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A Note on Overpartition Triples

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Abstract

Let *n* be a positive integer and $\overline{p}_3(n)$ denotes the number of overpartition triples. In this note, we prove two identities modulo 16 and 32 for $\overline{p}_3(n)$. We provide a new method to reprove a result of Lin Wang for completely determining and $\overline{p}_3(n)$ modulo 16. Also, we find and prove an infinite family of congruences modulo 32 for $\overline{p}_3(n)$. The new method relies on expanding the fourth power of the overpartition infinite product together with the help of Gauss' identity.

Keywords: Partitions, Overpartitions, Overpartition triples, Congruences, Sum of divisors.

مذكره حول التجزئات الفوقية الثلاثية

علي حمود فليح, انعام مطر شرقي

قسم الرياضيات, كلية العلوم, الجامعة المستنصرية, بغداد, العراق

الخلاصة

لنفترض ان *n* عددا صحيحا موجبا وان الداله $\overline{p}_3(n)$ تشير الى عدد التقسيمات الفوقيه الثلاثيه للعدد الصحيح *n* . في هذه المذكره نبرهن على اثبات متطابقتين من القياس 16 و 32 للداله $\overline{p}_3(n)$. قدمنا طريقه جديده لاعادة برهان نتيجة تعود الى لين وانغ والتي تحدد بصورة كامله $\overline{p}_3(n)$ للمقاس 16. هذا بالاضافة الى ايجاد وبرهان عائلة غير منتهيه من المتطابقات ذات القياس 32 للداله $\overline{p}_3(n)$. الاسلوب المتبع في البرهان يعتمد على توسيع دالة الاس الرابع للمضروب اللانهائي لدالة التجزئه الفوقية سوية مع استخدام محايدة كاوس.

1. Introduction

1.1 Partitions and Overpartitions. In 1964, Leibniz wrote to Bernoulli asking for the possible ways to deconstruct a positive integer into a sum of numbers (integers), later called parts [1]. We arrange these parts in a non-increasing order to specify such integer sums differently. Since that time, the history of integer partitions has begun. Euler investigated the number of ways to sum an integer n into m parts. Using the contemporary mathematical notation $D(n, m)$ which was introduced afterward to denote the number of partitioning n into m parts, Euler established the generating formula for such partitions and by taking m to infinity, the algebraic generating formula of the number of unrestricted partitions, known as $p(n)$, is identified by

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$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

The introduction of generating functions by Euler was without a doubt the most significant contribution in the historical record of partitions [1]. Several questions regarding $p(n)$ can be asked. In particular, about its mathematical properties. In 1920, Ramanujan [2] discovered remarkable arithmetic patterns for the partition function $p(n)$

$$p(ln+t) \equiv 0 \pmod{l},$$

where $(l,t) = (5,4), (7,5), (11,6)$. Following Ramanujan's discovery of these beautiful identities, the study of partitions has progressed far beyond $p(n)$. Identities of the form $f(l\,n + t)$ modulo powers of primes have become the focus of such research for a partition function, say $f(n)$, and for the celebration of Ramanujan's identities such equivalences are referred to as Ramanujan-type congruences. Later, an extension of partitions, overpartitions were introduced. ``An overpartition of a positive number is a partition in which the first occurrence of a part can be overlined" [3]. The function $\overline{p}(n)$ denotes the number of overpartitions of *n* and its generating function is noted by

$$\overline{P}(q) = \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n}.$$

After p(n), the function $\overline{p}(n)$ has become a prominent focus, and numerous number theorists have investigated its arithmetic properties and discovered a decent number of Ramanujan-type congruences. The works of Hirschhorn and Sellers [4,5], Kim [6,7], Lovejoy [8], Mahlburg [9] and Treneer [10] provide a wealth of information. In general, the proofs are approached using a variety of methods and techniques that include elementary number theory to modular forms. Many theorems and procedures for regular partitions have overpartition counterparts, which ought to come naturally. Several other generalizations, such as plane partitions and plane overpartitions have been introduced and investigated. For example, see [11],[12], [13] and [14].

