On Complete Intuitionistic Fuzzy Roh-Ideals in Roh-Algebras

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Abstract

In this paper, we shall investigate and study some kinds of ρ−ideals in an intuitionistic fuzzy setting, they are called complete intuitionistic fuzzy ρ−subalgebra, complete intuitionistic fuzzy ρ−ideal, and complete intuitionistic fuzzy ρ̅−ideal. In this study, we have also proposed some hypotheses to explain some of the relationships between these kinds of intuitionistic fuzzy ideals.

Keywords: Intuitionistic fuzzy ρ−subalgebra, Intuitionistic fuzzy ρ−ideal, intuitionistic fuzzy ρ̅-ideal.

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1. Introduction

The invention of the fuzzy subset of a set is investigated by Zadeh [1] in 1965. Scholars have been interested in extending the notions and outcomes of every concept in mathematics to the boarder framework of fuzzy setting. BCK-algebras were proposed by Imai and Iseki [2] as a generalization of the concept of set theoretic difference and propositional calculus. In the same year, Iseki [3] introduced BCI-algebra, which is a generalization of BCK-algebra. In BCK-algebra, Xi [4] introduced fuzzy ideals as well as the concept of fuzzy subalgebra. In 1996, the d-algebras class is given by Neggers and Kim [5] which is a generalization of BCK-algebras, and examined the relationship between them. Akram and Dar [6] considered the connotations of the fuzzy d-(algebra/subalgebra/ideal). In 2009, Kim [7] investigated the connotation of
fuzzy dot subalgebra in d-algebra. After that, the connotation of fuzzy dot d-ideals of a d-algebra is introduced by Al-Shehri [8]. Yun et al. [9] employed the connotation of intuitionistic fuzzy set in d-algebra and considered intuitionistic fuzzy topological d-algebra with intuitionistic fuzzy d-algebra. Mahmood & Abud Alradha [10] introduced the ρ-algebra. After that, many of structures and applications in ρ – algebras using fuzzy sets [11] and soft sets [12,13] are considered. Here, in this paper, we introduce the notion of complete intuitionistic fuzzy ρ – subalgebra, complete intuitionistic fuzzy ρ – ideal, and complete intuitionistic fuzzy ρ̅ – ideal of ρ – algebra. The goal of this work is to study and investigate new ideals in intuitionistic fuzzy ρ – ideals. In addition, several ideas are presented to explain some of the links between these intuitionistic fuzzy ρ – ideals.

2. Preliminaries and Some Results.

In the present section, we will recall some basic concepts and results that are necessary for this article.

Definition 2.1.[10] A ρ-algebra is a non-empty set U with a constant 0 and a binary operation “∩” which satisfies the following axioms:

(1) α∩α = 0,
(2) 0∩α = 0,
(3) α∩β = 0 = β∩α imply that α = β,
(4) For all α ⊈ β ∈ U – {0} imply that α∩β = nα ≠ 0.

Definition 2.2. [10] Let ∅ ≠ Y ⊆ U and (U, n, 0) be a ρ-algebra. We say Y is a ρ – subalgebra of U if α∩β ∈ Y for any α, β ∈ Y.

Definition 2.3.[10] A non-empty subset Y of a ρ – algebra U is called an ρ – ideal of U if the following conditions hold::

(1) α, β ∈ Y ⇒ α∩β ∈ Y,
(2) α∩β ∈ Y & β ∈ Y ⇒ α ∈ Y.

Remark 2.4[10]. If Y is any a ρ – ideal, then Y is ρ – subalgebra. However, the converse may be not holding.

Definition 2.5.[10] A non- empty subset Y of a ρ-algebra U is called an ρ̅ – ideal of U if it satisfies the following:

(1) 0 ∈ Y,
(2) α ∈ Y & β ∈ U ⇒ α∩β ∈ Y.

Proposition 2.6. [10] Assume that ∅ ≠ Y ⊆ U, where U is ρ-algebra. Then Y is a ρ- subalgebra of U if it is ρ̅ - Ideal.

Definition 2.7. [14] An Intuitionistic fuzzy set (briefly,IFS) over the universal U is defined by I = {< α, I_τ(α), I_ρ(α) >| α ∈ U}, where I_τ(α) and I_ρ(α): U → [0,1] are maps, with I_τ(α) and I_ρ(α) are real numbers and their values represent the degree of membership and non- membership of α to I, respectively.

