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Z-Small Quasi-Dedekind Modules

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Abstract
In this paper, we define and study z-small quasi-Dedekind as a generalization of small quasi-Dedekind modules. A submodule A of R-module M is called z-small $(A \ll_z M)$ if whenever $A + B = M, Z_2(M) \subseteq B$, then $B = M$. Also, M is called a z-small quasi-Dedekind module if for all $f \in End_R(M), f \neq 0$ implies $Kerf \ll_z M$. We also describe some of their properties and characterizations. Finally, some examples are given.

Keywords: Quasi-Dedekind module, Z-small quasi-Dedekind module, Second singular submodule.

المقاسات شبة الديكاندية الصغيرة من النمط z

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الخلاصة :
في هذا البحث عرفنا ودرسنا المقاس شبه الديكاندي الصغير من النمط z كاتعميم للمقاس شبه الديكاندي الصغير . يقال للمقاس الجزئي الصغير من النمط z اذا كان $A + B = M, Z_2(M) \subseteq B$ فان, $B = M$.
يقال للمقاس انه مقاس شبه ديكاندي صغير من النمط z اذا كان لكل تشاكل $0 \neq f \in End(M)$ فان نواة التشاكل تكون مقاس جزئي صغير من النمط z. واعطينا بعض خصائصها وتوصيفاتها وأمثلة عليها.

1. Introduction

Consider that R is a ring with identity and M is a unitary right module. A submodule of R is called a small submodule of M if $M = W + U$, where U is a submodule of M implies that $U = M$. It is known that a submodule A of an R-module M is said to be essential in M if $A \cap W \neq 0$ for every non-zero submodule W of M.

Recall that M is called a small quasi-Dedekind module if for all $f \in End_R(M), f \neq 0$ implies $Kerf \ll M$"[1]. Amina in [2] introduced the concept of z-small (briefy $A \ll_z M$) if whenever $A + B = M, Z_2(M) \subseteq B$, then $B = M$. $Z_2(M)$ is called the second singular

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submodule of M which is defined in [3] by $Z(\frac{M}{Z(M)}) = \frac{Z_2(M)}{Z(M)}$, where $Z(M) = \{x \in M : xI = 0$ for some essentially ideal I of $R\}$. Equivalently, $Z(M) = \{x \in M: ann_R(x) \leq_{ess} R\}$ M is called singular (non-singular) if $Z(M) = M(Z(M)) = 0$ [3]. Asgri [4] proved that $Z_2(M) = \{x \in M: xI = 0$, for some t-essentially ideal I of $R\}$. Equivalently $Z_2(M) = \{x \in M: ann(x)_R \leq_{tes} R\}$. It is worth that a submodule N of M is called t-essentially in M, $N \leq_{tes} M$ if $N \cap W \subseteq Z_2(M), W \leq M$, then $W \subseteq Z_2(M)$ [4].

Note that every small submodule is z-small, however, the converse is not true. This motivate us to introduce a generalization of small quasi-Dedekind, namely z-small quasi-Dedekind. A module M is called a z-small quasi-Dedekind module if for all $f \in End_R(M), f \neq 0$ implies $Kerf \ll_z M$ (i.e $Kerf$ is a z-small submodule in M). Every small quasi-Dedekind R-module is a z-small quasi-Dedekind R-module . However, the converse is not true. Many characterizations of z-small quasi-Dedekind modules are given.

2. Z-Small Quasi-Dedekind Modules

First, we begin with the following lemma and remarks that we need throughout this work.

Lemma (2.1)

Let $f: M \longrightarrow \acute{M}$ be a monomorphism. Then $f(Z_2(M)) = Z_2(f(M))$.

Proof

Let $y \in Z_2(f(M))$. Then $y = f(M)$ and $f(M)I = 0$ for some $I \leq_{tes} R$. So $f(MI) = 0$ since f is one to one, this implies $mI = 0$. Hence, $m \in Z_2(M)$. Thus $y = f(m) \in f(Z_2(M))$, then we get $Z_2(f(M)) \subseteq f(Z_2(M))$.
Conversely, let $y \in f(Z_2(M))$, then $y = f(x)$ and $x \in Z_2(M)$, that is $xI = 0$ for some $I \leq_{tes} R$. It follows that $f(x)I = 0$, i.e $y = f(x) \in Z_2(f(M))$. Thus, $f(Z_2(M)) \subseteq Z_2(f(M))$.