1.2 Overpartition Triples. Taking a higher power of the overpartition function and obtaining a wider class of overpartitions as tuples is a natural generalization to overpartitions, which makes sense to examine for a bigger class of arithmetic properties linked to overpartitions. An overpartition triple of a positive integer *n* is a 3-tuple of overpartitions $(\lambda_1, \lambda_2, \lambda_3)$ such that $|\lambda_1| + |\lambda_2| + |\lambda_3| = n$ [15]. The function $\overline{p}_3(n)$ counts all overpartition triples of *n* and we define $\overline{p}_3(0) := 1$. For example, there are 24 overpartition triples of $n = 2$ given by

$$(2,\phi,\phi), (\overline{2},\phi,\phi), (\phi,2,\phi), (\phi,\overline{2},\phi), (\phi,\phi,2), (\phi,\phi,\overline{2}),$$
$$(1,1,\phi), (\overline{1},1,\phi), (1,\overline{1},\phi), (\overline{1},\overline{1},\phi), (\phi,1,1), (\phi,\overline{1},1), (\phi,1,\overline{1}), (\phi,\overline{1},\overline{1}),$$
$$(1,\phi,1), (\overline{1},\phi,1), (1,\phi,\overline{1}), (\overline{1},\phi,\overline{1}), (1+1,\phi,\phi), (\overline{1}+1,\phi,\phi),$$
$$(\phi, 1+1,\phi), (\phi, \overline{1}+1,\phi), (\phi,\phi,1+1), (\phi,\phi,\overline{1}+1).$$

Thus, $\overline{p}_3(2) = 24$. The generating function for overpartition triples is given by

$$\overline{P}_3(q) = \sum_{n=0}^{\infty} \overline{p}_3(n) q^n = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^3 \qquad \ldots(1)$$

Recall Gauss' identity [16],

$$\prod_{n=1}^{\infty}\frac{1-q^n}{1+q^n}=\frac{1}{\overline{P}(q)}=\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}=1+2\sum_{n=1}^{\infty}q^{4n^2}-2\sum_{n=1}^{\infty}q^{(2n-1)^2}.$$

Thus, we get

$$\frac{1}{\overline{P}(q)}\equiv 1+2\sum_{n=1}^{\infty}q^{4n^2}+14\sum_{n=1}^{\infty}q^{(2n-1)^2}\ (mod\ 16) \qquad \ldots(2)$$

and

$$\frac{1}{\overline{P}(q)}\equiv 1+2\sum_{n=1}^{\infty}q^{4n^2}+30\sum_{n=1}^{\infty}q^{(2n-1)^2}\ (mod\ 32) \qquad \ldots(3)$$

1.3 The Sum of Divisors. The sum of divisors function, it is often known as $\sigma(n)$, is a widely used mathematical function that has been linked to a number of well-known functions and notable identities such as the Riemann zeta function and the Dirichlet series. Robin's criterion for the Riemann hypothesis involving $\sigma(n)$ drew a lot of attention to this function. For a positive integer n,

$$\sigma(n)=\sum_{d|n}d,$$

these numbers are generated by the q-series

$$\sum_{n=1}^{\infty}\sigma(n)q^n=\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

To explore the function $\sigma(n)$ and its relation to $p(n)$, one may look at the surprising nearly equivalent recursions for these functions which satisfy

$$f(n)=f(n-1)+f(n-2)-f(n-5)-f(n-7)+f(n-12)+\cdots,$$

in which the difference only occurs when $n=0$ where $p(0)$ is replaced by 1 and $\sigma(0)$ by n. Furthermore, $p(n)$ and $\sigma(n)$ have a combination relationship stated by

$$mp(m)=\sum_{n=0}^{m}\sigma(n)p(m-n).$$

See [17] for more details about these relations.

Throughout the main proofs, we will frequently refer to a well-known fact that $\sigma(n)\equiv 1(mod\ 2)$ if and only if $n=m^2$ or $n=2m^2$ for some integer m.

