Iraqi Journal of Science

ISSN: 0067-2904

Generalized Strong Commutativity Preserving Centralizers of Semiprime Rings

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Abstract:
Let R be a semiprime ring with center $Z(R)$ and U be a nonzero ideal of R. An additive mappings $f, g: R \rightarrow R$ are called right centralizer if $f(xy) = xf(y)$ and $g(xy) = xg(y)$ holds for all $x, y \epsilon R$. In the present paper, we introduce the concepts of generalized strong commutativity centralizers preserving and generalized strong cocommutativity preserving centralizers and we prove that R contains a nonzero central ideal if any one of the following conditions holds: (i) $f(x)x = xg(x)$, (ii) $[f(x), g(y)] = 0$, (iii) $[f(x), g(y)] = \pm[x, y]$, (iv) $f(x)og(y) = 0$, (v) $f(x)og(y) = \pm xoy$, (vi) $[f(x), g(y)] = \pm xoy$, (vii) $f(x)og(y) \pm xy \epsilon Z(R)$, (viii) $f(U) \subseteq Z(R)$ for all $x, y \epsilon U$.

Keywords: Semiprime Ring, Right (Left) Centralizer, Strong Commutativity Preserving.

التمركزات الحافظة للابدالية القوية المعممة على الحلقات شبه الاولية

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الخلاصة:
لتكن R حلقة شبة اولية مع المركز $Z(R)$ و U مثالي غير صفري في R. الدوال الجمعية $f, g: R \rightarrow R$ تسمى تمركز ايمن اذا $f(xy) = xf(y)$ و $g(xy) = xg(y)$ تتحقق لكل $x, y \epsilon R$. في بحثنا هذا سنقدم مفاهيم التمركزات الحافظة للابدالية القوية المعممة والحافظة اللاابدالية القوية المعممة و سنبرهن بان R تحتوي مثالي غير صفري مركزي اذا تحقق احد هذة الشروط: (1) $f(x)x = xg(x)$, (2) $[f(x), g(y)] = 0$, (3) $[f(x), g(y)] = \pm[x, y]$, (4) $f(x)og(y) = 0$, (5) $f(x)og(y) = \pm xoy$, (6) $[f(x), g(y)] = \pm xoy$, (7) $f(x)og(y) \pm xy \epsilon Z(R)$, (8) $f(U) \subseteq Z(R)$ لكل $x, y \epsilon U$.

1. Introduction:

Throughout the present paper, R will represent an associative ring with center $Z(R)$. The characteristic of R is the smallest positive integer n such that $nx = 0$ for all $x \epsilon R$. For any $x, y \epsilon R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol xoy will denote the anticommutator $xy + yx$. Recall that a ring R is prime if for any $a, b \epsilon R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$. A ring R is called semiprime if for any $a \epsilon R$, $aRa = \{0\}$ implies $a = 0$. A prime ring is clearly a semiprime. In [1], Zalar, introduce the concepts as, an additive maping $f: R \rightarrow R$ is called right (resp. left) centralizer if $f(xy) = xf(y)$ (resp. $f(xy) = f(x)y$) holds for all $x, y \epsilon R$, and f is called a centralizer if it is a right as well as a left centralizer, f is called a Jordan centralizer if $f(xy + yx) = f(x)y + yf(x) = xf(y) + f(y)x$ holds for all $x, y \epsilon R$, and f is called a right (resp. left) Jordan centralizer if $f(x^2) = xf(x)$ (resp. $f(x^2) = f(x)x$) holds for all $x \epsilon R$. In [1], it was shown a Jordan

