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The Operational Matrices Methods for Solving Falkner-Skan Equations

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Abstract
The method of operational matrices is based on the Bernoulli and Shifted Legendre polynomials which is used to solve the Falkner-Skan equation. The nonlinear differential equation converting to a system of nonlinear equations is solved using Mathematica®12, and the approximate solutions are obtained. The efficiency of these methods was studied by calculating the maximum error remainder (MER_m), and it was found that their efficiency increases as m increases. Moreover, the obtained approximate solutions are compared with the numerical solution obtained by the fourth-order Runge-Kutta method (RK4), which gives a good agreement.

Keywords: Falkner-Skan equation; Bernoulli polynomial; Legendre polynomial; operational matrix.

طرق المصفوفات التشغيلية لحل معادلات فالكنر–سكان

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الخلاصة
تم استخدام طريقة المصفوفات التشغيلية المستندة على متعددة حدود برنولي ومتعددة حدود لجندر المزاحة لحل معادلة فالكنر–سكان. يتم تحويل المعادلة التفاضلية غير الخطية الى نظام معادلات غير خطية والتي تحل باستخدام برنامج Mathematica®12, ويتم الحصول على الحلول التقريبية. تمت دراسة كفاءة هذه الطرق من خلال حساب الحد الاقصى للخطأ متبقي MER_m, ووجد ان كفاءتها تزداد مع زيادة m. بالاضافة الى ذلك, تتم مقارنة الحلول التقريبية التي تم الحصول عليها مع الحل العددي الذي تم الحصول عليه بواسطة طريقة رانج–كوتا من الدرجة الرابعة RK4, والتي تعطي توافقات جيدة.

1 Introduction

Nonlinear ordinary differential equations (NODEs) arise in many areas of engineering and applied science, such as chemistry, electrical circuits and fluid mechanics [1-2]. Therefore, it has become necessary to find approximate or numerical solutions to these types of equations. Some of these solutions are obtained by using iterative methods, such as the variational

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iteration method (VIM) [3], the Daftardar-Jafari method (DJM) [4], the Temimi-Ansari method (TAM) [5-6] and the modified decomposition method [7].
Also, there are other methods for solving the NODEs such as the extended differential transform method [8], the wavelet homotopy analysis method [9], the Runge-Kutta method [10] and the optimal Galerkin-homotopy asymptotic method [11]. In addition, the weighted residuals methods [12] are also used.

There are some applications in NODEs that have been solved by different methods, for example, the Emden-Fowler equations have been solved by the VIM [13]. The Jeffery-Hamel flow problem was solved using three iterative methods, namely: DJM, TAM and Banach contraction method (BCM) [14]. The quadratic Riccati differential equations were solved by the fourth-order Runge-Kutta method [15]. The nonlinear system of the smoking habit model was studied by analytical methods such as ADM and VIM, and numerically by the finite difference method, and the Runge-Kutta method [16]. Also, the epidemic model SJR was solved using DJM, TAM and BCM [17].

One of the approximation methods consists in simplifying the unknown function based on orthogonal polynomials and operational matrices (OM) so that it becomes a system of algebraic equations that can be easily solved. Many works have explored this method using OM based on some polynomials, for example, the Legendre polynomial [18], the Chebyshev polynomial [19], the Genocchi polynomial [20], the Bernoulli polynomial [21], the Bernstein polynomial [22] and the Wang-Ball polynomial [23].

Among the NODEs, especially the boundary value problems, there are the so-called boundary layer flow problems (BLFP), which occur in industrial processes such as metal plate cooling, plastic film drawing, and glass and polymer research [24-25].

The most famous application of BLFP is the Falkner-Skan equation, which has been solved by many researchers for example Al-Jawary and Adwan [26] use three iterative methods, namely the TAM, the DJM, and the BCM. Al-Jawary and Ibraheem [27] implemented two methods for operational matrices based on Bernstein and Chebyshev polynomials. Also, Yun [28] used a new iterative method. Temimi and Ben-Romdhane [29] also proposed an iterative finite difference method. Finally, Elnady et al. [30] used the Chebyshev series in matrix form.
The main aim of this paper is to use the Bernoulli operational matrix (BrOM) and the shifted Legendre operational matrix (LOM) to solve the Falkner-Skan equation.
This paper is organized as follows: In Section 2, the Falkner-Skan equation is introduced. In Section 3, the orthogonal polynomials based on the Bernoulli polynomial and the Legendre polynomial are given. In Section 4, the Falkner-Skan equation is solved by the proposed methods and the results are compared approximately with the Bernstein operational matrix method and numerically with the fourth-order Runge-Kutta method. Finally, the conclusions are presented in Section 5.