Definition 2.8[15] A complement an intuitionistic fuzzy set I^c over the universal U is defined by I^c = 1 – I = 1 – {< α, I_τ(α), I_ρ(α) >| α ∈ U}

= {< α, 1 – I_τ(α), 1 – I_ρ(α) >| α ∈ U}

= {< α, I_τ^c(α), I_ρ^c(α) >| α ∈ U}.

Definition 2.9.[14] Let I be an IFS over the universal U and t ∈ [0,1] then the set (< α, I_τ(α)
\[ t \leq t \mid \alpha \in \mathbb{U} \] is called an intuitionistic fuzzy (t-cut), (briefly, IF -t-cut) and denoted by \( I_t \).

**Definition 2.10.** [13] An IFS \( I \) in \( \mathbb{U} \) is called an intuitionistic fuzzy \( \rho \)-subalgebra (briefly, IF \(-\rho - SA\)) of \( \mathbb{U} \) if it satisfies the following conditions:

(i) \( I_T(\alpha \land \beta) \geq \min \{ I_T(\alpha), I_T(\beta) \} \),

(ii) \( I_F(\alpha \land \beta) \leq \max \{ I_F(\alpha), I_F(\beta) \} \), for any \( \alpha, \beta \in \mathbb{U} \).

**Definition 2.11.**[13] Let \((\mathbb{U}, ^\mu, 0)\) be a \( \rho \)-algebra and \( I \) be an IFS of \( \mathbb{U} \). We say that \( I \) is an intuitionistic fuzzy \( \rho \)-ideal of \( \mathbb{U} \) (briefly, IF \(-\rho - I\) ) if the following conditions hold:

(i) \( I_T(\alpha \land \beta) \geq \min \{ I_T(\alpha), I_T(\beta) \} \),

(ii) \( I_F(\alpha \land \beta) \leq \max \{ I_F(\alpha), I_F(\beta) \} \),

(iii) \( I_T(\alpha) \geq \min \{ I_T(\alpha), I_T(\beta) \} \),

(iv) \( I_F(\alpha) \leq \max \{ I_F(\alpha), I_F(\beta) \} \), for any \( \alpha, \beta \in \mathbb{U} \).

**Definition 2.12.**[13] Let \((\mathbb{U}, ^\mu, 0)\) be a \( \rho \)-algebra and \( I \) an IFS of \( \mathbb{U} \). We say that \( I \) is an intuitionistic fuzzy \( \bar{\rho} \)-ideal of \( \mathbb{U} \) (briefly, IF \(-\bar{\rho} - I\)) if the following conditions hold:

(i) \( I_T(0) \geq I_T(\alpha) \),

(ii) \( I_F(0) \leq I_F(\alpha) \),

(iii) \( I_T(\alpha \land \beta) \geq \min \{ I_T(\alpha), I_T(\beta) \} \),

(iv) \( I_F(\alpha \land \beta) \leq \max \{ I_F(\alpha), I_F(\beta) \} \), for any \( \alpha, \beta \in \mathbb{U} \).

**Remarks 2.13.**[13]:

1- Every (IF \(-\rho - I)\) is (IF \(-\rho - SA\)).

2-Let \( I \) be (IF \(-\bar{\rho} - I)\) then \( I \) is (IF \(-\rho - SA)\).

**Lemma 2.14.**[13] Let \( I \) be an IF \(-\rho - SA\) of \( \mathbb{U} \) then:

(i) \( I_T(0) \geq I_T(\alpha) \),

(ii) \( I_F(0) \leq I_F(\alpha) \), for any \( \alpha \in \mathbb{U} \).

3. The Complete Intuitionistic Fuzzy \( \rho \)-Subalgebra.

**Definition 3.1.** Let \( I \) be an IFS of \( \rho - \) algebra \((\mathbb{U}, ^\mu, 0), I_T(\alpha) = I_T(0), I_F(\alpha) = I_F(0)\) is a subset of \( \mathbb{U} \) and called an intuitionistic fuzzy \( \rho \)-kernel of an intuitionistic fuzzy set over \( \mathbb{U} \).