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centralizer in a semiprime ring is a left centralizer and each Jordan centralizer is a centralizer. A mapping $f: R \to R$ is called centralizing (resp. commuting) if $[f(x), x] \epsilon Z(R)$ (resp.$[f(x), x] = 0$) holds for all $x \epsilon R$, (see [2-5] for more information). In [6], Bresar, introduce the concept of generalized derivation, an additive mapping $f: R \to R$ is called a generalized derivation if there exists a derivation $d: R \to R$ such that $f(xy) = f(x)y + xd(y)$ holds for all $x, y \epsilon R$, and d is called the associated derivation of f. Obviously, generalized derivation with $d = 0$ covers the concept of left centralizer. In [7], Majeed, introduce the concept of cocentralizing (resp. cocommuting), two maps f and g are said to be cocentralizing (resp. cocommuting) on S, if $f(x)x - xg(x) \epsilon Z(R)$ (resp. $f(x)x - xg(x) = 0$) for all $x \epsilon S$, where S is a nonempty subset of R. In [8], Bell and Mason, introduce the concept of strong commutativity preserving maps, a mapping $f: R \to R$ is called strong commutativity preserving (SCP) (resp. commutativity preserving (CP)) if $[f(x), f(y)] = [x, y]$ (resp. $[f(x), f(y)] = 0$) holds for all $x, y \epsilon R$. In [9,Theorem 2], Golbasi, proved that if R is a semiprime ring, (f, d) and (g, h) be two generalized derivations of R. If $f(x)x = xg(x)$ for all $x \epsilon R$, then R has a nonzero central ideal and he generalize some known results for derivations to generalized derivations of semiprime rings. In [10], Golbasi and Koc, extended the results in [9] of a prime rings for a nonzero lie ideal. In [11], Shuliang, get many sufficient conditions of central lie ideal on generalized derivations when R is prime ring and U denote a lie ideal of R.

In the present paper we introduce some concepts, two maps $f, g: R \to R$ are called generalized strong commutativity preserving (GSCP) (resp. generalized commutativity preserving (GCP)) if $[f(x), g(y)] = [x, y]$ (resp. $[f(x), g(y)] = 0$) holds for all $x, y \epsilon R$. Two maps $f, g: R \to R$ are called generalized strong cocommutativity preserving (GSCCP) (resp. generalized cocommutativity preserving (GCCP)) if $f(x) o g(y) = x o y$ (resp. $f(x) o g(y) = 0$) holds for all $x, y \epsilon R$, and we prove that if R is a semiprime ring and U be a nonzero ideal of R and f, g be two nonzero right centralizer on R such that f and g are cocommuting on U, then R contains a nonzero central ideal. Also we prove many sufficient conditions.

2. Preliminaries:

In the following remark, we shall make extensive use of the basic commutator and anti - commutator identities.

Remark 1:

Let R be a ring and $x, y, z \epsilon R$, then

(i) $[xy, z] = x[y, z] + [x, z]y$,
(ii) (ii) $[x, yz] = y[x, z] + [x, y]z$,
(iii) $(xy) o z = x(y o z) - [x, z]y = (x o z)y + x[y, z]$,
(iv) $x o (yz) = (x o y)z - y[x, z] = y(x o z) + [x, y]z$,(

The following lemmas are well-known results which will be used to prove our results in the next section.

Lemma 1:[2]

Let R be a semiprime ring and U be any nonzero one sided ideal of R, then $Z(U) \subseteq Z(R)$.

Lemma 2:

Let R be a semiprime ring and U be a nonzero ideal of R. If f is a nonzero right centralizer on R then f is nonzero on U.

Proof:-

If $f(x) = f(xr)$ for all $x \epsilon U, r \epsilon R$.

It follows that,

$$\boldsymbol{xf(r) = 0}.$$

So,

$\boldsymbol{Uf(R) = \{0\}}$ **and hence,** $\boldsymbol{URf(R) = \{0\}}$. **(1)**

Since R is semiprime, it must contain a family $\mathrm{P} = \{P_\alpha : \alpha \epsilon \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$ (see [12] for more details). If P a typical member of Ƥ, then from (1), it follows that,

$$\boldsymbol{URf(R) = \{0\} = \cap P_\alpha} \textbf{ and hence, } \boldsymbol{URf(R) \subseteq P}.$$

By primeness of P,

either $\boldsymbol{U \subseteq P}$ **or** $\boldsymbol{f(R) \subseteq P}$. **(2)**

Now, using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{U = 0}$ **or** $\boldsymbol{f(R) = 0}$. **(3)**

Since *U* is nonzero ideal, then we get, $f(R) = 0$.

Lemma 3:[proposition 1.2.6. [13]]

Let *R* be a semiprime ring of characteristic different from 2, *U* be an ideal of *R* and $f: R \to R$ be an additive mapping which satisfies $f(x^2) = f(x)x$ for all *x* in *U*, then *R* contains a central ideal.

Lemma 4:[11]

A group cannot be a union of two of its proper subgroups.

3. The Main Results:

In the following theorem we extend Theorem 2 of [9] for a nonzero ideal and a nonzero right centralizer.