2. Main Results and Proofs

The goal of this paper is to provide two theorems for the overpartition triple function $\overline{p}_3(n)$ modulo 16 and 32. The first result fully characterizes $\overline{p}_3(n)$ modulo 16 and the second result provides an infinite family of congruences modulo 32. We define the following function throughout the proofs of the main theorems,

$$A(q):=\frac{q}{1-q}.$$

Also for all integer k such that $k\geq 1$, we recall that the standard notation for the k^{th} power of the overpartition generating function is defined by

$$\overline{P}(q)^k=\overline{P}_k(q).$$

In particular, we prove the following identity modulo 16 for $\overline{P}_4(q)$.

Theorem 2.1

$$\overline{P}_4(q) \equiv 1 + 8\sum_{n=1}^{\infty}\left(q^{n^2} + q^{2n^2}\right) \ (mod\ 16).$$

Proof.

$$\overline{P}_4(q) = \prod_{n=1}^{\infty}\left(1 + 2\frac{q^n}{1-q^n}\right)^4 = \prod_{n=1}^{\infty}\left(1 + 4A(q^n) + 4A^2(q^n)\right)^2$$

$$\equiv \prod_{n=1}^{\infty}\left(1 + 8A(q^n) + 8A^2(q^n)\right) \qquad (mod\ 16)$$

$$\equiv 1 + 8\sum_{n=1}^{\infty} A(q^n) + A^2(q^n) \qquad (mod\ 16).$$

Extending each of the terms in the preceding series in terms of q-series,

$$A(q) + A^2(q) = \frac{q}{1-q} + \frac{q^2}{(1-q)^2} = \sum_{m=1}^{\infty} q^m + q^2\sum_{m=1}^{\infty} mq^{m-1}$$

$$= \sum_{m=1}^{\infty} q^m + mq^{m+1} = \sum_{m=1}^{\infty} mq^m.$$

Replacing q by q^n and taking sum over all integers$n \geq 1$, we obtain

$$\sum_{n=1}^{\infty} A(q^n) + A^2(q^n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} mq^{nm} = \sum_{n=1}^{\infty}(q^n + 2q^{2n} + 3q^{3n} + \cdots)$$

$$= q + (1+2)q^2 + (1+3)q^3 + (1+2+4)q^4 + (1+5)q^5 + \cdots$$

$$= \sum_{n=1}^{\infty}\sum_{d|n} d\, q^n = \sum_{n=1}^{\infty} \sigma(n)q^n.$$

Recall that $\sigma(n) \equiv 1\ (mod\ 2)$ for $n = m^2$ or$n = 2m^2$, $m \geq 1$. Thus, we conclude that

$$1 + 8\sum_{n=1}^{\infty} A(q^n) + A^2(q^n) \equiv 1 + 8\sum_{n=1}^{\infty}\left(q^{n^2} + q^{2n^2}\right) (mod\ 16),$$

as it is desired. □

For the fourth power of the overpartition generating function, the next result establishes an equivalence modulo 32.

Theorem 2.2

$$\overline{P}_4(q) \equiv 1 + 8\sum_{n=1}^{\infty}\sigma(n)q^n + 16\sum_{n=1}^{\infty}\left(q^{2n^2} + q^{4n^2}\right) \qquad (mod\ 32)$$

Proof.

$$\overline{P}_4(q) = \prod_{n=1}^{\infty}\left(1 + 8A(q^n) + 24A^2(q^n) + 32A^3(q^n) + 16A^4(q^n)\right)$$

$$\equiv 1+\sum_{n=1}^{\infty}\big(8A(q^n)+24A^2(q^n)+16A^4(q^n)\big)\ (mod\ 32)$$
$$\equiv 1+\sum_{n=1}^{\infty}\big(8A(q^n)+8A^2(q^n)+16A^2(q^n)+16A^4(q^n)\big)\ (mod\ 32)$$
$$\equiv 1+\sum_{n=1}^{\infty}(8\sigma(n)q^n+16\sigma(n)q^{2n})\quad (mod\ 32). \tag{4}$$

Note that

$$16\sum_{n=1}^{\infty}\sigma(n)q^{2n}\equiv 16\sum_{n=1}^{\infty}\big(q^{2n^2}+q^{4n^2}\big)\ (mod\ 32). \tag{5}$$

Thus, by plugging (5) in (4), we obtain

$$\overline{P}_4(q)\equiv 1+\sum_{n=1}^{\infty}\big(8\sigma(n)q^n+16q^{2n^2}+16q^{4n^2}\big)\ (mod\ 32). \qquad \square$$

The following result was proved by Wang [15], however, we give a different proof which involves recalling Theorem 2.1.