2 The Falkner-Skan Equation

The Falkner-Skan equation is one of the 3^{rd} order NODEs. It was first discussed by Falkner and Skan in 1931 [31]. It plays an important role in fluid mechanics and in the development of boundary layer theory in it. It has many applications, such as aerodynamic extrusion of plastic sheets, insulating materials and polymer studies, where this is done on a continuously extended surface moving parallel to its plane and with a constant traction velocity [26], which can be defined as follows:

$$y'''(x) + y(x)\, y''(x) + \beta\, [\epsilon^2 - (y'(x))^2] = 0, \qquad \dots (1)$$

with the following boundary conditions:

$$y(0) = 0,\ y'(0) = 1 - \epsilon,\ y'(\infty) = \epsilon, \qquad \dots (2)$$

Where β is a pressure gradient parameter and ϵ is a velocity ratio parameter. When $\beta=0$, then Eq.(1) is called the Blasius equation, when $\beta=0.5$, then Eq.(1) is called the Homann flow problem, and when $\beta=1$, then Eq.(1) is called the Hiemenz flow problem [27].

In [26] the authors derived the initial condition $y''(0) = -0.832666$ from the boundary condition $y'(\infty)$ by applying the Padé approximation, this value will be used in this paper.

3 Orthogonal polynomials

Orthogonal polynomials play an important role in both pure and applied mathematics, as well as in numerical methods [32]. In this section, the Bernoulli and Legendre polynomials are introduced.

3.1 The Bernoulli polynomials (Br-polynomials)

The m^{th}Bernoulli polynomials on the interval [0,1] are defined as follows [33]:

$$Br_m(x) = \sum_{i=0}^{m} \binom{m}{i} Br_i\, x^{m-i}, \qquad \dots (3)$$

Where $Br_i = Br_i(0)$ is the Bernoulli number for all $i = 0,1,\dots,m$. To obtain these numbers, the following formula is used:

$$Br_i = -\sum_{k=1}^{i+1} \frac{(-1)^k}{k} \binom{i+1}{k} \sum_{j=1}^{k} j^i, \qquad \dots (4)$$

For $i \geq 0, i \neq 1$. If $i = 1$, then $Br_1 = -\frac{1}{2}$.

Due to the use of Br-polynomials in many branches of mathematics, this leads to an interest in the study of their properties such as derivation and integration, which are included in the following formulas:

$\frac{dBr_n(x)}{dx} = nBr_{n-1}(x), n \geq 1,$

and $\int_a^z Br_n(x)\, dx = \frac{Br_{n+1}(z) - Br_{n+1}(a)}{n+1}$.

Moreover, any $y(x) \in L^2[0,1]$ can be easily approximated by the linear combination of Br-polynomials as follows:

$$y(x) = \sum_{i=0}^{m} c_i\, Br_i(x) = C^T\, Br(x), \qquad \dots (5)$$

where $C^T = [c_0, c_1, \dots, c_m]$ and $Br(x) = [\, Br_0(x), Br_1(x), \dots, Br_m(x)]^T$.

While the m^{th} derivative of $Br(x)$ is:

$$\frac{d^m Br(x)}{dx^m} = (D_{Br})^m\, Br(x), \qquad \dots (6)$$

Where $D_{Br} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m & 0 \end{bmatrix}$ is the $(m+1) \times (m+1)$ OM of the derivative.

Therefore, the m^{th} derivative of $y(x)$ will be:

$$y^{(m)}(x) = C^T (D_{Br})^m\, Br(x). \qquad \dots (7)$$

After writing $y(x)$ and its derivatives in terms of operational matrices, put them into the ODE and its conditions. Also, replace each x in it with appropriate points computed from Chebyshev roots, called collocation nodes, defined as follows:

$$x_r = \frac{1}{2}\, (cos\frac{r\,\pi}{m} + 1),\ \ r = 1,\dots,m-1. \qquad \dots (8)$$

A system of algebraic equations then emerges that can be solved using computer programs such as MATLAB or MATHEMATICA. Thus, finding the value of the vector C^T in Eq.(5) leads to an approximate solution.