Theorem 1:

Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R* such that f and g are cocommuting on *U*, then *R* contains a nonzero central ideal.

Proof:-

By the given hypothesis, we have,

$\boldsymbol{f(x)x = xg(x)}$ **for all** $\boldsymbol{x \epsilon U}$. **(1)**

Replacing x by $x + y$ in (1), we get,

$\boldsymbol{f(x)y + f(y)x = xg(y) + yg(x)}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(2)**

Now, replacing x by yx in (2), we get,

$\boldsymbol{yf(x)y + f(y)yx = yxg(y) + y^2 g(x)}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(3)**

And so,

$$\boldsymbol{yf(x)y + f(y)yx = y(xg(y) + yg(x)\,)}.$$

Using (2), we get,

$$\boldsymbol{yf(x)y + f(y)yx = yf(x)y + yf(y)x}.$$

It follows that,

$$\boldsymbol{f(y)yx = yf(y)x} \text{ then, } \boldsymbol{[f(y), y]x = 0} \text{ for all } \boldsymbol{x, y \epsilon U}.$$

So,

$\boldsymbol{[f(y), y]U = 0}$ **and hence,** $\boldsymbol{[f(y), y]RU = \{0\}}$ **for all** $\boldsymbol{y \epsilon U}$. **(4)**

Since *R* is semiprime, it must contain a family $\mathrm{P} = \{P_\alpha : \alpha \epsilon \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$ (see [12] for more details). If *P* is a typical member of Ƥ, then from (4), it follows that,

$[f(y), y]RU = \{0\} = \cap P_\alpha$ and hence, $[f(y), y]RU \subseteq P$.

By primeness of *P*,

either $\boldsymbol{U \subseteq P}$ **or** $\boldsymbol{[f(y), y] \epsilon P}$ **for all** $\boldsymbol{y \epsilon U}$. **(5)**

Now, using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{U = 0}$ **or** $\boldsymbol{[f(y), y] = 0}$ **for all** $\boldsymbol{y \epsilon U}$. **(6)**

Since *U* is nonzero ideal, then $[f(y), y] = 0$ for all $y \epsilon U$.

From the above relation, we get,

$$\boldsymbol{f(y)y = yf(y)}.$$

And so,

$$\boldsymbol{f(y^2) = yf(y) = f(y)y}.$$

Thus, we have,

$\boldsymbol{f(y^2) = f(y)y}$ **for all** $\boldsymbol{y \epsilon U}$. **(7)**

Therefore, *R* contains a nonzero central ideal by Lemma 3.

The following Corollaries are the immediate consequences of the above Theorem.

Corollary 1:

Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $f(x)y = xg(y)$ for all $x, y \in U$, then *R* contains a nonzero central ideal.

Corollary 2:

Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R* such that f and g are cocommuting on *U*, then *R* is comm- utative.

Corollary 3:

Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $f(x)y = xg(y)$ for all $x, y \in U$, then *R* is commutative**.**

Now, we will introduce some definitions.

Definition 1:

Let R be a ring , two maps $f,\ g : R \to R$ are called generalized strong commutativity preserv-ing (GSCP) (resp. generalized commutativity preserving (GCP)) if $[f(x), g(y)] = [x, y]$ (resp. $[f(x), g(y)] = 0$) for all $x, y \in R$.

Definition 2:

Let R be a ring, two maps $f,\ g : R \to R$ are called generalized strong cocommutativity prese- rving (GSCCP) (resp. generalized cocommutativity preserving (GCCP)) if $f(x)og(y) = xoy$ (resp. $f(x)og(y) = 0$) for all $x, y \in R$.

Example:

Let $R = \{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in N$ where N is the ring of integer$\}$ be a ring of 2x2 matrices with respect to the usual addition and multiplication.

Let $U = \{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in N\}$ be an ideal of R. Define maps $f, g: R \to R$ by $f\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $g\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$.

Then f , g is a right centralizer which is satisfy GSCP and GSCCP on U.

Theorem 2:

Let R be a semiprime ring of characteristic different from 2, U be a nonzero ideal of R and f, g be two nonzero right centralizer on R such that f and g are GCP on U, then R contains a nonzero central ideal.