Theorem 2.3 (Wang, [15]). For all positive integers n, we have

$$\overline{p}_3(n)=\begin{cases}10 & if\ n\ is\ twice\ a\ square,\\ 8 & if\ n\ is\ an\ odd\ square,\\ 6 & if\ n\ is\ an\ even\ square,\\ 0 & otherwise.\end{cases}\qquad (mod\ 16)$$

Proof. By the help of Theorem 2.1 and equation (2), we get

$$\overline{P}_3(q)=\frac{\overline{P}(q)^4}{\overline{P}(q)}\equiv\frac{1+8\sum_{n=1}^{\infty}\big(q^{n^2}+q^{2n^2}\big)}{\overline{P}(q)}$$
$$\equiv\left(1+8\sum_{n=1}^{\infty}\big(q^{n^2}+q^{2n^2}\big)\right)\left(1+2\sum_{n=1}^{\infty}q^{4n^2}+14\sum_{n=1}^{\infty}q^{(2n-1)^2}\right)$$
$$\equiv 1+2\sum_{n=1}^{\infty}q^{4n^2}+14\sum_{n=1}^{\infty}q^{(2n-1)^2}+8\sum_{n=1}^{\infty}q^{n^2}+8\sum_{n=1}^{\infty}q^{2n^2}\ (mod\ 16).$$

Expanding the exponents into even and odd integers in the following series

$$\sum_{n=1}^{\infty}q^{n^2}=\sum_{n=1}^{\infty}q^{4n^2}+\sum_{n=1}^{\infty}q^{(2n-1)^2},$$

we obtain the following congruence modulus 16,

$$\overline{P}_3(q)\equiv 1+6\sum_{n=1}^{\infty}q^{(2n-1)^2}+8\sum_{n=1}^{\infty}q^{2n^2}+10\sum_{n=1}^{\infty}q^{4n^2}.$$

Extracting all terms of the form $q^{(2n-1)^2}$, we get

$$\overline{p}_3((2n-1)^2)\equiv 6\ (mod\ 16).$$

Similarly, extracting all terms of the forms q^{4n^2} and q^{2n^2}, we find

$$\overline{p}_3(4n^2) \equiv 10 \ \ (mod\ 16),$$
$$\overline{p}_3(2n^2) \equiv 8 \ \ (mod\ 16).$$

The final case of q^n is not among the forms $q^{(2m-1)^2}, q^{4m^2}, q^{2m^2}$ which provides

$$\overline{p}_3(n) \equiv 0 \ \ (mod\ 16).$$

By combining all the previous cases, the proof is completed. □

The next result yields an infinite family of congruences modulo 32 for overpartition triples.

Theorem 2.4. For all integer$k, n \geq 0$, we have

$$\overline{p}_3\left(2^k(8n+7)\right) \equiv 0 \ \ \ (mod\ 32).$$

Proof. By the help of Theorem 2.2 and equation (3), thus the modulus 32,

$$\overline{P}_3(q) = \frac{\overline{P}(q)^4}{\overline{P}(q)}$$
$$\equiv \left(1 + 8\sum_{n=1}^{\infty} \sigma(n) q^n + 16 \sum_{n=1}^{\infty}\left(q^{2n^2} + q^{4n^2}\right)\right)\left(1 + 2\sum_{n=1}^{\infty} q^{4n^2} + 30 \sum_{n=1}^{\infty} q^{(2n-1)^2}\right)$$
$$\equiv \ 1 + 8\sum_{n=1}^{\infty} \sigma(n) q^n + 16\sum_{n=1}^{\infty}\left(q^{2n^2} + q^{4n^2}\right) + 2\sum_{n=1}^{\infty} q^{4n^2} + 30\sum_{n=1}^{\infty} q^{(2n-1)^2}$$
$$+ 16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sigma(n) q^{n+4m^2} + 16 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sigma(n) q^{n+(2m-1)^2} \ \ (mod\ \ 32) \ldots . (6)$$