3.2 The Legendre polynomials (L-polynomials)

The Legendre polynomials of m^{th} order defined on the interval [-1,1] as [34]:

$$L_0(t) = 1,\ L_1(t) = t,\ \ldots,\ L_{m+1}(t) = \frac{(2\,m+1)\,t\,L_m(t) - m\,L_{m-1}(t)}{m+1},\ m = 1,2,\ldots. \quad \ldots (9)$$

Inserting $t = 2x - 1$, we obtained L-polynomials of m^{th} order defined on the interval [0,1], they are called shifted L-polynomials, denoted by $P_m(x)$ and calculated as follows:

$$P_0(x) = 1,\ P_1(x) = 2x - 1,\ \ldots,\ P_{m+1}(x) = \frac{(2m+1)(2x-1)\,P_m(x) - m\,P_{m-1}(x)}{m+1}, m = 1,2,\ldots. \quad (10)$$

Furthermore, the shifted L-polynomial can be used to describe any function $y(x) \in L^2[0,1]$ as follows:

$$y(x) = \sum_{i=0}^{\infty} c_i\,P_i(x). \quad \ldots (11)$$

In practice, only the first $(m+1)$ terms from Eq.(11) are considered, thus:

$$y(x) = \sum_{i=0}^{m} c_i\,P_i(x) = C^T\phi(x), \quad \ldots (12)$$

where $C^T = [c_0, c_1, \ldots, c_m], \phi(x) = [P_0(x), P_1(x), \ldots, P_m(x)]^T$.

The derivatives of $\phi(x)$ can be defined as:

$$\frac{d^m\phi(x)}{dx^m} = (D_L)^m\phi(x), \quad \ldots (13)$$

where D_L is the $(m + 1) \times (m + 1)$ OM of the derivative, defined as:

$$D_L = (d_{ij}) = \begin{cases} 2\,(2j+1), & j = i - k, \quad \begin{cases} k = 1,3,\ldots,m, & \text{if } m \text{ odd}, \\ k = 1,3,\ldots,m-1, & \text{if } m \text{ even}, \end{cases} \\ 0 & \text{otherwise}. \end{cases} \quad \ldots (14)$$

Therefore, the derivatives of $y(x)$can be approximated as follows:

$$y^{(m)}(x) = C^T(D_L)^m\,\phi(x). \quad \ldots (15)$$

Then, complete the solution by finding the value of the vector C^T in Eq.(12), using the same procedures as the previous method.

4 Numerical results and discussions

In this section, the proposed BrOM and Shifted LOM methods are applied to solve the Falkner-Skan equation:

$$y'''(x) + y(x)\,y''(x) + \beta\,[\epsilon^2 - (y'(x))^2] = 0, \quad \ldots (16)$$

with the following conditions: $y(0) = 0,\ y'(0) = 1 - \epsilon,\ y''(0) = -0.832666.$ … (17)

Also, a comparison is made with the Bernstein operational matrix method available in the literature from [27].

To solve this equation using the BrOM method, we follow the explanation in Section 3.1 with $m = 3,\ \beta = 0.5,$ and $\epsilon = 0.1$, to illustrate the technique of this method, we assume the approximate solution as:

$$y(x) = c_0 Br_0(x) + c_1 Br_1(x) + c_2 Br_2(x) + c_3 Br_3(x) = C^T\,Br(x), \quad \ldots (18)$$

Where $Br_0(x) = 1,\ Br_1(x) = x - \frac{1}{2},\ Br_2(x) = x^2 - x + \frac{1}{6},\ Br_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$

Here, from Eq.(6) we have the OM:

$$D_{Br} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix},\ (D_{Br})^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix},\ (D_{Br})^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}.$$

After converting each $y(x)$ and its derivatives in Eqs.(16) and (17) in terms of operational matrices, we substitute the collocation nodes Eq.(8) instead of each x, giving us the following system of nonlinear algebraic equations:

$0.005 - 0.5\,c_1^2 + 2c_0c_2 - 0.16667\,c_2^2 + 6c_3 + 1.5c_0c_3 + 0.4375\,c_1c_3 - 0.09375\,c_2c_3 - 0.072265625c_3^2 = 0,$

$c_0 - \frac{c_1}{2} + \frac{c_2}{6} = 0,$

$c_1 - c_2 + \frac{c_3}{2} = 0.9,$

$2c_2 - 3c_3 = -0.832666.$ … (19)

Finally, by solving (19) we obtain:

$c_0 = 0.325123,\ \ c_1 = 0.53927,\ \ c_2 = -0.332929,\ \ c_3 = 0.055603.$

So we can write the approximate solution:

$$y(x) = [0.325123, \quad 0.53927, \quad -0.332929, \quad 0.055603]\begin{bmatrix} 1 \\ x - \frac{1}{2} \\ x^2 - x + \frac{1}{6} \\ x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \end{bmatrix},$$

$$= -6.938894 \times 10^{-18} + 0.9x - 0.416333x^2 + 0.055603x^3.$$

Thus, until $m = 11$, the approximate solution is as follows:

$y(x) = -2.054732523449481 \times 10^{-17} + 0.8999999999999999x - 0.41633300000000006x^2 + 0.06666669642395842x^3 - 2.287188864465855 \times 10^{-7}x^4 - 0.0029989197220697955x^5 + 0.00045927147583849636x^6 + 0.00016795229137371606x^7 - 0.00008385099697427289x^8 + 0.000010664854550909739x^9 + 0.0000015586944340273339x^{10} - 4.715367587551143 \times 10^{-7}x^{11}.$

Moreover, this equation is solved using the shifted LOM method, and the description in Section 3.2 is followed when $m = 3,\ \beta = 0.5,$ and $\epsilon = 0.1$. First, the approximate solution is assumed as follows:

$y(x) = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) = C^T\,\phi(x),$ … (20)

Where $P_0(x) = 1,\ \ P_1(x) = 2x - 1,\ \ P_2(x) = 6x^2 - 6x + 1,$

$P_3(x) = 20x^3 - 30x^2 + 12x - 1.$

From Eq.(14) we have the OM:

$$D_L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 2 & 0 & 10 & 0 \end{bmatrix},\ (D_L)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 \end{bmatrix},\ (D_L)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 120 & 0 & 0 & 0 \end{bmatrix}.$$

After writing $y(x)$ and its derivatives in Eqs.(16) and (17) in terms of operational matrices, the collocation nodes Eq.(8) are substituted instead of each x to obtain the system of nonlinear algebraic equations:

$0.0050000000000000001 - 2.\,c_1^2 + 12c_0c_2 - 6.\,c_2^2 + 120c_3 + 30.\,c_0c_3 + 13.5c_1c_3 - 11.25c_2c_3 - 13.40625c_3^2 = 0,$

$c_0 - c_1 + c_2 - c_3 = 0,$

$2c_1 - 6c_2 + 12c_3 = 0.9,$

$12c_2 - 60c_3 = -0.832666.$ … (21)

Finally, solving (21), we obtain:
$c_0 = 0.325123, \;\; c_1 = 0.266855, \;\; c_2 = -0.055488, \;\; c_3 = 0.00278.$
Thus, the approximate solution is as follows:

$$y(x) = [0.325123, \quad 0.266855, \quad -0.055488, \quad 0.00278] \begin{bmatrix} 1 \\ 2x-1 \\ 6x^2-6x+1 \\ 20x^3-30x^2+12x-1 \end{bmatrix},$$
$$= 1.3877788 \times 10^{-17} + 0.9x - 0.4163329x^2 + 0.055603x^3.$$

Consequently, until $m = 11$ the approximate solution will be as follows:
$y(x) =$
$4.932117349520156 \times 10^{-12} + 0.8999999994721597x - 0.4163329862335926x^2 + 0.06666654435521716x^3 + 6.475009681821894 \times 10^{-7}x^4 - 0.0030018091478452493x^5 + 0.00046486808426081296x^6 + 0.00016190153784680777x^7 - 0.00008115875419114989x^8 + 0.000011665951985557817x^9.$

The exact solution of the Falkner-Skan equation is unknown. Therefore, the maximum error remainder (MER_m) is calculated to determine the accuracy of the proposed method. The MER_m is given by:

$$MERm = max_{0\le x\le 1}|y'''(x) + y(x)\,y''(x) + \beta[\epsilon^2 - (y'(x))^2]|. \qquad \ldots (22)$$

Figure 1 shows the logarithmic plots for MER_m of the approximate solutions obtained by the two proposed methods at all m iterations ($m = 3$ to 11).