Proposition (2.2)

1) Let M be an R-module, $f \in End_R(M)$, there exists f which is monomorphism , $N \leq M$ if $f(N) \ll_z f(M)$, then $N \ll_z M$.
2) Let $f \in End_R(M), f$ is monomorphism if $N \ll_z f(M)$ then $f^{-1}(N) \ll_z M$.
3) Let $\frac{A}{C} \ll_z \frac{M}{C}$, $C \ll_z M$, then $A \ll_z M$.

Proof:

1) Let $N + B = M$, $B \supseteq Z_2(M)$. Then $f(N) + f(B) = f(M)$ since $B \supseteq Z_2(M)$, we get $f(B) \supseteq f(Z_2(M))$. Hence, $f(B) = f(M)$ by Lemma(2.1). Since $f(N) \ll_z f(M)$. But, $f(B) = f(M)$ implies $B = M$ since f monomorphism. Thus $N \ll_z M$.
2)Let $f^{-1}(N) + L = M$, $L \supseteq Z_2(M)$. Then $f(f^{-1}(N) + f(L) = f(M)$. As $L \supseteq Z_2(M), f(L) \supseteq f(Z_2(M))$ and since f is monomorphism , $f(L) \supseteq Z_2(f(M))$. But $f(f^{-1}(N)) \subseteq N$ so that $N + f(L) = f(M)$. As $N \ll_z f(M), f(L) = f(M)$. But f is a monomorphism , hence $L = M$. Thus, $f^{-1}(N) \ll_z M$.
3)Let $A + B = M$, $B \supseteq Z_2(M)$. Then $\frac{A}{C} + \frac{B+C}{C} = \frac{M}{C}$. But $\frac{B+C}{C} \supseteq \frac{Z_2+C}{C} \supseteq Z_2(\frac{M}{C})$, since for any $y \in Z_2\left(\frac{M}{C}\right), y = m + c$ and $(m + c)I = 0$ for some $I \leq_{tes} R$. Hence $mI = 0$ and so $m \in Z_2(M)$. Thus $m + c \in \frac{Z_2(M)}{C}$. It follows that $\frac{B+C}{C} = \frac{M}{C}$ since $\frac{A}{C} \ll_z \frac{M}{C}$. Thus $B + C = M$. But $C \ll_z M$ and $B \supseteq Z_2(M)$ so that $B = M$. Thus $A \ll_z M$.

Definition (2.3)

An R-module M is called a z-small quasi-Dedekind module if for all $f \in End_R(M), f \neq 0$ implies that $Kerf \ll_z M$(i.e $Kerf$ is a z-small submodule in M).
Note*: M denoted z-small quasi-Dedekind module.

Remarks and Examples (2.4)

1) It is clear that every small quasi-Dedekind R-module is a z-small quasi-Dedekind R-module. But the converse is not true in general.
2) For example: Z_6 as Z-module is not small quasi-Dedekind, since there is, $f: Z_6 \longrightarrow Z_6$ which is defined by $f(\overline{x}) = 3\overline{x}, \overline{x} \in Z_6$. So $f \neq 0$, but $Kerf = \{\overline{x} \in Z_6: f(\overline{x}) = \overline{x}\} = \{\overline{x} \in Z_6: 3\overline{x} = \overline{0}\} = (\overline{2})$ *is not small in* Z_6. However, Z_6 is M Z-module. Since for each $f \in Z_6, f \neq 0, Kerf \leq Z_6$. But Z_6 as Z-module is singular, so $Z_2(Z_6) = Z_6$ and so $Kerf \ll_z Z_6$.
3) $Z \oplus Z$ is not M Z-module, $\exists f: Z \oplus Z \longrightarrow Z \oplus Z \ni f(x, y) = (x, 0); x, y \in Z$. So $f \neq 0$, but $Kerf = (0) \oplus Z$ *is not* $z-smll$ $Z \oplus Z$. Since $Kerf = 0 \oplus Z$ and $(0 \oplus Z) + (Z \oplus 0) = Z \oplus Z$ and $Z \oplus (0) \supseteq Z_2(Z \oplus Z) = 0$, but $Z \oplus (0) \neq Z \oplus Z$.
4) If $M = 0$, then M is M in note *
5) Any integral domain R is an M R-module, but the converse is not true in general, for example:
Z_4 as Z_4-module is a M in note *, but it is not an integral domain.

