ISSN: 0067-2904

Generalized-hollow lifting modules

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Abstract
Let R be any ring with identity, and let M be a unitary left R-module. A submodule K of M is called generalized coessential submodule of N in M, if $\frac{N}{K} \subseteq$ Rad$(\frac{M}{K})$. A module M is called generalized hollow-lifting module, if every submodule N of M with $\frac{M}{N}$ is a hollow module, has a generalized coessential submodule of N in M that is a direct summand of M. In this paper, we study some properties of this type of modules.

Keywords: generalized coessential submodule, generalized strong supplement submodule, generalized hollow-lifting module.

مقاسات الرفع المجوفه– المعممة

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الخلاصة:
لتكن R حلقة ذات عنصر محايد و ليكن M مقاسا أحاديا أيسرا على R. يقال عن المقاس M بأنه مقاس رفع مجوف– معمم أذا كان لكل مقاس جزئي N في M بحيث أن $\frac{M}{N}$ أجوف فأن N له مقاس جزئي معمم رديف جوهريا و يكون جمع مباشر. في هذا البحث سوف ندرس خواص هذا النوع من المقاسات و نبرهن بعض النتائج التي تعتبر تعميم لمقاسات الرفع المجوفه.

1. Introduction:

Throughout this paper R is a ring with identity, and every R-module is a unitary left R-module, N⊆M denotes N is a submodule of M.

Let M be an R-module, and let N⊆M, N is called a small submodule of M (denoted by N≪M), if for every K⊆M, M= N+K implies K = M, [1]. A nonzero module M is called hollow, if every proper submodule of M is small, [1]. A submodule K of M is called coessential submodule of N in M (denoted by $K \subseteq_{ce} M$), if $\frac{N}{K} \ll \frac{M}{K}$.

Let M be a module and N, K⊆M. N is a supplement of K in M, if M = N+K and N∩K≪N, [1]. And N is called a generalized supplement of K in M, if M = N+K and N∩K⊆ Rad(N), where Rad(N) is the Jacobson radical of N, [2]. N is called strong supplement of K in M, if N is a supplement of K in M and N∩K is a direct summand of K, [3].

An R-module M is called lifting or satisfies (D_1), if for every submodule N of M, there exists a direct summand K of M, such that K is coessential of N in M, [1]. M is called hollow-lifting, if for every submodule N of M with $\frac{M}{N}$ is hollow has a coessential submodule in M that is a direct summand of M, [4]. Clearly every lifting module is hollow-lifting, while the converse does not hold in general, see [4].

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In section two of this paper, we introduce generalized coessential submodule of M. A submodule K of M is called a generalized coessential of N in M, if $\frac{N}{K} \subseteq \mathrm{Rad}(\frac{M}{K})$. We also introduce generalized hollow-lifting module as a generalization of hollow-lifting module, [1]. An R-module M is called a generalized-hollow lifting module (for short, G-hollow lifting module), if for every submodule N of M, with $\frac{M}{N}$ is a hollow module, N has a generalized coessential submodule of M that is a direct summand of M. We prove some properties of G-hollow lifting modules. In fact, we prove for an indecomposable module M, M is a G-hollow lifting module if and only if M is hollow or else M has no hollow factor module. We also prove that for N⊆M, N has a generalized strong supplement in M if and only if N has a generalized coessential submodule that is a direct summand of M, therefore M is a G-hollow-lifting module if and only if for every submodule N of M, with $\frac{M}{N}$ is hollow has a generalized strong supplement in M.

In section three, we prove that for fully invariant submodule N of M, if M is a G-hollow lifting module, then $\frac{M}{N}$ is a G-hollow lifting module. In fact, we give sufficient condition for direct sum of two G-hollow lifting module to be G-hollow lifting. We prove if $M = M_1 \oplus M_2$ is a duo module, then M is a G-hollow lifting module, if and only if M_1 and M_2 are G-hollow lifting modules.

2. Some properties of G-hollow lifting modules

In this section, we introduce G-hollow lifting module as a generalization of hollow lifting module, and study some properties of this type of modules.

Recall that an R-module M is called lifting or satisfies (D_1), if for every submodule N of M, there exists a direct summand K of M such that K is coessential of N in M, [5].

As a generalization of coessential submodule, we introduce the following.

Definition 2.1: A submodule K of M is called generalized coessential submodule of N in M denoted **by $K \subseteq_{GCe} N$, if $\frac{N}{K} \subseteq \mathrm{Rad}(\frac{M}{K})$.**

It is clear that, if K is coessential submodule of N in M, then K is generalized coessential submodule of N in M. However the converse in general is not true, for example $0 \subseteq_{GCe}$ in Q as Z-module, but 0 is not coessential of Q.