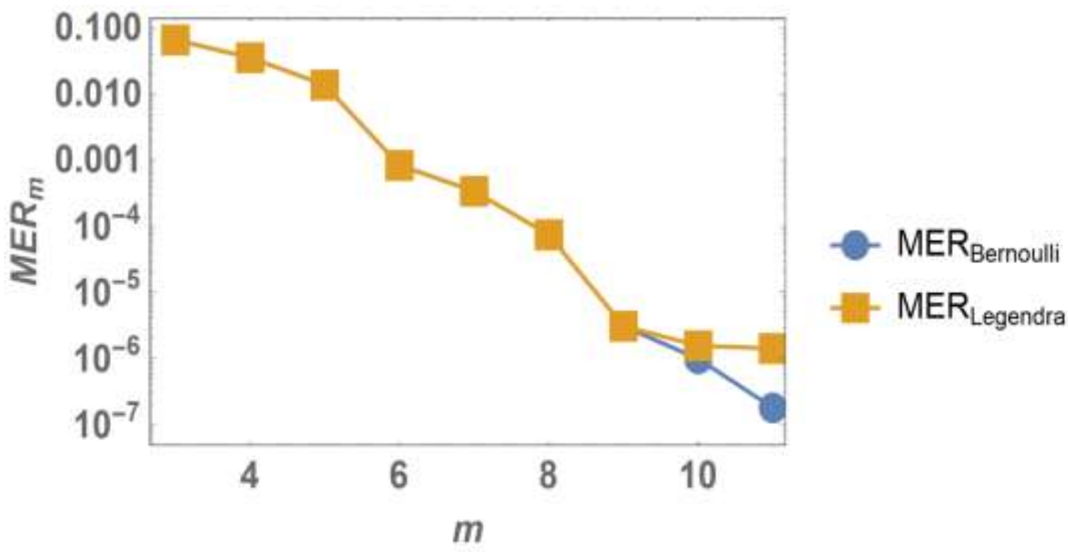

Figure 1: Logarithmic plots for the $MERm$ of BrOM, and LOM when $\beta = 0.5$ *and* $\epsilon = 0.1$.

In addition, Table 1 shows the MER_m values for all m by the proposed methods.

Table 1: The comparison of MER_m between the BrOM and LOM methods for $m = 3$ to11.

m	BrOM	LOM
3	0.06638210579917353	0.06638210579917331
4	0.03585217953692066	0.035852179536900675
5	0.013793149969654195	0.013793149969654916
6	0.0008372318648852684	0.000837231864886101
7	0.00034011248679716	0.0003401124867997707
8	0.0000744189940117268	0.00007441899400839613
9	0.000003072797600423005	0.000003072797582992503
10	$9.87645466521414 \times 10^{-7}$	0.000001560741745953997
11	$1.785437506174325 \times 10^{-7}$	0.000001412178463611191

From Figure 1 and Table 1 it can be concluded that the error values decrease as m increases. Thus, the solution using the BrOM method provides better accuracy of the Falkner-Skan equation than the solution using the shifted LOM method.
If we fix $\beta = 0.5$ and change $\epsilon = 0.1, 0.2, 0.3, 0.4$, we find that the accuracy increases, as shown in Figure 2(a) and Figure 2(b).

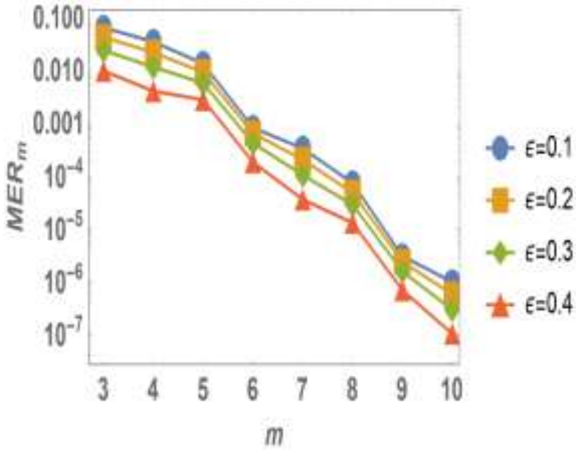

Figure 2(a): Logarithmic plots for the $MERm$ of BrOM when $= 0.5$ and $\epsilon = 0.1, 0.2, 0.3, 0.4$.