**Example 3.2.** Let \( \mathbb{U} = \{0,1,2,3\} \), define \( ^\mu \) on the set \( \mathbb{U} \) as in Table 1. Then \((\mathbb{U}, ^\mu, 0)\) is a \( \rho \)-algebra, we define an (IFS) \( I \) in \( \mathbb{U} \) as follows:

\[
I_T = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 0.4 & 0 & 0.4
\end{pmatrix},
I_F = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 0.3 & 0 & 0.3
\end{pmatrix},
K(I) = \{0,2\}.
\]

**Table 1:** \( K(I) = \{0,2\} \).
\[
\begin{array}{c|ccc|c}
\alpha & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 & 2 \\
3 & 3 & 2 & 2 & 0 \\
\end{array}
\]

**Proposition 3.3.** If I is \((IF - \rho - SA)\) of \((U, \sqcup, 0)\), then \(K(I)\) is a \((\rho - SA)\).

**Proof:** Let \(\alpha, \beta \in K(I)\), then \(I_T(\alpha) = I_T(\beta) = I_T(0), I_F(\alpha) = I_F(\beta) = I_F(0)\). Also, since I is \((IF - \rho - SA)\), we obtain \(I_T(\alpha) \geq \min\{I_T(\alpha), I_T(\beta)\} = \min\{I_T(0), I_T(0)\} = I_T(0)\). From Lemma (2.14), we get \(I_T(\alpha \beta) = I_T(0)\). Also \(I_F(\alpha \beta) \leq \max\{I_F(\alpha), I_F(\beta)\} = \max\{I_F(0), I_F(0)\} = I_F(0)\), and from Lemma (2.14), we get \(I_F(\alpha \beta) = I_F(0)\). This implies \(\alpha \beta \in K(I)\). Hence, \(K(I)\) is \((\rho - SA)\).

**Definition 3.4.**

Let \((U, \sqcup, 0)\) be a \(\rho\)-algebra and I be an IFS of U. We say that I is an intuitionistic \(\rho\)-constant of U if all maps \(I_T, I_F : U \rightarrow [0, 1]\) are constant maps.

**Example 3.5.** Let \(U = \{\alpha, \beta, \gamma, \delta\}\), define \(\sqcup\) on the set \(U\) as in Table 2. Then \((U, \sqcup, \alpha)\) is a \(\rho\)-algebra, we define an (IFS) I in \(U\) as follows:

\[
I_T = I_F = \left(\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\
0.5 & 0.5 & 0.5 & 0.5 \\
\end{array}\right).
\]

Hence, I is intuitionistic \(\rho\)-constant.

**Table 2:** I is an intuitionistic \(\rho\)-constant.

<table>
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<tr>
<th>(\alpha)</th>
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**Lemma 3.6.** Let I be \((IF - \rho - SA)\) of \(U\) then:

(i) \(I_T(\alpha) \geq I_T(0)\),

(ii) \(I_F(\alpha) \leq I_F(0)\) for any \(\alpha \in U\).

**Proof:** Let I be \((IF - \rho - SA)\), then from Lemma (2.14)

We obtain: \(I_T(0) \geq I_T(\alpha), I_F(0) \leq I_F(\alpha)\), for any \(\alpha \in U\).

Since, \(I^C = 1 - I\), thus

\(I_T(\alpha) = 1 - I_T(\alpha) \geq 1 - I_T(0) = I_T(0)\),

\(I_F(\alpha) = 1 - I_F(\alpha) \leq 1 - I_F(0) = I_F(0)\), This completes the proof.

**Proposition 3.7.** Let I be an IFS of \(\rho\)-algebra \((U, \sqcup, 0)\), then I is \((IF - \rho - SA)\) if it is \(I = \{<\alpha, I_T(\alpha) = I_T(0), I_F(\alpha) = I_F(0) > | \alpha \in U\}\).

**Proof:** Let I be an IFS and \(I_T(\alpha) = I_T(0), I_F(\alpha) = I_F(0)\) for any \(\alpha \in U\). Now, let \(\alpha, \beta \in U\), then \(I_T(\alpha \beta) = I_T(0) = \min\{I_T(0), I_T(0)\} = \min\{I_T(\alpha), I_T(\beta)\}\), thus \(I_T(\alpha \beta) \geq \min\{I_T(\alpha), I_T(\beta)\}\), \(I_F(\alpha \beta) = I_F(0) = \max\{I_F(0), I_F(0)\} = \max\{I_F(\alpha), I_F(\beta)\}\), thus \(I_F(\alpha \beta) \leq \max\{I_F(\alpha), I_F(\beta)\}\). Hence, I is \((IF - \rho - SA)\).