Definition 2.2: An R-module M is called generalized lifting or satisfies (GD_1), if for every submodule N of M, there exists a direct summand K of M, such that $K \subseteq_{GCe} N$ in M.

It is clear that every lifting module is a generalized lifting module.

An R-module M is called hollow lifting, if every submodule N of M such that $\frac{M}{N}$ is hollow has a coessential submodule that is a direct summand of M, [6].

It is clear that every lifting module is a hollow lifting module.

As a generalization of hollow lifting module, we introduce the following.

Definition 2.3: An R-module M is called generalized-hollow lifting module (for short G-hollow lifting), if for every submodule N of M with $\frac{M}{N}$ is hollow N has a generalized coessential submodule in M that is a direct summand in M.

Proposition 2.4: Let M_1 and M_2 be hollow modules, if $M = M_1 \oplus M_2$ then the following are equivalent:

1. M is G-hollow lifting.

2. M is G-lifting.

Proof: 1→ 2) Let $N \subseteq M$, let $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$. If $\pi_1(N) \neq M_1$ and $\pi_2(N) \neq M_2$, then $\pi_1(N) \ll M_1$ and $\pi_2(N) \ll M_2$.Thus $\pi_1(N) \oplus \pi_2(N) \ll M_1 \oplus M_2$. [1]

Now let $n \in N$, then $n \in M = M_1 \oplus M_2$, hence $n = m_1 + m_2$, where $m_1 \in M_1$, $m_2 \in M_2$

$\pi_1(n) = \pi_1 (m_1 + m_2) = m_1$ and $\pi_2(n) = \pi_2(m_1+m_2) = m_2$, thus $n = \pi_1(n) + \pi_2(n)$,this implies that $N \subseteq \pi 1(N) \oplus \pi 2(N)$, therefore $N \ll M$. Assume that $\pi_1(N) \neq M_1$ then $M = N + M_2$, thus $M / N = N + M_2 / N \cong M2/N \cap M2$ but M_2 is hollow, hence $M_2 / N \cap M_2$ is a hollow module this implies that M / N is hollow, therefore $\exists\, K \subseteq \oplus M$ such that $N / K < \mathrm{Rad}(M / K)$,hence M is a generalized lifting.

2 → 1) Clear.

Remark 2.5: It is clear that every module has no hollow factor module is a G-hollow lifting module. However, if M is indecomposable we have the following:
Proposition 2.6: Let M be an indecomposable module, then the following are equivalent:
1. M is a G-hollow lifting module.
2. M is hollow or else M has no hollow factor module.

Proof: 1→2 Suppose that M has a hollow factor module, then ∃ N ⊊ M, such that $\frac{M}{N}$ is hollow. Since M is a G-hollow lifting module, then ∃ K $\subseteq_{\oplus}$M, for K⊆M. But M is indecomposable, then K = 0 and hence N⊆Rad(M).
2→1 Clear

Let R be any ring, and M is an R-module. Let N, K be two submodules of M, K is called strong supplement of N in M, if K is a supplement of N in M, and K∩N is a direct summand of N, [3]
As a generalization of strong supplement submodule, we introduce the following:
Definition 2.7: Let N, K be submodules of M. K is called a generalized strong supplement of N (for short G-strong supplement of N), if M = N+K with K∩N⊆Rad(K) and K∩N$\subseteq_{\oplus}$N.
Remark 2.8: In semisimple modules, every submodule is G-strong supplement.
Proposition 2.9: Let N⊆M, then the following are equivalent:
1. N has a G-strong supplement in M.
2. N has a G-coessential submodule that is a direct summand of M.
Proof: 1→2 Let K be a G-strong supplement of N in M, then M = N+K, N∩K⊆Rad(M) and N∩K$\subseteq_{\oplus}$N, hence ∃L⊆N such that (N∩K)⊕L = N, then M = L⊕K. Now $\frac{N}{L} = \frac{(N\cap K)\oplus L}{L} \subseteq \frac{Rad(M)+L}{L} \subseteq Rad(\frac{M}{L})$.
2→1 Let K be a G-coessential of N in M, that is a direct summand of M, then $\frac{N}{K} \subseteq Rad(\frac{M}{K})$ and M = K⊕L for L⊆M. Thus N = N∩(K⊕L) = K⊕(N∩L) and N+L = M. $\frac{N}{K} = \frac{(N\cap L)\oplus K}{K} \subseteq Rad(\frac{M}{K}) \cong \frac{N\cap L}{K\cap(N\cap L)} \subseteq Rad(\frac{K\oplus L}{K}) \cong Rad(\frac{L}{L\cap K})$. Thus N∩L⊆Rad(L) ⊆Rad(M).
Corollary 2.10: Let M be any R-module, then the following are equivalent:
1. M is a G-hollow lifting module.
2. Every submodule N of M, with $\frac{M}{N}$ is hollow, has a G-strong supplement in M.
Proposition 2.11: Let M be a finitely generated module over a commutative local ring. Then the following are equivalent:
1. M is a G-hollow lifting module.
2. M is a G-lifting module.
Proof: 1→2 Clear
2→1 Let N⊆M such that $\frac{M}{N}$ is cyclic, since R is local, then $\frac{M}{N}$ is local. Hence by corollary 2.10, N has a G-strong supplement, and by prop.2.9, N has a G-coessential submodule that is a direct summand in M.