Proof:-

By the given hypothesis, we have,

$[\boldsymbol{f(x), g(y)}] = \mathbf{0}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(1)**

Replacing x by yx in (1), we get,

$[\boldsymbol{f(yx), g(y)}] = \mathbf{0}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(2)**

So,

$$[\boldsymbol{y f(x), g(y)}] = \mathbf{0}.$$

From Remark 1, we have,

$$\boldsymbol{y}[\boldsymbol{f(x), g(y)}] + [\boldsymbol{y, g(y)}]\boldsymbol{f(x)} = \mathbf{0}.$$

Using (1), we get,

$[\boldsymbol{y, g(y)}]\boldsymbol{f(x)} = \mathbf{0}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(3)**

Replacing x by zx in (3), we get,

$[\boldsymbol{y, g(y)}]\boldsymbol{f(zx)} = \mathbf{0}$ **for all** $\boldsymbol{x, y, z \epsilon U}$. **(4)**

And hence,

$$[\boldsymbol{y, g(y)}]\boldsymbol{z f(x)} = \mathbf{0} \text{ for all } \boldsymbol{x, y, z \epsilon U}.$$

So,

$$[\boldsymbol{y, g(y)}]\boldsymbol{U f(x)} = \mathbf{0}.$$

And hence,

$[\boldsymbol{y, g(y)}]\boldsymbol{U R f(x)} = \{\mathbf{0}\}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(5)**

Since R is semiprime, it must contain a family $Ƥ = \{P_\alpha : \alpha \epsilon \Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$ (see [12] for more details). If P is a typical member of Ƥ, then from (5), it follows that,

$$[\boldsymbol{y, g(y)}]\boldsymbol{U R f(x)} = \{\mathbf{0}\} = \cap \boldsymbol{P_\alpha} \text{ and hence, } [\boldsymbol{y, g(y)}]\boldsymbol{U R f(x)} \subseteq \boldsymbol{P}.$$

By primeness of P,

either $\boldsymbol{f(x) \epsilon P}$ **or** $[\boldsymbol{y, g(y)}]\boldsymbol{U} \subseteq \boldsymbol{P}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(6)**

Now, using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{f(x)} = \mathbf{0}$ **or** $[\boldsymbol{y, g(y)}]\boldsymbol{U} = \mathbf{0}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(7)**

Since f is nonzero on R, then by Lemma 2, f is nonzero on U.

So,

$$[\boldsymbol{y, g(y)}]\boldsymbol{U} = \mathbf{0} \text{ and hence, } [\boldsymbol{y, g(y)}]\boldsymbol{R U} = \{\mathbf{0}\} = \cap \boldsymbol{P_\alpha}.$$

Then,

either $\boldsymbol{U \subseteq P}$ **or** $[\boldsymbol{y, g(y)}]\boldsymbol{\epsilon P}$ **for all** $\boldsymbol{y \epsilon U}$. **(8)**

Using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{U} = \mathbf{0}$ **or** $[\boldsymbol{y, g(y)}] = \mathbf{0}$ **for all** $\boldsymbol{y \epsilon U}$. **(9)**

Since *U* is nonzero ideal, then $[y, g(y)] = 0$ for all $y \epsilon U$.
From the above relation, we get,

$$\boldsymbol{y g(y) = g(y) y}.$$

And so,

$$\boldsymbol{g(y^2) = y g(y) = g(y) y}.$$

Thus, we have,
$\boldsymbol{g(y^2) = g(y) y}$ **for all** $\boldsymbol{y \epsilon U}$. **(10)**
Therefore, *R* contains a nonzero central ideal by Lemma 3.

Corollary 4:
Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R* such that f and g are GCP on *U*, then *R* is commutative.

Theorem 3:
Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R* such that f and g are GSCP on *U*, then *R* contains a nonzero central ideal.

Proof:-
By the given hypothesis, we have,
$\boldsymbol{[f(x), g(y)] = [x, y]}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(1)**
Replacing x by *yx* in (1), we get,
$\boldsymbol{[f(yx), g(y)] = [yx, y]}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(2)**
So,

$$\boldsymbol{[y f(x), g(y)] = [yx, y]} \textbf{ for all } \boldsymbol{x, y \epsilon U}.$$

From Remark 1, we have,

$$\boldsymbol{y[f(x), g(y)] + [y, g(y)] f(x) = y[x, y] + [y, y] x}.$$

Using (1), we get,
$\boldsymbol{[y, g(y)] f(x) = 0}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(3)**
By the same method in Theorem 2, we complete the proof.