**Proposition 3.8.** Let I be an IFS of \(\rho\)-algebra \((U, \sqcup, 0)\), then \(I^C\) is \((IF - \rho - SA)\). If \(I^C = \{<\alpha, I_T(\alpha) = I_T(0), I_F(\alpha) = I_F(0) > | \alpha \in U\}\)

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Let \( I^c = \{ \alpha \mid \leq 1, \leq \rho \} \) and let \( \alpha, \beta \in U \), then
\[ I^c(\alpha \wedge \beta) = I^c(0) = \min\{I^c(\alpha), I^c(\beta)\} \]
and
\[ I^c(\alpha \vee \beta) = I^c(0) = \max\{I^c(\alpha), I^c(\beta)\}. \]
Thus, \( I^c(\alpha \wedge \beta) \leq \max\{I^c(\alpha), I^c(\beta)\} \).

Proposition 3.9. If \( (IF - \rho - SA) \) of \( (U, \oplus, 0) \), then \( K(I^c) \) is a \( (\rho - SA) \).

Proof: Assume that \( I \) is \( (IF - \rho - SA) \) of \( I \) and \( \alpha, \beta \in I \), then \( I(\alpha, 0) \geq t, I(\alpha) \leq t \) and \( (I^c(\alpha) = I(\alpha)) \).

Proposition 3.10. Let \( I \) be \( (IF - \rho - SA) \) then \( I \) is \( (\rho - SA) \).

Proof: Assume that \( I \) is \( (IF - \rho - SA) \) and \( I \), then \( I(\alpha, 0) \geq t, I(\alpha) \leq t \) and \( (I^c(\alpha) = I(\alpha)) \).

Proposition 3.11. Let \((U, \oplus, 0)\) be a \( \rho \)-algebra and \( I \) be an \( IFS \) of \( U \). Then \( I \) is \((IF - \rho - SA)\) if it is an intuitionistic fuzzy \( \rho \)-constant.

Proof: Assume that \( I \) is a constant. Then for all \( \alpha, \beta \in U \), \( I(\alpha) = I(\alpha) = I(\beta) = I(\beta) \), and \( I(\alpha \wedge \beta) \geq I(\alpha \wedge \beta) \).

Proposition 3.12. Let \( I \) be \((IF - \rho - SA)\). Then \( 0 \in I \), if \( I \neq \emptyset \).

Proof: Assume that \( I \) is \((IF - \rho - SA)\) and \( I \neq \emptyset \) then there is at least \( \alpha \in I \). From Lemma 2.14 and Definition 2.9, we obtain \( I(0) \geq I(\alpha) \geq t, I(0) \leq I(\alpha) \leq t \), this means \( 0 \in I \).

Corollary 3.13. If \( I \) is an intuitionistic fuzzy \( \rho \)-constant then \( I \) is \((\rho - SA)\).

Proof: It is directly obtained from the proof of Proposition 3.11 and Proposition 3.10.

Definition 3.14. Let \( I \) be an \( IFS \) in \( U \). We say that \( I \) is a complete intuitionistic fuzzy \( \rho \)-subalgebra (briefly, \( CIF - \rho - SA \)) of \( U \) if it satisfies the following:

(i) \( I(\alpha \wedge \beta) \leq \max\{I(\alpha), I(\beta)\} \),

(ii) \( I(\alpha \vee \beta) \geq \min\{I(\alpha), I(\beta)\} \) for any \( \alpha, \beta \in U \).

Example 3.15. Let \( U = \{0, \mu, \nu, \xi\} \), and define \( \oplus \) on the set \( U \) as in Table 3. Then \((U, \oplus, 0)\) is a \( \rho \)-algebra, we define an \((IFS) I \) in \( U \) as follows:
\[
I_T = \begin{pmatrix} 0 & 0.4 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.7 & 0.7 \end{pmatrix},
I_F = \begin{pmatrix} 0 & 0.4 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.7 & 0.7 \end{pmatrix}.
\]
Hence, \( I \) is \((CIF - \rho - SA)\).

Table 3: \( I \) is \((CIF - \rho - SA)\).
Lemma 3.16. Let $I$ be $(\text{CIF} - \rho - \text{SA})$ of $\mathcal{U}$ then:
(i) $I_T(0) \leq I_T(\alpha)$,
(ii) $I_F(0) \geq I_F(\alpha)$, for any $\alpha \in \mathcal{U}$.