Proposition (2.5)

Let M be an R-module. Then M is in note * if and only if $Hom(M/N, M) = 0$ for all $N \ll_z M$.

Proof

⇒Suppose that there exists N *which is not* z-small of $M \ni Hom(M/N, M) \neq 0$, then there exists $\emptyset: M/N \longrightarrow M, \emptyset \neq 0$. Hence, $\emptyset \circ \pi \in End_R(M)$, where $\emptyset \circ \pi \neq 0$ and π is the canonical projection which implies $Ker(\emptyset \circ \pi) \ll_z M$ but $N \subseteq Ker(\emptyset \circ \pi)$, so $N \ll_z M$ by[2] that is a contradiction.
⇐Assume that there is $f: M \longrightarrow M, f \neq 0 \ni Kerf$ is not z-small in M. Define $g: M / Kerf \longrightarrow M$ by $g(m + Kerf) = f(m)$, for all $m \in M$. So g is well-defined and $g \neq 0$. Hence, $Hom(M/Kerf, M) \neq 0$ that is a contradiction.

Proposition(2.6)

Let I is an ideal of $R \ni I \subseteq ann_R(M)$, M be an R-module and $\overline{R} = R/I$, then M in note * R-module iff M in note * $\overline{R}$-module.

Proof

⇒We have $Hom_R(M/K, M) = Hom_{\overline{R}}(M/K, M)$, for all $K \leq M$, by [5 , p.51]. Thus, if M is in note * R-module, then $Hom_R(M/K, M) = 0$ for all $K \nleq_{z-s} M$, so $Hom_{\overline{R}}(M/K, M) = 0$ for all $K \nleq_{z-s} M$, thus M is a z-small quasi-Dedekind $\overline{R}$-module.
⇐ By the way, the converse can be proved.

Proposition(2.7)

Let M_1, M_2 be R-modules $\ni M_1 \cong M_2$. M_1 is M in note * R-module if and only if M_2 is M in note * R-module.

Proof

⇒Let : $M_2 \longrightarrow M_2$, $f \neq 0$. To prove ker f $\ll_z M_2$. Since $M_1 \cong M_2$,there exists an isomorphism $g: M_1 \longrightarrow M_2$. Consider the following: $M_1 \xrightarrow{g} M_2 \xrightarrow{f} M_2 \xrightarrow{g^{-1}} M_1$. Hence, $h = g^{-1} \circ f \circ g \in End_R(M_1), h \neq 0$. So $Kerh \ll_z M_1$ (since M_1 is z-small quasi-Dedekind), then $g(Kerh) \ll_z M_2$ by [2, Proposition 2.15]. But, we can show that $g(Kerh) = Kerf$ as follows: let $y \in g(Kerh)$, so $y = g(x), x \in Kerh$. So $h(x) = 0$; that is $g^{-1} \circ f \circ g(x) = 0$, then $g^{-1}f(y) = 0$, so $g^{-1}(f(y)) = 0$ and hence $f(y) = 0$, since g^{-1} is monomorphism , so that $y \in Kerf$, hence $g(Kerh) \subseteq Kerf$. Now, let $y \in Kerf$, then $f(y) = 0$, but $y \in M_2$, so there exists an $x \in M_1 \ni y = g(x)$, because g is onto. Hence, $f(g(x)) = 0$ and so $g^{-1}(f((g(x))) = 0$, that is $h(x) = 0$. Therefore, $x \in Kerh$ .Hence, $y = g(x) \in g(Kerh)$. Therefore, $Kerf = g(Kerh) \ll_z M_2$, hence $Kerf \ll_z M_2$.
⇐ The proof of the converse of the proposition is similar.