Proposition 2.12: Let M be a G-hollow lifting module, then every submodule N of M such that $\frac{M}{N}$ is hollow, can be written as N = K⊕L, where K is a direct summand of M and N∩L⊆Rad(M).

Proof: Let N⊆M, with $\frac{M}{N}$ is hollow, since M is a G-hollow lifting module, then ∃K⊆M, K⊆N and $\frac{N}{K} \subseteq Rad(\frac{M}{K})$, let L⊆M with M = K⊕L then N = K⊕ (L∩N). Now $\frac{N}{K} = \frac{(K\oplus(L\cap N)}{K} \cong \frac{N\cap L}{K\cap(N\cap L)} \cong N \cap L$
But N / K ⊆ Rad (M / K) = Rad (K⊕L) / K ≅ Rad(L / (L ∩ K)) = Rad (L)
Thus N∩L ⊆ Rad(L) ⊆ Rad(M).

3. The direct sum of G-hollow lifting module

In this section, we prove under certain condition, an R-module $M = M_1 \oplus M_2$ is a G-hollow lifting module if and only if M_1 and M_2 are G-hollow lifting modules.

We start by the following:

Proposition 3.1: Let $M_1, M_2, \ldots, M_n$ be R-modules having no hollow factor modules. Then $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ is a G-hollow lifting module.

Proof: Suppose that M has a submodule N such that $\frac{M}{N}$ is hollow. Since $\frac{M_1+N}{N} + \frac{M_2+N}{N} + \ldots + \frac{M_n+N}{N} = \frac{M}{N}$, then $\exists$ i∈{1,…,n} such that $\frac{M_i+N}{N} = \frac{M}{N}$ *which* is hollow, so $\frac{M_i+N}{N}$ is hollow. Hence M_i has a hollow factor, a contradiction. Then by Remark 1.5, M is a G-hollow lifting module.

Remark 3.2: From prop.2.6, it is clear that every indecomposable module M which has no factor module is a G-hollow lifting module, but it is not lifting.

Proposition 3.3: If $M = N \oplus K$, where N is indecomposable having no hollow factor module, K is semisimple. Then M is a G-hollow lifting module.

Proof: Let $L \subseteq M$, such that $\frac{M}{L}$ is hollow. Then M = N+L or M = K+L. Since N has no hollow factor modules, and $\frac{N+L}{L} \cong \frac{N}{N \cap L}$, thus K+L = M. But K is semisimple, then $\exists$ $K_1 \subseteq K$, such that $K = K_1 \oplus (K \cap L)$. Hence $M = K \oplus L$, therefore M is a G-hollow lifting, but not lifting.

Remark 3.4: The direct sum of two G-hollow lifting modules need not be a G-hollow lifting as the following example show:

Example: Let P be any prime integer, and let $M = \frac{Z}{PZ} \oplus \frac{Z}{p^3Z}$ as Z-module, it is not G-hollow lifting module .While both of Z / PZ and Z / P^3Z are G- hollow lifting modules.

Lemma 3.5 [3]: Let M be any R-module, if $M = M_1 \oplus M_2$, then $\frac{M}{N} = \frac{M_1+N}{N} \oplus \frac{M_2+N}{N}$ for every fully invariant submodule N of M.

Proposition 3.6: Let M be any R-module, if M is a G-hollow lifting module, then $\frac{M}{N}$ is a G-hollow lifting module, for every fully invariant submodule N of M.