Proof: Let $I$ be $(\text{CIF} - \rho - \text{SA})$ then,
(i) $I_T(0) = I_T(\alpha \lor \beta) \leq \max \{I_T(\alpha), I_T(\beta)\} = I_T(\alpha)$.
(ii) $I_F(0) = I_F(\alpha \lor \beta) \geq \min \{I_F(\alpha), I_F(\beta)\} = I_F(\alpha)$. This completes the proof.

Proposition 3.17. If $I$ is a $(\text{CIF} - \rho - \text{SA})$, then $K(I)$ is $(\rho - \text{SA})$.

Proof: Let $\alpha, \beta \in K(I)$, then $I_T(\alpha) = I_T(\beta) = I_T(0), I_F(\alpha) = I_F(\beta) = I_F(0)$. Also,
$I_T(\alpha \lor \beta) = \max \{I_T(\alpha), I_T(\beta)\} = I_T(0), I_F(\alpha \lor \beta) = \min \{I_F(\alpha), I_F(\beta)\} = I_F(0)$, and from Lemma 3.16 $I_T(0) \leq I_T(\alpha \lor \beta)$, $I_F(0) \geq I_F(\alpha \lor \beta)$. Thus,
$I_T(\alpha \lor \beta) = I_T(0), I_F(\alpha \lor \beta) = I_F(0), \text{ and } \alpha \lor \beta \in K(I)$ hence $K(I)$ is $(\rho - \text{SA})$.

Proposition 3.18. Let $I$ be an IFS then $I$ is $(\text{IF} - \rho - \text{SA})$ if and only if $I^c$ is $(\text{CIF} - \rho - \text{SA})$.

Proof: Let $I$ be $(\text{IF} - \rho - \text{SA})$, then $I_T(\alpha \lor \beta) \geq \min \{I_T(\alpha), I_T(\beta)\}, I_F(\alpha \lor \beta) \leq \max \{I_F(\alpha), I_F(\beta)\}$, for any $\alpha, \beta \in \mathcal{U}$. Now, $I_T(\alpha \lor \beta) = 1 - I_T(\alpha \lor \beta) \leq 1 - \min \{I_T(\alpha), I_T(\beta)\} = \max \{1 - I_T(\alpha), 1 - I_T(\beta)\} = \max \{1 - I_T(\alpha) \lor 1 - I_T(\beta)\} = 1 - \min \{1 - I_T(\alpha), 1 - I_T(\beta)\} = \min \{I_T(\alpha), I_T(\beta)\}$. Hence $I^c$ is $(\text{CIF} - \rho - \text{SA})$.

Conversely, let $I^c$ be $(\text{CIF} - \rho - \text{SA})$, then $I_T(\alpha \lor \beta) \leq \max \{I_T(\alpha), I_T(\beta)\}, I_F(\alpha \lor \beta) \geq \min \{I_F(\alpha), I_F(\beta)\}$, for any $\alpha, \beta \in \mathcal{U}$. Now, $I_T(\alpha \lor \beta) = 1 - I_T(\alpha \lor \beta) \geq 1 - \max \{I_T(\alpha), I_T(\beta)\}, I_F(\alpha \lor \beta) = 1 - I_F(\alpha \lor \beta) \leq 1 - \min \{I_F(\alpha), I_F(\beta)\} = \max \{1 - I_F(\alpha), 1 - I_F(\beta)\} = \max \{I_F(\alpha), I_F(\beta)\}$. Hence, $I$ is $(\text{IF} - \rho - \text{SA})$.

Corollary 3.19.
1- Let $I^c$ be a $(\text{CIF} - \rho - \text{SA})$, then $I_t$ is $(\rho - \text{SA})$.
2- Let $I$ be an intuitionistic fuzzy $\rho$-constant then $I_t$ is $(\rho - \text{SA})$.

Proof 1: From Proposition (3.18) and Proposition (3.10), the proof is obtained. Or we can also get the proof from Proposition (3.11) and Proposition (3.10).

Corollary 3.20.
1- Let $I$ be $(\text{IF} - \rho - I)$ then $I_t$ is $(\rho - \text{SA})$.
2- Let $I$ be $(\text{IF} - \rho - I)$ then $I^c$ is $(\text{CIF} - \rho - \text{SA})$.