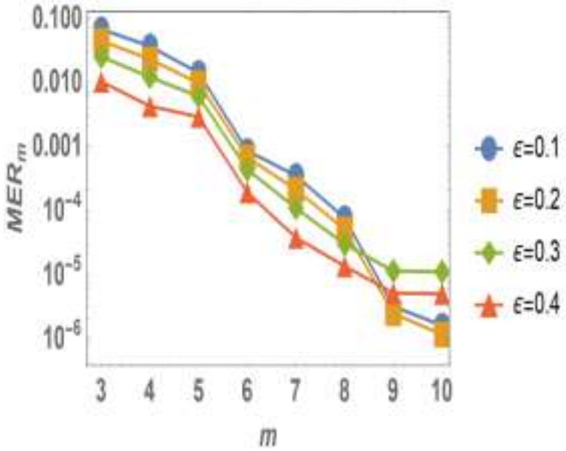

Figure 2(b): Logarithmic plots for the $MERm$ of LOM when $= 0.5$ and $\epsilon = 0.1, 0.2, 0.3, 0.4$.

If we set $\epsilon = 0.1$ and change $\beta = 0.5, 1, 1.5, 2$, the errors will decrease, as shown in Figure 3(a) and Figure 3(b).

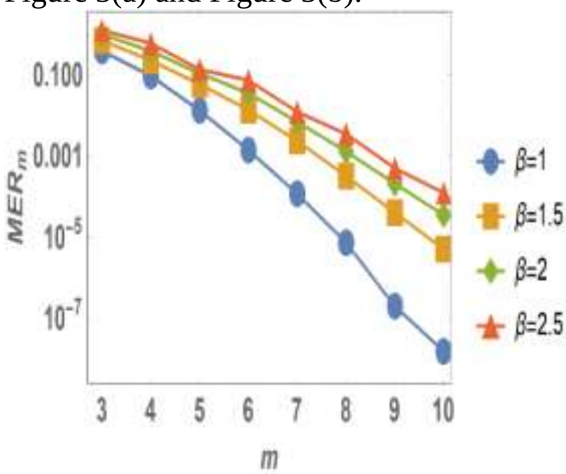

Figure 3(a): Logarithmic plots for the $MERm$ of BrOM when $= 0.1$ *and* $\beta = 0.5, 1, 1.5, 2$.

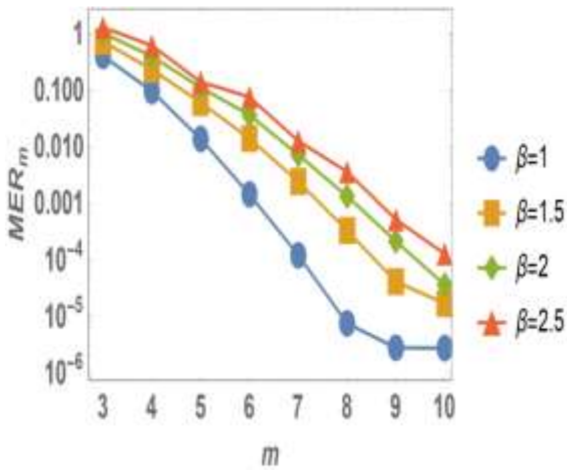

Figure 3(b): Logarithmic plots for the $MERm$ of LOM when $= 0.1$ *and* $\beta = 0.5, 1, 1.5, 2$.

Moreover, a comparison was made between the approximate solutions obtained by the proposed methods for $m = 11$ and the numerical solution obtained by the Range-Kutta method (RK4). This can be seen in Figure 4, which shows good agreement between the two.

