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Complete (k, r)-Caps From Orbits In $PG(3, 11)$

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Abstract

The purpose of this article is to partition $PG(3,11)$ into orbits. These orbits are studied from the view of caps using the subgroups of $PGL(4,11)$ which are determined by nontrivial positive divisors of the order of $PG(3,11)$. The τ_i-distribution and c_i-distribution are also founded for each cap.

Keywords: Cap, Complete cap, Companion matrix, Projective space, Singer group.

أغطية−(k, r) التامة من مدارات في $PG(3, 11)$

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الخلاصة

الغرض من هذا البحث هو تقسيم PG (3,11) إلى مدارات ودراسة هذه المدارات من منظور الاغطية باستخدام الزمر الجزئية في PGL (4،11) والتي يتم تحديدها بواسطة القواسم الموجبة غير التافهة لي رتبة $PG(3,11)$. أيضًا، تم أيجاد توزيع−τ_i و توزيع−c_i لكل غطاء.

1. Introduction

The idea of construction caps of different sizes and degrees in projective space has been studied by many researchers in different ways. Some researchers studied the subject through the standard frame points [1] [2] [3]. The others studied the subject by group action on the projective plane [4] [5] as well as studied in three dimensional projective space over the fields of orders 8 and 23 [6] [7]. Also, some articles have been presented that are focused on the size of complete caps as in [8] [9] [10] [11] [12].

In $PG(3, q)$ the projective space of three dimension and order q has $\theta(3,11) = q^3 + q^2 + q + 1$ points, and by the duality has $q^3 + q^2 + q + 1$ planes, $(q^2 + 1)(q^2 + q + 1)$ lines and every plane contains $q^2 + q + 1$ lines, every lines contains $q + 1$ points, any point of the space has the quadruple form $[x_1, x_2, x_3, x_4]$. Also, there exists five points such that no four of them are on the line, for example, the points $[1,0,0,0], [0,1,0,0], [0,0,1,0], [0,0,0,1], [1,1,1,1]$ which are called the standard frame of $PG(3,11)$. The points of $PG(3, q)$ have a unique forms which are

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$[1,0,0,0\,]$, $[x,1,0,0]$, $[x,y,1,0\,]$, $[x,y,z,1]$ for all x,y,z in F_q. A plane π in $PG(3,q)$ is the set of all points $[x_1\,,x_2\,,x_3\,,x_4\,]$ satisfying linear equation $u_1x_1+u_2x_2+\ u_3\ x_3+u_4x_4=0$. This plane denote by $\pi[u_1,u_2\,,u_3\,,u_4\,]$, where u_1,u_2,u_3,u_4 are elements in F_q with $F_q\backslash\{0\}$.

2. Basic definitions and background

Definition 1: [13] A (k,r)-cap in $PG(n\geq 3,q)$ is a set of k points such that no $r+1$ points are collinear, but at most r points of which lie in any line. Here r is called the degree of the (k,r)-cap.

Definition 2: [13] The (k,r)-cap is called complete cap if it is not contained in $(k+1,r)$-cap.The maximum size of the cap of degree r is denoted by $m_r(n,q)$ and the smallest size of a complete cap of degree r is denoted by $t_r(n,q)$.

Definition 3: [13] Let K be a cap of degree r, an i-secant of a K in $PG(n,q)$ is a line such that $|k\ \cap\pi|=i$. The number of i-secants of K is denoted byτ_i.

Let Q be a point that is not on the (k,r)-cap, K. The number of i-secant of K passing through Q is denoted by $\sigma_i(Q)$.The number $\sigma_r(Q)$ of r-secants is called the *index of Q with respect to K*.The set of all points of index i will be denoted by C_i and the cardinality of C_i is denoted by c_i. The sequence $(t_0\,,\ldots,t_r)$ will represent to the secant distribution and the sequences $(c_0,\ldots,c_d)$ refer to the index distribution.