Corollary 5:
Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R* such that f and g are GSCP on *U*, then *R* is commutative.
The following theorem is proved as in Theorem 3 with necessary variations.

Theorem 4:
Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $[f(x), g(y)] + [x, y] = 0$ for all $x, y \epsilon U$, then *R* contains a nonzero central ideal.

Corollary 6:
Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $[f(x), g(y)] + [x, y] = 0$ for all $x, y \epsilon U$, then *R* is com-mutative.

Theorem 5:
Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R* such that f and g are GCCP on *U*, then *R* contains a nonzero central ideal.

Proof:-
By the given hypothesis, we have,
$\boldsymbol{f(x) o g(y) = 0}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(1)**
Replacing y by *xy* in (1), we get,
$\boldsymbol{f(x) o g(xy) = 0}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(2)**
So,

$$\boldsymbol{f(x) o x g(y) = 0}.$$

From Remark 1, we have,

$$\boldsymbol{x(f(x) o g(y)) + [f(x), x] g(y) = 0}.$$

Using (1), we get,
$\boldsymbol{[f(x), x] g(y) = 0}$ **for all** $\boldsymbol{x, y \epsilon U}$. **(3)**
Replacing y by *zy* in (3), we get,

$[f(x), x]g(zy) = 0$ **for all** $x, y, z\epsilon U$. **(4)**
And hence,

$$[f(x), x]zg(y) = 0 \text{ for all } x, y, z\epsilon U.$$

So,

$$[f(x), x]Ug(y) = 0.$$

And hence,
$[f(x), x]URg(y) = \{0\}$ **for all** $x, y\epsilon U$. **(5)**
Since *R* is semiprime, it must contain a family $\text{Ƥ} = \{P_\alpha : \alpha\epsilon\Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$ (see [12] for more details). If *P* is a typical member of Ƥ, then from (5), it follows that,

$$[f(x), x]URg(y) = \{0\} = \cap P_\alpha \text{ and hence, } [f(x), x]URg(y) \subseteq P.$$

By primeness of *P*,
either $g(y)\epsilon P$ **or** $[f(x), x]U \subseteq P$ **for all** $x, y\epsilon U$. **(6)**
Now, using the fact that $\cap P_\alpha = \{0\}$, we conclude that,
either $g(y) = 0$ **or** $[f(x), x]U = 0$ **for all** $x, y\epsilon U$. **(7)**
Since g is nonzero on *R*, then by Lemma 2, g is nonzero on *U*.
So,

$$[f(x), x]U = 0 \text{ and hence, } [f(x), x]RU = \{0\} = \cap P_\alpha.$$

Then,
either $U \subseteq P$ **or** $[f(x), x]\epsilon P$ **for all** $x\epsilon U$. **(8)**
Using the fact that $\cap P_\alpha = \{0\}$, we conclude that,
either $U = 0$ **or** $[f(x), x] = 0$ **for all** $x\epsilon U$. **(9)**
Since *U* is nonzero ideal, then $[f(x), x] = 0$ for all $x\epsilon U$.
From the above relation, we get,

$$f(x)x = xf(x).$$

And so,

$$f(x^2) = xf(x) = f(x)x.$$

Thus, we have,
$f(x^2) = f(x)x$ **for all** $x\epsilon U$. **(10)**
Therefore, *R* contains a nonzero central ideal by Lemma 3.

Corollary 7:

Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R* such that f and g are GCCP on *U*, then *R* is commutative.

Theorem 6:

Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R* such that f and g are GSCCP on *U*, then *R* contains a nonzero central ideal.

Proof:-

By the given hypothesis, we have,
$f(x)og(y) = xoy$ **for all** $x, y\epsilon U$. **(1)**
Replacing y by *xy* in (1), we get,
$f(x)og(xy) = xo(xy)$ **for all** $x, y\epsilon U$. **(2)**
So,

$$f(x)oxg(y) = xo(xy).$$

From Remark 1, we have,

$$x\big(f(x)og(y)\big) + [f(x), x]g(y) = x(xoy) + [x, x]y.$$

Using (1), we get,
$[f(x), x]g(y) = 0$ **for all** $x, y\epsilon U$. **(3)**
By the same method in Theorem 5, we complete the proof.