Remark (2.8)

Suppose $N \leq M$, $f \in End_R(M), f \neq 0$, and $f(N) \ll_z f(M)$ then it is not necessarily $N \ll_z M$.

Proposition (2.9)

Let M be in note * R-module, $f \in End_R(M), f \neq 0$, and $f(Z_2(M)) = Z_2(f(M)), N \leq M$. If $f(N) \ll_z f(M)$ then $N \ll_z M$.

Proof

Let $B \leq M$ and $N + B = M$ and $B \supseteq Z_2(M)$ then $f(N) + f(B) = f(M)$ $f(B) \supseteq f(Z_2(M))$ so $f(B) \supseteq Z_2(f(M))$ by hypothesis, since $f(N) \ll_z f(M)$ implies $f(B) = f(M)$. Now, we can show that $Kerf + B = M$. Let $m \in M$, hence $f(m) \in f(M) = f(B)$. As a result, there is $b \in B \ni f(m) = f(b)$, hence $m - b \in Kerf$. It follows that $m = (m - b) + b$, thus $M \subseteq Kerf + B$. Thus $Kerf + B = M$, but M is a z-small quasi-Dedekind R-module , so $Kerf \ll_z M$ this indicates $B = M$. Therefore $N \ll_z M$.

Proposition (2.10)

Let Mbe note * R-module , $f \in End_R(M), f \neq 0 , N \leq M$. If $N \ll_z f(M)$ then $f^{-1}(N) \ll_z M$.

Proof

It is clear that $Kerf \subseteq f^{-1}(N)$, we will prove $\frac{f^{-1}(N)}{Kerf} \ll_z \frac{M}{Kerf}$. Let $\frac{f^{-1}(N)}{Kerf} + \frac{L}{Kerf} = \frac{M}{Kerf}$, where $\frac{L}{Kerf} \leq \frac{M}{Kerf}$. Then $f^{-1}(N) + L = M$, hence $f(f^{-1}(N)) + f(L) = f(M)$ but $f(f^{-1}(N)) \subseteq N$, then $f(M) = f(f^{-1}(N)) + f(L) \subseteq Nf(L)$, also , we have $N \subseteq f(M)$ and $f(L) \subseteq f(M)$, so $N + f(L) \subseteq f(M)$ and thus $N + f(L) = f(M)$. Since $N \ll_z f(M)$, and $L \supseteq Z_2(M) \Rightarrow f(L) \supseteq f(Z_2(M))$. Hence $f(L) \supseteq Z_{2(}f(M))$ by hypothesis then $f(L) = f(M)$. We claim that $L = M$. Let $x \in M$, then $f(x) \in f(M) = f(L)$, hence $f(x) = f(l)$ for some $l \in L$. Consequently $x - l \in Kerf \subseteq L$ and hence $x \in L$, so $M \subseteq L$. Thus $M = L$ which implies $\frac{L}{Kerf} = \frac{M}{Kerf}$, so $\frac{f^{-1}(N)}{Kerf} \ll_z \frac{M}{Kerf}$. But $Kerf \ll_z M$, so by Proposition(2.2), $f^{-1}(N) \ll_z M$.

Proposition (2.11)

Let M be in note * and quasi-injective R-module, $N \leq M$ ∋for every $U \leq N, U \ll_z M$ implies $U \ll_z N$.Then N is as in note * R-module.

Proof

Let $f: N \longrightarrow N, f \neq 0$. To show $Kerf \ll_z N$.Since M is a quasi-injective R-module , there exists $g: M \longrightarrow M \ni g \circ i = i \circ f$, where i is the inclusion mapping .

Then $g(N) = f(N) \neq 0$; that is $g \neq 0$. So that $Kerg \ll_z M$, since M is z-small quasi-Dedekind. But $Kerf \subseteq Kerg$, hence $Kerf \ll_z M$ .On the other hand $Kerf \leq N$, so $Kerf \ll_z N$. Thus N is a R-module in note *.

Recall that a submodule N of a module M is called Z-coclosed in M(briefly $N \ll_{z-cc} M$, if whenever $H \leq N$ with $\frac{N}{H} \ll_z \frac{M}{H}$ implies that $N = H$.[2]

Corollary(2.12)

Let M be quasi-injective R-module, let $N \leq M$. If N is a z-coclosed submodule, then N is a z-small quasi-Dedekind R-module.