Definition 4: [14] The group of projectivities of $PG(n.q)$ is called the projective general linear group $PGL\ (n+1,q)$. The elements of $PGL(n+1,q)$ are non-singular matrices of dimension $n+1$, and its cardinality is

$$\frac{(q^{n+1}-1)(q^{n+1}-q)\ldots(q^{n+1}-q^n)}{(q-1)}.$$

Definition 5: [14] Let $f(x)=x^{n+1}-a_nx^n-\cdots-a_1x-a_0$ be primitive polynomial over F_q of degree $n+1$. A companion matrix for f is a $(n+1)\times(n+1)$ matrix

$$C_f=\begin{pmatrix} 0 & & I_n \\ 0 & & \\ \vdots & & \vdots \\ a_0 & \cdots & a_n \end{pmatrix}.$$

The points and hyperplanes of $PG(n,q)$ are found by the formula:

$P(i)=\pi_0C_f^i$,

where $\pi_0=[1,0,\cdots,0]$ and i from 0 to $\theta(n,q)-1$. The companion matrix C_f forms a cyclic subgroup of $PGL(n+1,q)$ that is called the *Singer group*.

3. Procedures of construction complete caps and main results

To start the procedures of construction the following lemma is important.

Lemma 6:

(i) There exist 14 non-trivial cyclic subgroups of $PGL(4,11)$ of order t divided $\theta(3,11)$=1464**.**

(ii) There exist 14 equivalence classes up to projectivity space $PG(3,11)$ of order t in y such that $t\ \cdot i=\emptyset\ (3,11)$.

Proof:
(i) Let $Y = \{2, 3, 4, 6, 8, 12, 24, 61, 122, 183, 244, 366, 488, 732\}$ be the set of non-trivial factors of $\theta(3,11)$. The companion matrix C_f has order $\theta(3,11)$, which also gives a cyclic subgroup, $\langle C_f \rangle$ of $PGL(4,11)$ such that $PG(3,11)$ is invariant with respect to it. All elements of Y divided the order of C_f by Lagrange Theorem and give cyclic subgroups of $\langle C_f \rangle$ denoted by $S_i = \langle {C_f}^i \rangle, i \in Y$. In $PGL(4,11)$, any other cyclic subgroups of an order divided $\theta(3,11)$ will be a copy isomorphic to S_i for $i \in Y$.
(ii) For any $i \in Y$, the action of the subgroups S_i on projective space $PG(3,11)$ will divide the space points into i orbits; that is, equivalence classes of order $t \in Y$; that is, $t = \frac{\theta(3,11)}{i}$. The i equivalence classes will be projectively equivalent by C_f.

To construct the complete caps in the projective space $PG(3,11)$ using the action of the subgroups S_i in Lemma 6, the following algorithm is used:
1. Finding the points of $PG(3,11)$ by formula P_i=$(1,0,0,0)C(f)^i$, $i = 0,1,2, \ldots, 1463$, where $C(f)$=$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a^8 & a & 1 & a^2 \end{pmatrix}$ and $f(x)$ be primitive polynomial over F_{11} since $f(x) \neq 0$ for all $x \in F_{11}$, $f(x) = x^4 - a^2x^3 - x^2 - ax - a^8$, $a = 2$ be primitive element of $F_{11} = \{0, a, a^2, a^3, \cdots, a^9 | a^{10} = 1\}$.
By duality, the lines constructed using the formula $L_i = L_0\, C(f)^i, i = 0,1,2, \ldots, 1463$.
Here L_0 is the line that passes through a point whose fourth coordinate is zero.
2. Depending on the nontrivial divisors of 1464 which are $\{2, 3, 4, 6, 8, 12, 24, 61, 122, 183, 244, 366, 488, 732\}$ generated subgroup of $PGL(4,11)$, Lemma 6(i).
3. Finding the orbits of each H_i. Let $O_i[i,t]$ be representative of these orbits where $i \cdot t = \theta(3,11)$
4. Check, if $O_i\,[i,t]$ is cap. If yes, then find the degree of $O_i[i,t]$ and check it, complete or not.
5. If $O_i[i,t]$ is not a complete cap, then the following steps are used to make it complete.
Step i: Determine the set of index zero points of the cap, $O_i[i,t] = B$.
Step ii: Add points of index zero c_0 to make it complete.
Step iii: If step ii succeeds, then we stop.
Let $C_0^{O_i}$ be the set of adding points to $O_i\,[i,t]$. A complete (k,r)-cap will be $A = O_i[i,t] \cup C_0^B$. The number 1464 has 14 non trivial positive divisors which are 2,3,4,6,8,12,24, 61,122,183,244,366,488,732, so we have 14 orbits.
The above procedure is executed by GAP (Groups-Algorithms-programing) a system for computational Discrete Algebra [15].