Corollary 8:

Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R* such that f and g are GSCCP on *U*, then *R* is commutative.
Now, using the similar techniques as in the above Theorem, we get the following theorems.

Theorem 7:

Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $f(x)og(y) + xoy = 0$ for all $x, y\epsilon U$, then *R* contains a nonzero central ideal.

Corollary 9:

Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $f(x)og(y) + xoy = 0$ for all $x, y\epsilon U$, then *R* is commu-tative.

Theorem 8:

Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $[f(x), g(y)] = xoy$ for all $x, y\epsilon U$, then *R* contains a nonzero central ideal.

Corollary 10:

Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $[f(x), g(y)] = xoy$ for all $x, y\epsilon U$, then *R* is commutat-ive.

Theorem 9:

Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $[f(x), g(y)] + xoy = 0$ for all $x, y\epsilon U$, then *R* contains a nonzero central ideal**.**

Corollary 11:

Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $[f(x), g(y)] + xoy = 0$ for all $x, y\epsilon U$, then *R* is comm-utative.

Theorem 10:

Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $f(x)og(y) - xy\epsilon Z(R)$ for all $x, y\epsilon U$, then *R* contains a nonzero central ideal**.**

Proof:-

By the given hypothesis, we have,

$\boldsymbol{f(x)og(y) - xy\epsilon Z(R)}$ **for all** $\boldsymbol{x, y\epsilon U}$. **(1)**

This implies that,

$\boldsymbol{[f(x)og(y) - xy, r] = 0}$ **for all** $\boldsymbol{x, y\epsilon U, r\epsilon R}$. **(2)**

Replacing *y* by *xy* in (2), we get,

$\boldsymbol{[f(x)og(xy) - x^2y, r] = 0}$ **for all** $\boldsymbol{x, y\epsilon U, r\epsilon R}$. **(3)**

So,

$$\boldsymbol{[f(x)oxg(y) - x^2y, r] = 0}.$$

From Remark 1, we have,

$$\boldsymbol{[x(f(x)og(y)) + [f(x), x]g(y) - x^2y, r] = 0}.$$

And hence,

$$\boldsymbol{[x(f(x)og(y)) - x^2y, r] + [\,[f(x), x]g(y), r] = 0}.$$

Using (2), we get,

$\boldsymbol{[\,[f(x), x]g(y), r] = 0}$ **for all** $\boldsymbol{x, y\epsilon U, r\epsilon R}$. **(4)**

Replacing *y* by *zy* in (4), we get,

$\boldsymbol{[\,[f(x), x]g(zy), r] = 0}$ **for all** $\boldsymbol{x, y, z\epsilon U, r\epsilon R}$. **(5)**

So,

$\boldsymbol{[\,[f(x), x]zg(y), r] = 0}$ **for all** $\boldsymbol{x, y, z\epsilon U, r\epsilon R}$. **(6)**

From Remark 1, we have,

$\boldsymbol{[f(x), x]z[g(y), r] + [\,[f(x), x]z, r]g(y) = 0}$ **for all** $\boldsymbol{x, y, z\epsilon U, r\epsilon R}$. **(7)**

Replacing *z* by $zg(y)$ in (7), we get,

$\boldsymbol{[f(x), x]zg(y)[g(y), r] + [\,[f(x), x]zg(y), r]g(y) = 0}$ **for all** $\boldsymbol{x, y, z\epsilon U, r\epsilon R}$. **(8)**

Using (6), we get,

$$\boldsymbol{[f(x), x]zg(y)[g(y), r] = 0} \text{ for all } \boldsymbol{x, y, z\epsilon U, r\epsilon R}.$$

So,

$$\boldsymbol{[f(x), x]Ug(y)[g(y), r] = 0}.$$

And hence,

$\boldsymbol{[f(x), x]URg(y)[g(y), r] = \{0\}}$ **for all** $\boldsymbol{x, y\epsilon U, r\epsilon R}$. **(9)**

Since *R* is semiprime, it must contain a family $\mathrm{Ƥ} = \{P_\alpha : \alpha\epsilon\Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$ (see [12] for more details). If *P* is a typical member of Ƥ, then from (9), it follows that,

$$\boldsymbol{[f(x),x]URg(y)[g(y),r] = \{0\} = \cap P_\alpha}.$$

And hence,

$$\boldsymbol{[f(x),x]URg(y)[g(y),r] \subseteq P}.$$

By primeness of *P*,

either $\boldsymbol{[f(x),x]U \subseteq P}$ **or** $\boldsymbol{g(y)[g(y),r]\epsilon P}$ **for all** $\boldsymbol{x,y\epsilon U, r\epsilon R}$. **(10)**