Proof 1: From Remarks (2.13)-1 and Proposition (3.10), the proof is obtained. Or we can also get the proof from Remarks (2.13)-1 and Proposition (3.18).

Lemma 3.21. Let $I$ be $(\text{IF} - \rho - I)$ of $\mathcal{U}$ then:
(i) $I_T(0) \geq I_T(\alpha)$,
(ii) $I_F(0) \geq I_F(\alpha)$, for any $\alpha \in \mathcal{U}$.

Proposition 3.22. Let $I$ be $(\text{IF} - \rho - I)$ then $K(I)$ is $-\text{ideal}$.
\[ \rho_{i+1} = I_{T}(0) \geq \max\{I_{T_{c}}(\alpha), I_{T_{c}}(\beta)\} = \max\{1, -1\} = 1 \]

This implies \( \alpha \geq \beta \) in Iₜ, hence \( I \) is \( \rho \)-ideal.

**Definition 3.25.** Assume that \((\mathbb{U}, \rho, 0)\) is a \( \rho \)-algebra and let \( I \) be an IFS of \( \mathbb{U} \). We say \( I \) is a complete an intuitionistic fuzzy \( \rho \)-ideal of \( \mathbb{U} \) (briefly, CIF \( \rho \)-ideal) if the following conditions hold:

(i) \( I_{T}(\alpha \rho \beta) \leq \max\{I_{T}(\alpha), I_{T}(\beta)\} \),

(ii) \( I_{F}(\alpha \rho \beta) \geq \min\{I_{F}(\alpha), I_{F}(\beta)\} \),

(iii) \( I_{T}(\alpha) \leq \max\{I_{T}(\alpha \rho \beta), I_{T}(\beta)\} \),

(iv) \( I_{F}(\alpha) \geq \min\{I_{F}(\alpha \rho \beta), I_{F}(\beta)\} \), for any \( \alpha, \beta \in \mathbb{U} \).

**Example 3.26.** Let \( \mathbb{U} = \{p, q, r, s\} \) and define \( \rho \) on the set \( \mathbb{U} \) as in Table 4. Then \((\mathbb{U}, \rho, 0)\) is a \( \rho \)-algebra, we define an (IFS) \( I \) in \( \mathbb{U} \) as follows:

\[
I_{T} = \begin{pmatrix}
p & q & r & s \\
0.1 & 0.3 & 0.2 & 0.2 \\
\end{pmatrix},
I_{F} = \begin{pmatrix}
p & q & r & s \\
0.6 & 0.5 & 0.4 & 0.4 \\
\end{pmatrix}.
\]

Hence, I is (CIF \( \rho \)-ideal).

**Table 4:** I is (CIF \( \rho \)-ideal).
Proposition 3.27. Let $I$ be an IFS. Then $I$ is $(\text{IF} - \rho - I)$ if and only if $I^c$ is $(\text{CIF} - \rho - I)$. 

Proof: Let $I$ be $(\text{IF} - \rho - I)$. From proof of Proposition (3.18), we obtain $I_I(\alpha, \beta) \leq \min \{I_I(\alpha), I_I(\beta)\}$. Now, $I_I(\alpha) = 1 - I_I(\alpha) \leq 1 - \min \{I_I(\alpha), I_I(\beta)\}$, $I_I(\beta) = \max \{1 - I_I(\alpha), 1 - I_I(\beta)\} = \max \{1 - I_I(\alpha), I_I(\beta)\}$. Hence $I^c$ is $(\text{CIF} - \rho - I)$.

Conversely: Let $I^c$ be $(\text{CIF} - \rho - I)$ then from proof of Proposition (3.18), we obtain $I_I(\alpha, \beta) \leq \max \{I_I(\alpha), I_I(\beta)\}$. Now, $I_I(\alpha) = 1 - I_I(\alpha) \leq 1 - \max \{I_I(\alpha), I_I(\beta)\}$, $I_I(\beta) = \min \{1 - I_I(\alpha), I_I(\beta)\} = \min \{1 - I_I(\alpha), I_I(\beta)\}$, $I_I(\alpha) = 1 - I_I(\alpha) \leq 1 - \min \{I_I(\alpha), I_I(\beta)\}$. Hence, $I$ is $(\text{IF} - \rho - I)$.