Theorem 7: The 14 equivalence classes $O_i[i,t]$ up to projective space $PG(3,11)$ are divided into complete and incomplete caps as follows:
(i) Four complete caps of degrees 7,6,4,2.
(ii) Ten incomplete caps of degrees 2,3,4,6,12.
Proof:
Let $T = C(f)$.
(i) Complete caps:
1. The action of the cyclic subgroup $\langle T^3 \rangle$ on $PG(3,11)$ gets 3 orbits $O_3[3,488]$ of size 488 points. This orbit will be cap of degree 7 since it's τ_i-distribution of this orbit

is $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7)$=(976,2928,976,6466,976,2928,976) and c_i values are $c_5 = 976$, $c_i = 0$ for $i \neq 5$.

2. The action of the cyclic subgroup $\langle T^4 \rangle$ on $PG(3,11)$ gets 4 orbits $O_4[4,366]$ of size 366 points. This orbit will be cap of degree 6 since it's τ_i-distribution of this orbit is $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_7) = (793, 1464, 5124, 1464, 5124, 1464, 793)$ and c_i values are $c_4 = 732$, $c_5 = 366$ $c_i = 0$ for $i \neq 4, 5$.

3. The action of the cyclic subgroup $\langle T^6 \rangle$ on $PG(3,11)$ get 6 orbits $O_6[6,244]$ of size 244 points. This orbit will be cap of degree 4 since it's τ_i- distribution of this orbit is $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4) = (2989, 1464, 7320, 1464, 2989)$ and c_i values are $c_{18} = 244$, $c_{20} = 976$, $c_i = 0$ for $i \neq 18, 20$.

4. The action of the cyclic subgroup $\langle T^{12} \rangle$ on $PG(3,11)$ get 12 orbits $O_{12}[12,122]$ of size 122 points. This orbit will be cap of degree 2 since it's τ_i-distribution of this orbit is $(\tau_0, \tau_1, \tau_2,)$ = $(7381, 1464, 7381)$ and c_i values are $c_{55} = 1342$, $c_i = 0$ for $i \neq 55$.

(ii) Incomplete caps:

1. The action of the cyclic subgroup $\langle T^2 \rangle$ on $PG(3,11)$ gives 2 orbits $O_2[2,732]$ of size 732 points. This orbit will be cap of degree 12 since it's τ_i-distribution of this orbit is $(\tau_0, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9, \tau_{12})$=$(61, 1464, 1464, 2928, 4392, 2928, 1464, 1464, 61)$, and it has $c_0 = 732$. Since it has degree 12, then it has some lines as subsets, so to be complete we have to add the 732 points of index zero to the orbit.

2. The cyclic subgroup $\langle T^8 \rangle$ acts on $PG(3,11)$ and gives 8 orbits $O_8[8,183]$ of size 183 points. This orbit is incomplete cap of degree 4 since it's τ_i-distribution of this orbit is $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4)$ = $(7321, 4392, 5490, 1525, 1098)$, and it has $c_0 = 183$, $c_6 = 366$, $c_9 = 732$. The orbit $O_8[8,183]$ will be complete cap of degree 12 when adding 19 points to it.

3. The cyclic subgroup $\langle T^{24} \rangle$ acts on $PG(3,11)$ and gives 24 orbits $O_{24}[24,61]$ of size 61 points. This orbit is incomplete cap of degree 2 since it's τ_i-distribution of this orbit is (τ_0, τ_1, τ_2)=$(9943, 4453, 5490, 1830)$, and it has $c_0 = 1452$. The orbit $O_{24}[24,61]$ will be complete of degree 2 when adding 61 points to it.