Now, using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{[f(x),x]U = 0}$ **or** $\boldsymbol{g(y)[g(y),r] = 0}$. **(11)**

Let $U_1 = \{x\epsilon U \mid [f(x),x]U = 0\}$ and $U_2 = \{y\epsilon U, r\epsilon R \mid g(y)[g(y),r] = 0\}$. Then U_1, U_2 are both additive subgroups of *U*, and $U_1 \cup U_2 = U$.

Thus by using Lemma 4, we have either $U = U_1$ or $U = U_2$.

If $U = U_1$, we have,

$[f(x),x]U = 0$ and hence, $[f(x),x]RU = \{0\} = \cap P_\alpha$.

Then,

either $\boldsymbol{U \subseteq P}$ **or** $\boldsymbol{[f(x),x]\epsilon P}$. **(12)**

Using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{U = 0}$ **or** $\boldsymbol{[f(x),x] = 0}$. **(13)**

Since *U* is nonzero ideal, then $[f(x),x] = 0$ for all $x\epsilon U$.

From the above relation, we get,

$$\boldsymbol{f(x)x = xf(x)}.$$

And so,

$$\boldsymbol{f(x^2) = xf(x) = f(x)x}.$$

Thus, we have,

$\boldsymbol{f(x^2) = f(x)x}$ **for all** $\boldsymbol{x\epsilon U}$. **(14)**

Therefore, *R* contains a nonzero central ideal by Lemma 3.

On the other hand, if $U = U_2$, we have,

$\boldsymbol{g(y)[g(y),r] = 0}$ **for all** $\boldsymbol{y\epsilon U, r\epsilon R}$. **(15)**

Replacing *r* by *z* in (15), we get,

$\boldsymbol{g(y)[g(y),z] = 0}$ **for all** $\boldsymbol{y,z\epsilon U}$. **(16)**

Now, replacing *z* by *zy* in (16), we get,

$\boldsymbol{g(y)[g(y),zy] = 0}$ **for all** $\boldsymbol{y,z\epsilon U}$. **(17)**

From Remark 1, we have,

$$\boldsymbol{g(y)z[g(y),y] + g(y)[g(y),z]y = 0}.$$

Using (16), we get,

$$\boldsymbol{g(y)z[g(y),y] = 0} \textbf{ for all } \boldsymbol{y,z\epsilon U}.$$

So,

$$\boldsymbol{g(y)U[g(y),y] = 0} \textbf{ and hence, } \boldsymbol{g(y)RU[g(y),y] = \{0\} = \cap P_\alpha}.$$

Then,

either $\boldsymbol{g(y)\epsilon P}$ **or** $\boldsymbol{U[g(y),y] \subseteq P}$ **for all** $\boldsymbol{y\epsilon U}$. **(18)**

Using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{g(y) = 0}$ **or** $\boldsymbol{U[g(y),y] = 0}$ **for all** $\boldsymbol{y\epsilon U}$. **(19)**

Since g is nonzero on *R*, then by Lemma 2, g is nonzero on *U*.

So,

$$\boldsymbol{U[g(y),y] = 0} \textbf{ and hence, } \boldsymbol{UR[g(y),y] = \{0\} = \cap P_\alpha}.$$

Then,

either $\boldsymbol{U \subseteq P}$ **or** $\boldsymbol{[g(y),y]\epsilon P}$ **for all** $\boldsymbol{y\epsilon U}$. **(20)**

Using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{U = 0}$ **or** $\boldsymbol{[g(y),y] = 0}$ **for all** $\boldsymbol{y\epsilon U}$. **(21)**

Since *U* is nonzero ideal, then $[g(y),y] = 0$ for all $y\epsilon U$.

From the above relation, we get,

$$\boldsymbol{g(y)y = yg(y)}.$$

And so,

$$\boldsymbol{g(y^2) = yg(y) = g(y)y}.$$

Thus, we have,

$\boldsymbol{g(y^2) = g(y)y}$ **for all** $\boldsymbol{y \epsilon U}$. **(22)**

Therefore, *R* contains a nonzero central ideal by Lemma 3.