Proof

Let N be z-coclosed for all $U \leq N, U \ll_z M$ implies $U \ll_z N$. [2, Reamers and examples 2.2(5)]. Then the result follows by Proposition (2.11).

Corollary(2.13)

Let M be an R-module, there exist $\overline{M}$ is a in note * quasi injective R-module , and for all $U \leq M, U \ll_z \overline{M}$ implies $U \ll_z M$. Then M is a R-module in note *.

Proof

Since $\overline{M}$ is a R-module in note * and quasi-injective R-module, then M is a R-module in note * by Proposition 2.11**.**

Proposition(2.14)

Let M be an R-module in note * . Then $ann_R(N) = ann_R(M)$ for all N is not z-small of M.

Proof

Since M is an R-module in note *, so by Theorem(2.5), $Hom(M/_N, M) = 0$ for all N is not $\text{z} - \text{small of}$ M which implies N is a quasi-invertible submodule for all N is not z-small in M. Thus by [1, Proposition 0.1.3 $ann_R(N) = ann_R(M)$ for all $N\ not\ z - small\ in\ M$.

Proposition(2.15)

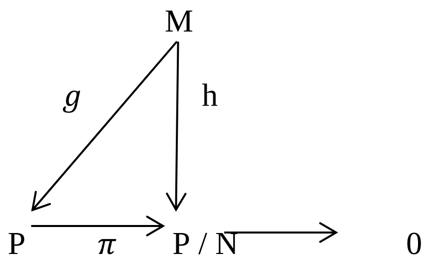
Let M be an R-module ∋ $M/_U$ is projective for all U is not z-small in M. If M is an R-module in note *, then $M/_N$ is R-module in note * $\forall N \leq M$.

Proof

Let $K/_N$ is not z-small in $M/_N$, so by [1, Corollary 2.1.4], K is not z-small in M. Suppose that $Hom(\frac{M/_N}{K/_N},\frac{M}{N}) \neq 0$, but $Hom(\frac{M/_N}{K/_N},\frac{M}{N}) \cong Hom(\frac{M}{K},\frac{M}{N})$, so there exists $f: M/_K \longrightarrow M/_N$, $f \neq 0$. Since $M/_K$ is projective, then there exists $g: M/_K \longrightarrow M \ni \pi \circ g = f$, where π is the canonical projection.

Hence $\pi \circ g(M/_K) = f(M/_K) \neq 0$, so $g \neq 0$, but $g \in Hom(M/_K, M)$, Ks not z-small in M. Thus $Hom(M/_K, M) \neq 0$, K s not z-small in M; that is M is not R-module in note *, that is a contradiction. Hence, $M/_N$ is an R-module note *.

Let M and P be modules, then M is called P-projective in the case for all $N \leq P$ and any homomorphism $h: M \longrightarrow P/_N$, there exists a homomorphism $g: M \longrightarrow P \ni \pi \circ g = h$ where π is the natural epimorphism. Hence, the following diagram is commutive , [6].



An R-module M is called quasi-projective if , M is M-projective; that is for each $N \leq M$ and every homomorphism $h: M \longrightarrow M/_N$, there exists a homomorphism $g: M \longrightarrow M \ni \pi \circ g = h$, where π is the natural epimorphism[7].

Proposition(2.16)

Let M be a quasi-projective R-module and $N \leq M \ni g^{-1}(N) \ll_z M$ for each $g \in End_R(M)$, then $M/_N$ is a z-small quasi-Dedekind R-module.

Proof

Assume $f: M/_N \longrightarrow M/_N \ni f \neq 0$, because M is quasi-projective, there exists a homomorphism $g: M \longrightarrow M \ni \pi \circ g = f \circ \pi$ (π canonical projection).

Let $Kerf = L/_N = \{x + N: f(x + N) = N\}$
$= \{x + N: f \circ \pi(x) = N\} = \{x + N: \pi \circ g(x) = N\}$
$= \{x + N: g(x) + N = N\} = \{x + N: g(x) \in N\}$
$= \{x + N: x \in g^{-1}(N)\}$.

