s_2X_2 + s_3X_3, k_1X_1 + k_2X_2 + k_3X_3]$$

$$ad_A(B) = (s_2k_3 - s_3k_2)X_1 + 2(s_1k_2 - s_2k_1)X_2 + 2(s_3k_1 - s_1k_3)X_3$$

$$ad_A(B) = \begin{pmatrix} s_2k_3 - s_3k_2 & 2(s_1k_2 - s_2k_1) \\ 2(s_3k_1 - s_1k_3) & s_3k_2 - s_2k_3 \end{pmatrix} \quad (4,1)$$

Now, the complexification representation $\boldsymbol{ad}_{₵}: \mathbf{sl(2,₵)} \to gl(\mathbf{sl(2,₵)})$ can be written using definition 2.1 as follows:

For each $\mu, \tau = A + iB \in \text{sl}(2, ₵)$,$\boldsymbol{ad_{₵\mu}(\tau) = ad_{\mu}(A) + i\, ad_{\mu}(B) = [\mu, A] + i[\mu, B]}$

Note that μ can be written as $\mu = h_1X_1 + h_2X_2 + h_3X_3$ for some $h_i \in ₵$, $i = 1,2,3$, from (4.1) we have;

$$\boldsymbol{ad_{₵\mu}(\tau)} = \begin{pmatrix} h_2s_3 - h_3s_2 & 2(h_1s_2 - h_2s_1) \\ 2(h_3s_1 - h_1s_3) & h_3s_2 - h_2s_3 \end{pmatrix} + i\begin{pmatrix} h_2k_3 - h_3k_2 & 2(h_1k_2 - h_2k_1) \\ 2(h_3k_1 - h_1k_3) & h_3k_2 - h_2k_3 \end{pmatrix}$$

$\boldsymbol{ad_{₵\mu}(\tau)} = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{pmatrix}$ Where

$$\left.\begin{aligned} \sigma_1 &= h_2s_3 - h_3s_2 + i(h_2k_3 - h_3k_2), \\ \sigma_2 &= 2(h_1s_2 - h_2s_1) + i2(h_1k_2 - h_2k_1), \\ \sigma_3 &= 2(h_3s_1 - h_1s_3) + i2(h_3k_1 - h_1k_3), \\ \sigma_4 &= h_3s_2 - h_2s_3 + i(h_3k_2 - h_2k_3). \end{aligned}\right\} \quad (4.2)$$

5. Tensor representation of $\boldsymbol{so(3)_₵ \otimes sl(2, ₵)_₵}$

Definition 5.1 Tensor product of two representations say, ρ_1 ,ρ_2 of Lie algebras g_1, g_2 respectively denoted by $\rho_1 \otimes \rho_2$ is a representation of their direct sum $g_1 \oplus g_2$ defined by:

$\rho_1 \otimes \rho_2(\theta, \lambda) = \rho_1(\theta) \otimes I + I \otimes \rho_2(\lambda)$ for all $\theta \in g_1$, $\lambda \in g_2$.

Therefore, considering the complexification representations $\boldsymbol{ad_{₵z}}$ and $\boldsymbol{ad_{₵\mu}}$ founded in section 3 and 4, we get the tensor representation $\boldsymbol{ad_{₵z} \otimes ad_{₵\mu}}$ of $\boldsymbol{so(3)_₵ \oplus sl(2, ₵)_₵}$ as follows;

$$\boldsymbol{ad_{₵z} \otimes ad_{₵\mu}(\psi, \tau)} = \boldsymbol{ad_{₵z}}(\psi) \otimes I + I \otimes \boldsymbol{ad_{₵\mu}(\tau)}$$

$$= \begin{pmatrix} 0 & \omega_1 & \omega_2 \\ \omega_3 & 0 & \omega_4 \\ \omega_5 & \omega_6 & 0 \end{pmatrix} \otimes I_2 + I_3 \otimes \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_3 & \sigma_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \omega_1 & 0 & \omega_2 & 0 \\ 0 & 0 & 0 & \omega_1 & 0 & \omega_2 \\ \omega_3 & 0 & 0 & 0 & \omega_4 & 0 \\ 0 & \omega_3 & 0 & 0 & 0 & \omega_4 \\ \omega_5 & 0 & \omega_6 & 0 & 0 & 0 \\ 0 & \omega_5 & 0 & \omega_6 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \sigma_1 & \sigma_2 & 0 & 0 & 0 & 0 \\ \sigma_3 & \sigma_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_1 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & \sigma_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1 & \sigma_2 \\ 0 & 0 & 0 & 0 & \sigma_3 & \sigma_4 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1 & \sigma_2 & \omega_1 & 0 & \omega_2 & 0 \\ \sigma_3 & \sigma_4 & 0 & \omega_1 & 0 & \omega_2 \\ \omega_3 & 0 & \sigma_1 & \sigma_2 & \omega_4 & 0 \\ 0 & \omega_3 & \sigma_3 & \sigma_4 & 0 & \omega_4 \\ \omega_5 & 0 & \omega_6 & 0 & \sigma_1 & \sigma_2 \\ 0 & \omega_5 & 0 & \omega_6 & \sigma_3 & \sigma_4 \end{pmatrix}$$

Where $\omega_i, \sigma_j \in ₵ \ \forall\ 1 \le i \le 6, 1 \le j \le 4$ described in (3.2) and (4.2).

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