Corollary 12:

Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be be two nonzero right centralizer on *R*. If $f(x)og(y) - xy\epsilon Z(R)$ for all $x, y\epsilon U$, then *R* is com-mutative.

The following theorem is proved as in Theorem 10 with necessary variations.

Theorem 11:

Let *R* be a semiprime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $f(x)og(y) + xy\epsilon Z(R)$ for all $x, y\epsilon U$, then *R* contains a nonzero central ideal.

Corollary 13:

Let *R* be a prime ring of characteristic different from 2, *U* be a nonzero ideal of *R* and f, g be two nonzero right centralizer on *R*. If $f(x)og(y) + xy\epsilon Z(R)$ for all $x, y\epsilon U$, then *R* is commu-tative.

Finally, we prove the following theorem.

Theorem 12:

Let *R* be a semiprime ring, *U* be a nonzero ideal of *R* and f be a nonzero right centralizer on *R* such that $f(U) \subseteq Z(R)$, then $U \subseteq Z(R)$.

Proof:-

Since $\boldsymbol{f(U) \subseteq Z(R)}$**, then** $\boldsymbol{[f(x), y] = 0}$ **for all** $\boldsymbol{x, y\epsilon U}$. **(1)**

Replacing x by zx in (1), we get,

$\boldsymbol{[f(zx), y] = 0}$ **for all** $\boldsymbol{x, y, z\epsilon U}$. **(2)**

So,

$$\boldsymbol{[zf(x), y] = 0}.$$

From Remark 1, we have,

$$\boldsymbol{z[f(x), y] + [z, y]f(x) = 0}.$$

Using (1), we get,

$\boldsymbol{[z, y]f(x) = 0}$ **for all** $\boldsymbol{x, y, z\epsilon U}$. **(3)**

Replacing y by xy in (3), we get,

$\boldsymbol{[z, xy]f(x) = 0}$ **for all** $\boldsymbol{x, y, z\epsilon U}$. **(4)**

From Remark 1, we have,

$$\boldsymbol{x[z, y]f(x) + [z, x]yf(x) = 0}.$$

Using (3), we get,

$$\boldsymbol{[z, x]yf(x) = 0} \textbf{ for all } \boldsymbol{x, y, z\epsilon U}.$$

So,

$$\boldsymbol{[z, x]Uf(x) = 0}.$$

And hence,

$\boldsymbol{[z, x]URf(x) = \{0\}}$ **for all** $\boldsymbol{x, z\epsilon U}$. **(5)**

Since *R* is semiprime, it must contain a family $\Upsilon = \{P_\alpha : \alpha\epsilon\Lambda\}$ of prime ideals such that $\cap P_\alpha = \{0\}$ (see [12] for more details). If *P* is a typical member of Υ, then from (5), it follows that,

$[z, x]URf(x) = \{0\} = \cap P_\alpha$ and hence, $[z, x]URf(x) \subseteq P$.

By primeness of *P*,

either $\boldsymbol{f(x)\epsilon P}$ **or** $\boldsymbol{[z, x]U \subseteq P}$ **for all** $\boldsymbol{x, z\epsilon U}$. **(6)**

Now, using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{f(x) = 0}$ **or** $\boldsymbol{[z, x]U = 0}$ **for all** $\boldsymbol{x, z\epsilon U}$. **(7)**

Since f is nonzero on *R*, then by Lemma 2, f is nonzero on *U*.

So,

$[z, x]U = 0$ and hence, $[z, x]RU = \{0\} = \cap P_\alpha$.

Then,

either $\boldsymbol{U \subseteq P}$ **or** $\boldsymbol{[z, x]\epsilon P}$ **for all** $\boldsymbol{x, z\epsilon U}$. **(8)**

Using the fact that $\cap P_\alpha = \{0\}$, we conclude that,

either $\boldsymbol{U = 0}$ **or** $\boldsymbol{[z, x] = 0}$ **for all** $\boldsymbol{x, z\epsilon U}$. **(9)**

Since *U* is nonzero ideal, then $[z, x] = 0$ for all $x, z\epsilon U$.

So, *U* is commutative and hence, $U \subseteq Z(R)$ by Lemma 1.

Corollary 14:

Let R be a prime ring, U be a nonzero ideal of R and f be a nonzero right centralizer on R such that $f(U) \subseteq Z(R)$, then R is commutative.

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