4. The action of the cyclic subgroup $\langle T^{61} \rangle$ on $PG(3,11)$ gave 61 orbits $O_{61}[61,24]$ of size 24 points. This orbit will be cap of degree 12 since it's τ_i-distribution of this orbit is $(\tau_0, \tau_1, \tau_2, \tau_{12})$ = $(13200, 2880, 144, 2)$, and it has $c_0 = 1440$. Since it has degree 12, then it has some lines as subsets, so to be complete we have to add the 1440 points of index zero to the orbit.

5. The action of the cyclic subgroup $\langle T^{122} \rangle$ on $PG(3,11)$ gives 122 orbits $O_{122}[122,12]$ of size 12 points. This orbit will be cap of degree 12 since it's τ_i-distribution of this orbit is $(\tau_0, \tau_1, \tau_{12})$ = $(14641, 1584, 1)$, and it has $c_0 = 1452$. Since it has degree 12, then it has some lines as subsets, so to be complete we have to add the 1452 points of index zero to the orbit.

6. The cyclic subgroup $\langle T^{183} \rangle$ acts on $PG(3,11)$ and gives 183 orbits $O_{183}[183,8]$ of size 8 points. This orbit is incomplete cap of degree 4 since it's τ_i-distribution of this orbit is $(\tau_0, \tau_1, \tau_2, \tau_4)$ = $(15184, 1024, 16, 2)$, and it has $c_0 = 144$. This orbit $O_{183}[183,8]$ will be complete of degree 4 when adding 198 points to it.

7. The cyclic subgroup $\langle T^{244} \rangle$ acts on $PG(3,11)$ and gives 244 orbits $O_{244}[244,6]$ of size 6 points. This orbit is incomplete cap of degree 6 since it's τ_i-distribution of this orbit is (τ_0, τ_1, τ_6) = $(15433, 792, 1)$, and it has $c_0 = 1452$. This orbit $O_{244}[244,6]$ will be complete of degree 6 when adding 350 points to it.

8. The cyclic subgroup $\langle T^{366} \rangle$ acts on $PG(3,11)$ and gives 366 orbits $O_{366}[366,4]$ of size 4 points. This orbit is incomplete cap of degree 4 since it's τ_i-distribution of this orbit is (τ_0, τ_1, τ_4) =(15697, 528, 1), and it has $c_0 = 1452$, $c_1 = 8$. This orbit $O_{366}[366,4]$ will be a complete cap of degree 4 when adding 200 points to it.

9. The cyclic subgroup $\langle T^{488}\rangle$ acts on $PG(3,11)$ and gives 488 orbits $O_{488}[488,3]$ of size 3 points. This orbit is an incomplete cap of degree 3 since it's τ_i-distribution of this orbit is (τ_0, τ_1, τ_3)= (15829, 396, 1), and it has $c_0 = 1452$. This orbit $O_{488}[488,3]$ will be a complete cap of degree 3 when adding 204 points to it.

10. The cyclic subgroup $\langle T^{732}\rangle$ acts on $PG(3,11)$ and gives 732 orbits $O_{732}[732,2]$ of size 2 points. This orbit is incomplete cap of degree 2 since it's τ_i-distribution of this orbit is (τ_0, τ_1, τ_2) = (15961, 264, 1), and it has $c_0 = 1452$, $c_1 = 10$. This orbit $O_{732}[732,2]$ will be complete when adding 1462 points to it.

Examples 8:

1. (244,6)-cap is complete of degree 4.

The orbit $O_6[6,244]$ has 244 points and the τ_i-distribution of $O_6[6,244]$ is $(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4)$=(2989, 1464, 7320, 1464, 2989), so $O_6[6,244]$ is (244,4)-cap. Also, it has $c_{18} = 244, c_{20} = 976$; that is, $c_0 = 0$, thus it is complete cap.

2. (488,7)-cap is a complete of degree 7.