Thus $Kerf = g^{-1}(N)/_N$, but $g^{-1}(N) \ll_z M$, so by [2,Corollary 2. 4], $g^{-1}(N)/_N \ll_z M/_N$; that is $Kerf \ll_z M/_N$.

Corollary(2.17)

Let M be a quasi-projective R-module ∋for each $N \leq M, N \ll_z h(M)$ for all $h \in End_R(M)$. Then M is R-module in note * iff $M/_N$ is a z-smallquasi-Dedekind R-module.

Proof

⇒If $N = (0)$, we get the result.

⇐By Proposition 2.6, $N \ll_z h(M)$ implies that $h^{-1}(N) \ll_z M$. As a result, the following is the result: by the previous theorem.

an R-submodule N of an R-module M is invariant if $f(N) \subseteq N$ for each $f \in End_R(M)$. A fully invariant submodule is a term used by authors to describe an invariant submodule [8].

Proposition(2.18)

Let M be an R-module .Then M is R-module in note * if and only if there exists $N \ll_z M$, N is full invariant such that for each $f \in End_R(M), f \neq 0, f(M) \not\subset N$ and $M/_N$ is in note *.

Proof

⇒Choose $N = (0)$ implies $N \ll_z M$ and N is fully invariant and for all $f \in End_R(M), f \neq 0$, hence $f(M) \not\subset (0) = N$ and $M/_N = M/_{(0)} \cong M$ is z-small quasi-Dedekind.

⇐If $N = 0$, then M is in note *. Assume that $N \neq (0), N \leq_{z-s} M$. Let $f \in End_R(M), f \neq 0$. To prove $Kerf \ll_z M$. Define $g: M/_N \longrightarrow M/_N$ by $g(m + N) = f(m) + N$ for all $m \in M. g$ is well-defined , since if $m_1 + N = m_2 + N$ where $m_1, m_2 \in M$, then $m_1 - m_2 \in N$ and $f(m_1 - m_2)f(N) \subseteq N$, since N is fully invariant. This implies $f(m_1) - f(m_2) \in N$; that is:

$f(m_1) + N = f(m_2) + N$, thus

$g(m_1 + N) = g(m_2 + N). g \neq 0$, because if $g = 0$ then $g(M/_N) = N = 0_{M/_N}$. Hence, $f(M) + N = N$, it follows that $f(M) \subseteq N$ which is a contradiction with the hypothesis. Thus, $Kerg \ll_z M/_N$, since $M/_N$ is a z-small quasi-Dedekind R-module. Assume that $Kerg = L/_N \ll_z M/_N$, but $N \leq_{z-s} M$, so by [Remark 2..2(3)] , $L \ll_z M$. It is easy to prove the other direction to see that $Kerf \subseteq L$, so $Kerf \ll_z M$, hence M is an R-module in note *.

Recall that an R-module M is called multiplication if for each submodule N of M, there exists an ideal I of R such that $N = MI$

Corollary(2.19)

If M a multiplication R-module. Then M is R-module in note * if and only if there exists $N \ll_z M$ such that for all $f \in End_R(M), f \neq 0, f(M) \not\subset N$ and $M/_N$ is R-module in note *.

Proof

Since M is a multiplication R-module, every proper submodule of M is fully invariant. As a consequence, the result is attained by Theorem 2.14.

Proposition(2.20)

Every direct summand of an M is an R- module in note *.

Proof

Let $M = N \oplus K$ such that M is R-module in note *. To prove that K is a z-small quasi-Dedekind R-module. Let $f: K \longrightarrow K$, $f \neq 0$. Consider that $M \xrightarrow{\rho} K \xrightarrow{f} K \xrightarrow{i} M$, where ρ is the natural projection , and i is the inclusion mapping. So $h = i \circ f \circ \rho \in End_R(M), h \neq 0$ since there exists $y \in K$, $f(y) = x \neq 0$. Hence for each $z \in N$ then $h(z + y) = i \circ f \circ \rho(z + y) = i \circ f(y) = i(f(y)) = i(x) = x \neq 0$. Hence $Kerh \ll_z M$, since M is z-small quasi-Dedekind. But $Kerf \subseteq Kerh$, so $Kerf \ll_z M$. On the other hand $Kerf \leq K$ implies by [2, Remark and Examples 2.2(5)]. $Kerf \ll_z K$. Thus K is R-module in note *.