Proof: Let N be a fully invariant submodule of M, and let $\frac{K}{N} \subseteq \frac{M}{N}$ such that $\frac{M/N}{K/N} \cong \frac{M}{K}$ is hollow. Since M is G-hollow lifting, then $\exists$ $L \subseteq_{\oplus} M$, such that $L \subseteq K$, $\frac{K}{L} \subseteq \mathrm{Rad}\,(\frac{M}{L})$ and $M = K_1 \oplus L$ for $K_1 \subseteq M$, clearly $N+L \subseteq K$, then $\frac{L+N}{N} \subseteq \frac{K}{N}$. Define f: $\frac{M}{L} \to \frac{M}{N+L}$ by f(m+ L) = m+(L+N), ∀ m∈M. It is clear that f is an epimorphism, $f(\frac{K}{L}) \subseteq \mathrm{Rad}\,(\frac{M}{N+L})$, then $K+(L+N) \subseteq \mathrm{Rad}\,(\frac{M}{N+L})$, hence $\frac{K}{L+N} \subseteq_{GCe} \frac{M}{L+N}$. Now $\frac{M}{N} = \frac{K_1 \oplus L}{N} = \frac{K_1+N}{N} \oplus \frac{L+N}{N}$, hence L + N / N $\subseteq \oplus$ $\frac{M}{N}$, thus $\frac{M}{N}$ is a G-hollow lifting module.

Remark 3.7: If N is not fully invariant and M is a G-hollow lifting module, then $\frac{M}{N}$ need not be a G-hollow lifting module.

Example: Consider the Z-module $M = \frac{Z}{4Z} \oplus \frac{Z}{8Z}$, let $N = \frac{2Z}{4Z} \oplus \langle 0 \rangle$, clearly that M is G-hollow lifting module, since it is lifting but $\frac{M}{N}$ is not, since $\frac{M}{N} = \frac{\frac{Z}{4Z} \oplus \frac{Z}{8Z}}{\frac{2Z}{4Z} \oplus \langle 0 \rangle} \cong \frac{\frac{Z}{4Z}}{\frac{2Z}{4Z}} \oplus \frac{Z}{8Z}$. Then $\frac{M}{N} \cong \frac{Z}{2Z} \oplus \frac{Z}{8Z}$ which is not G-hollow lifting.

Recall that an R-module M is called duo module, if every submodule of M is fully invariant, [1].

Corollary 3.8: Let $M = M_1 \oplus M_2$ be a duo module. If M is a G-hollow lifting module, then M_1 and M_2 are G-hollow lifting module.

Corollary 3.9: Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ be a duo module, if M is a G-hollow lifting module, then M_i is G- hollow lifting module, $\forall$ i= 1,…,n.

Proposition 3.10: Let M b a duo module such that $M = M_1 \oplus M_2$, if M_1 and M_2 are G-hollow lifting modules, then M is a G-hollow lifting module.

Proof: Let N⊆M with $\frac{M}{N}$ is hollow, then $N = (N\cap M_1)\oplus(N\cap M_2)$. Hence $\frac{M}{N} = \frac{M_1 \oplus M_2}{(N\cap M_1)\oplus(N\cap M_2)} \cong \frac{M_1}{N\cap M_1} \oplus \frac{M_2}{N\cap M_2}$, thus $\frac{\frac{M}{N}}{\frac{M_1}{N\cap M_1}} \cong \frac{M_2}{N\cap M_2}$ is hollow, and similarly $\frac{M_1}{N\cap M_1}$ is hollow. Since M_1 and M_2 are G-hollow lifting module, then $\exists\, K_1 \subseteq_{\oplus} M_1$ with $K_1 \subseteq N\cap M_1$ and $\frac{N\cap M_1}{K_1} \subseteq Rad\,(\frac{M_1}{K_1})$, $M_1 = K_1 \oplus L_1$, $L_1 \subseteq M_1$ and $\exists\, K_2 \subseteq_{\oplus} M_2$ with $K_2 \subseteq N\cap M_2$ and $\frac{N\cap M_2}{K_2} \subseteq Rad\,(\frac{M_2}{K_2})$, $M_2 = K_2 \oplus L_2$, $L_2 \subseteq M_2$. Thus $K_1+K_2 \subseteq (N\cap M_1)+ (N\cap M_2) = N$ and $K_1+K_2 \oplus L_1+ K_2= M_1 \oplus M_2 = M$. Thus $K_1 \oplus K_2 \subseteq_{\oplus} M$. Now, $\frac{N}{K_1+K_2} = \frac{(N\cap M_1)\oplus(N\cap M_2)}{K_1 \oplus K_2} \cong \frac{N\cap M_1}{K_1} \oplus \frac{N\cap M_2}{K_2} \subseteq Rad\,(\frac{M_1}{K_1}) + Rad(\frac{M_2}{K_2}) \subseteq Rad\,(\frac{M}{K_1+K_2})$. Then $K_1+K_2 \subseteq_{GCe} N$, and hence M is G-hollow lifting module.

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