The orbit $O_3[3,488]$ has 488 points and the τ_i-distribution of $O_3[3,488]$ is $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7)$=(976, 2928, 976, 6466,976, 2989,976), so $O_3[3,488]$ is (488,7)-cap. Also, it has $c_5 = 976$; that is, $c_0 = 0$, thus it is complete cap.

Corollary 9:

(i) $O_2[2,732]$ is the union of 61 disjoint lines..

(ii) $O_{61}[61,24]$ is the union of tow disjoint lines.

(iii) $O_{122}[122,12]$ is just a line.

Proof:

(i) The class $O_2[2,732]$ is the union of 61 disjoint lines of the following order depending on the formula $L_i = L_0\, C(f)^i$, $i = 1, \dots, 60$.

(ii) The class O_{61} [61,24] is the union of $L_{83} \cup L_{6676}$.

(iii) The class $O_{122}[122,12]$ =L_{83}.

Note 10: All the results in this paper can be transformed into results in linear codes and Graphs, see [16] [17] [18].

4. Conclusions

The action of the cyclic groups of $PGL(4,11)$, $\langle T^2\rangle$, $\langle T^3\rangle$, $\langle T^4\rangle$, $\langle T^6\rangle$, $\langle T^8\rangle$, $\langle T^{12}\rangle$, $\langle T^{24}\rangle$, $\langle T^{61}\rangle$, $\langle T^{122}\rangle$, $\langle T^{183}\rangle$, $\langle T^{244}\rangle$, $\langle T^{366}\rangle$, $\langle T^{488}\rangle$, $\langle T^{732}\rangle$ on $PG(3,11)$ gave 14 orbits and these orbits gave complete and incomplete caps. The details as follows:

1. Complete caps:

Table 1: Details about the complete caps

	Orbit	τ_i-distribution	c_i-distribution
1	$O_3[3,488]$	$(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7)$	(c_5)
		(976,2928,976,6466, 976,2928,976)	(976)
2	O_4 [4,366]	$(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6)$	(c_4, c_5)
		(793,1464,5124,1464, 5124,793)	(732,366)
3	$O_6[6,244]$	$(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4)$	(c_{18}, c_{20})
		(2928,1464,7320,1464	(244,976)

		,2928)	
4	O_{12} [12,122]	(τ_0, τ_1, τ_2)	(c_{55})
		(7381,1464,7381)	(1342)

Let # denote the number of adding points to the orbit to be complete.

2. Incomplete caps.

Table 2: Details about the incomplete caps

	Orbit	τ_i-distribution	c_i-distribution	#
1	$0_2[2,732]$	$(\tau_0, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9, \tau_{12})$	(c_0)	732
		(61,1464,1464,2928,4392 ,2928,1464,1462,61)	732	
2	$O_8[8,183]$	$(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4)$	(c_o, c_6, c_9)	19
		(3721,4392,5490,1525,1098)	(183,366,732)	
3	$O_{24}[24,61]$	(τ_0, τ_1, τ_2)	(c_0)	61
		(9943,4453,1830)	(1452)	
4	$O_{61}[61,24]$	$(\tau_0, \tau_1, \tau_2, \tau_{12})$	(c_0)	1440
		(13200,2880,144,2)	(1440)	
5	$O_{122}[122,12]$	$(\tau_0, \tau_1, \tau_{12})$	(c_0)	1452
		(14641,1584,1)	(1452)	
6	O_{183} [183,8	$(\tau_0, \tau_1, \tau_2, \tau_4)$	(c_o, c_1)	198
		(15184,1024,16,2)	(1440,16)	
7	$O_{244}[244,6]$	(τ_0, τ_1, τ_6)	(c_o, c_1)	350
		(15433,792,1)	(1452,6)	
8	$O_{366}[366,4]$	(τ_0, τ_1, τ_4)	(c_o, c_1)	200
		(15697,528,1)	(1452,8)	
9	$O_{488}[488,3]$	(τ_0, τ_1, τ_3)	(c_o, c_1)	204
		(15829,396,1)	(1452,9)	
10	$O_{732}[732,2]$	(τ_0, τ_1, τ_2)	(c_o, c_1)	1462
		(15961,264,1)	(1452,10)	

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