Proposition(2.21)

If M an R-module, and $N, L \leq M \ni N + L = M$, and $\frac{M}{N \cap L}$ is a z-small quasi-Dedekind R-module, then $\frac{M}{N}$ and $\frac{M}{L}$ are an R-modules in note *.

Proof

Since $\frac{M}{N \cap L} = \frac{N+L}{N \cap L} = \frac{N}{N \cap L} \oplus \frac{L}{N \cap L}$ .Hence, by Theorem 2.20, $\frac{N}{N \cap L}$ and $\frac{L}{N \cap L}$ are an R-modules in note *. But, $\frac{N}{N \cap L} \cong \frac{N+L}{L} = \frac{M}{L}$, also $\frac{L}{N \cap L} \cong \frac{N+L}{N} = \frac{M}{N}$. So $\frac{M}{N}$ and $\frac{M}{L}$ are R-modules in note *.

Definition(2.22)

Let M , N be R-modules. M is said to be an R-module in note *relative to N if , for all $f \in Hom(M, N), f \neq 0$ implies $Kerf \ll_z M$.

Remarks and Examples(2.23)

1) M is R- module in note * if and only if M is R-module in note * relative to M.

2) Z is not module in note * relative to Z_4 , because there exists $f: Z \longrightarrow Z_4 \ni f(x) = \overline{x} \; \forall \, x \in Z$. It is clear that $f \neq 0$, but $Kerf = 4Z \not\ll\not\ll_z Z$.

Proposition(2.24)

Let M_1 , M_2 be R-modules and let $M = M_1 \oplus M_2$. If M is a z-small quasi-Dedekind R-module, then M_i is a module in note * relative to M_j for all $i, j = 1,2$.

Proof

Since $M = M_1 \oplus M_2$ is a Module in note *, so by Proposition 2.20, M_1 and M_2 are z-small quasi-Dedekind. Hence, by Remarks and Examples 2.23(1), $M_1$is a z-small quasi-Dedekind relative to M_1, and M_2 is a z-small quasi-Dedekind relative to M_2. Now, to prove that $M_1$is a z-small quasi-Dedekind relative to M_2. Assume $f: M_1 \longrightarrow M_2$, $f \neq 0$ and $M \xrightarrow{\rho} M_1 \xrightarrow{f} M_2 \xrightarrow{i} M$, ρ natural projection, i inclusion mapping.

Let $h = i \circ f \circ \rho \in End_R(M)$, since $f \neq 0$, there exists $0 = x \in M_1$ and $f(x) \neq 0$. This implies $0 \neq (x, 0) \in M$ and $h(x, 0) = h(x, 0) = i \circ f \circ \rho(x, 0) = i \circ f(x) = f(x) \neq 0$, thus $h \neq 0$. It follows that $Kerh \ll_z M = M_1 \oplus M_2$, since a z-small quasi-Dedekind R-module . But $Kerf \oplus (0) \subseteq Kerh$. since for $(x, 0) \in Kerf \oplus (0)$ implies $h(x, 0) = i \circ f \circ \rho(x, 0) = i \circ f(x) = f(x) = 0$ so $(x, 0) \in Kerh$, thus $Kerf \oplus (0) \ll_z M = M_1 \oplus M_2$, and hence by [2, Proposition 2.7] $Kerf \ll_z M_1$, so M_1 is a module in note * relative to M_2.

In general, the converse of Proposition 2.24 is not true, consider the following example.

Example(2.25)

Z as Z-module is in note * , so Z is a M in note *relative to Z. But $M = Z \oplus Z$ is not as Z-module ,see Remarks and Examples 2.2(3).

Conclusions:

In this work, A z-small quasi-Dedekind as a generalization of small quasi-Dedekind modules is defined and studied. In addition, some of their properties and characterizations are described. Finally, some examples and significant results are given.

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