We observe that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sigma(n) q^{n+4m^2} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sigma(n) q^{n+(2m-1)^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sigma(n) q^{n+m^2}$$
$$\equiv \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q^{n^2+m^2} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q^{2n^2+m^2} \ (\ mod\ 2).$$

Because the number of ways of representing *n* as a sum of two squares is even where $n = x^2 + y^2 = y^2 + x^2$ counted as two different representations, we obtain

$$16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q^{n^2+m^2} + 16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q^{2n^2+m^2}$$
$$\equiv 16\sum_{n=1}^{\infty} q^{2n^2} + 16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q^{2n^2+m^2} \ (mod\ 32) \qquad \ldots . (7)$$

By plugging (7) in (6), we get

$$\overline{P}_3(q) \equiv 1 + 8\sum_{n=1}^{\infty} \sigma(n) q^n + 18\sum_{n=1}^{\infty} q^{4n^2} + 30\sum_{n=1}^{\infty} q^{(2n-1)^2}$$
$$+ 16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q^{2n^2+m^2} \ \ \ (mod\ 32).$$

Note that it is easy to show that $2^k(8n+7)$ is not a square or twice a square. Also, by the three-squares theorem, no number of the form $4^k(8n+7) = 2^{2k}(8n+7)$ is a sum of three squares. Indeed, we only need to show that $2^{2k-1}(8n+7)$ is not of form $2x^2+y^2$. Suppose not, thus for some x and y, $2^{2k-1}(8n+7)=2x^2+y^2$ which provides that y must be even and so we rewrite the last equation as $2^{2k-1}(8n+7)=2x^2+4y^2$ which implies $4^{k-1}(8n+7)=x^2+2y^2$ contradicting the three-squares theorem. Thus, the terms of the form $q^{2^k(8n+7)}$ can be extracted only from the series

$$\sum_{n=1}^{\infty}\sigma(n)q^n.$$

Together with the multiplicative property of $\sigma(n)$ and $\sigma(8n+7)\equiv 0\ (mod\ 4)$, we obtain $\overline{p}_3\left(2^k(8n+7)\right)\equiv 8\sigma\left(2^k(8n+7)\right)=8\sigma(2^k)\sigma(8n+7)\equiv 0\ (mod\ 32)$. □

By setting $k=1$ in Theorem 2.4, we obtain a theorem that is proved by L. Wang [15] as follows.

Corollary 2.5 ([15], Theorem 2.2). For any integer $n\geq 0$, we have

$$\overline{p}_3(16n+14)\equiv 0\ (mod\ 32).$$

3. Final Remarks and Conclusions

In a recent study, the first author has studied a two-dimensional generalized concept of overpartitions, named k-rowed plane overpartitions [12] where the number of rows is bounded by k. In terms of overpartition triples, the 3-rowed plane overpartition generating formula is as follows:

$$\sum_{n=0}^{\infty}\overline{pl}_3(n)q^n=\left(\frac{(1-q)^2(1-q^2)}{(1+q)^2(1+q^2)}\right)\sum_{n=0}^{\infty}\overline{p}_3(n)q^n,$$

where $\overline{pl}_3(n)$ denotes the number of 3-rowed plane overpartitions of n. The link between these two generating functions may lead to the discovery of new congruences modulo powers of 2 for $\overline{pl}_3(n)$ using the same technique adopted in this study.

Although the same procedures may be extended to a greater power of 2, the results gained in this study are limited to small powers of 2. Also, identities modulo powers of odd primes involving the overpartition function $\overline{p}(n)$ and the sum of divisors function $\sigma(n)$ would be interesting to find. There are also divisor functions that are similar to $\sigma(n)$, such as the sums of odd, even, and proper divisors in which other sorts of overpartitions, such as overpartitions with distinct parts or odd parts, may be linked to such functions. Many questions remain unanswered to have a better understanding of the overpartition function, and its connection related to squares and sum of squares that often appear in overpartition congruences modulo powers of two.

The literature on overpartition congruences is highly rich, and several overpartition-type functions have been introduced including the powers of overpartition function to achieve a larger class and more generalized findings. Thus, this paper will be followed by a few publications that will investigate various types of overpartition-type functions modulo small powers of two.

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