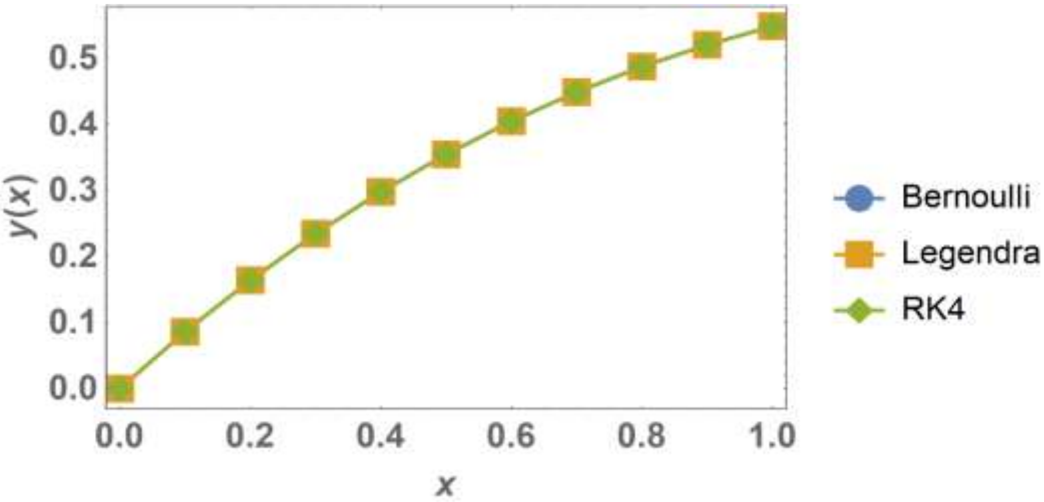

Figure 4: The comparison of the solutions of proposed methods and RK4.

Moreover, AL -Jawary and Ibraheem [27] solved this equation using the Bernstein operational matrix method (BOM). The comparison of the logarithmic plots for MER_m of the proposed methods and the MER_m of BOM at $m=3$ to 11 is shown in Figure 5.

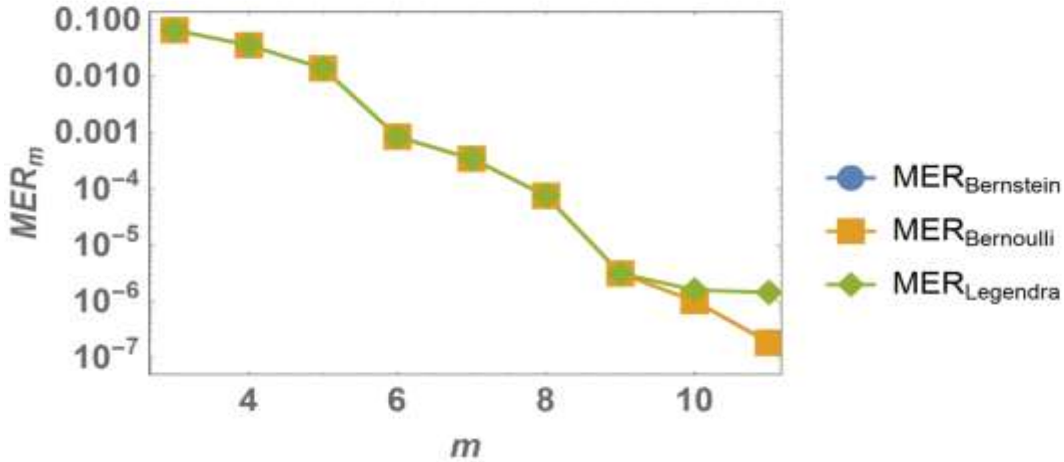

Figure 5: The comparison of the logarithmic plots for the $MERm$ of BOM, BrOM and LOM.

Further investigations were performed and the MER_m values are shown in Table 2. It can be seen that the results of BOM and BrOM are slightly similar, and both are better than LOM.

Table 2: The comparison of $MERm$ between the BOM, BrOM, and LOM methods for $m = 3$ to 11.

m	BOM (Al-Jawary & Ibraheem[27])	BrOM	LOM
3	0.06638210579917359	0.06638210579917353	0.06638210579917331
4	0.03585217953692854	0.03585217953692066	0.035852179536900675
5	0.013793149919500758	0.013793149969654195	0.013793149969654916
6	0.0008372318648590671	0.0008372318648852684	0.000837231864886101
7	0.0003401124904391928	0.000340112486798716	0.0003401124867997707
8	0.0000744189718077104	0.0000744189940117268	0.00007441899400839613
9	0.0000030727767437177	0.000003072797600423005	0.000003072797582992503
10	$9.873673916249714 \times 10^{-7}$	$9.87645466521414 \times 10^{-7}$	0.0000015607417459539997
11	$1.825806691391207 \times 10^{-7}$	$1.785437506174325 \times 10^{-7}$	0.0000014121784636111191

5 Conclusions

This paper uses the operational matrices of differentiation based on: Bernoulli and Legendre polynomials to solve the Falkner-Skan equation. We found that the Bernoulli operational matrix method has better accuracy than the Legendre operational matrix method. We also compared the results numerically with the Range-Kutta method, a good agreement has been achieved. Moreover, the maximum error remainder value for the proposed approximation methods was calculated. In addition, a comparison was made between the proposed methods and the Bernstein operational matrix method available in the literature. The errors obtained by Bernstein and Bernoulli were slightly identical.

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