Corollary 3.28. Let $I^c$ be $(\text{CIF} - \rho - I)$. Then,
1- $I_I$ is $(\rho - SA)$.
2- $I^c$ is $(\text{CIF} - \rho - SA)$.
3- $I^c$ is $(\rho - \text{ideal})$.

Proof 1: From Proposition (3.27) and Corollary (3.20)-1, we get the proof. We can also get the proof from Proposition (3.27) and Corollary (3.20)-2. Further, we can obtain the proof from Proposition (3.27) and Proposition (3.24).

Corollary 3.29. Let $I$ be $(\text{IF} - \rho - I)$ then:
1- $I_I$ is $(\rho - SA)$.
2- $I^c$ is $(\text{CIF} - \rho - SA)$.

Proof 1: From Remarks (2.13)-2 and Proposition (3.10), the proof is got. We can also get the proof from Remarks (2.13)-2 and Proposition (3.18).

Proposition 3.30. Let $I$ be $(\text{IF} - \rho - I)$ then $I_I$ is $(\text{IF} - \rho - I)$.

Proof: Assume that $I$ is $(\text{IF} - \rho - I)$ and $\alpha, \beta \in I_I$, then $I_I(\alpha, \beta) = \min \{I_I(\alpha), I_I(\beta)\} \leq t$, $I_I(\alpha, \beta) = \max \{1 - I_I(\alpha), I_I(\beta)\} \leq t$, this implies $\alpha, \beta \in I_I$. Now, since $I$ is $(\text{IFS} - \rho - I)$, and $I_I = \{< \alpha, I_I(\alpha) \geq t, I_I(\alpha) = t > | \alpha \in \mathbb{U} \}$, we obtain that $I_I(0) \geq I_I(\alpha) \geq t$, $I_I(0) = I_I(\alpha) = t$, thus $0 \in I_I$, hence $I_I$ is $(\text{IF} - \rho - I)$.

Definition 3.31. Let $(\mathbb{U}, \delta, 0)$ be a $\rho$-algebra and $I$ be an IFS of $\mathbb{U}$. We say that $I$ is a complete an intuitionistic fuzzy $\bar{\rho}$-ideal of $\mathbb{U}$ (briefly, CIF $- \bar{\rho} - I$) If the following conditions hold:
(i) $I_I(0) \leq I_I(\alpha)$,
(ii) $I_I(0) \geq I_I(\alpha)$,
(iii) $I_I(\alpha, \beta) \leq \max \{I_I(\alpha), I_I(\beta)\}$,
(iv) $I_I(\alpha, \beta) \leq \min \{I_I(\alpha), I_I(\beta)\}$, for any $\alpha, \beta \in \mathbb{U}$.

Example 3.32. Let $\mathbb{U} = \{x, y, z, w\}$ and define $\delta$ on the set $\mathbb{U}$ as in Table 5. Hence, $(\mathbb{U}, \delta)$ is a $\rho$-algebra. We define a (IFS) $I$ in $\mathbb{U}$ as follows:
\[ I_T = \begin{pmatrix} x & y & z & w \\ 0.1 & 0.2 & 0.2 & 0.2 \end{pmatrix}, I_F = \begin{pmatrix} x & y & z & w \\ 0.6 & 0.4 & 0.4 & 0.4 \end{pmatrix}. \text{Then, } I \text{ is } (\text{CIF } -\bar{\rho} - I). \]

**Table 5:** \( I \) is \((\text{CIF } -\bar{\rho} - I)\).

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**Lemma 3.33.** If \( I \) is \((\text{CIF } -\bar{\rho} - I)\), then \( I \) is \((\text{CIF } -\rho - \text{SA})\).

**Proposition 3.34.** Let \( I \) be \((\text{CIF } -\bar{\rho} - I)\), then \( K(1) \) is \((\rho - \text{SA})\).

**Proof:** Let \( \alpha, \beta \in K(1) \), then \( I_T(\alpha) = I_T(\beta) = I_T(0), I_F(\alpha) = I_F(\beta) = I_F(0) \). Also, \( I_T(\alpha \cap \beta) \leq \max \{ I_T(\alpha), I_T(\beta) \} = \max \{ I_T(0), I_T(0) \} = I_T(0) \), \( I_F(\alpha \cap \beta) \geq \min \{ I_F(\alpha), I_F(\beta) \} = \min \{ I_F(0), I_F(0) \} = I_F(0) \), and \( I_T(0) \leq I_T(\alpha \cap \beta), I_F(0) \geq I_F(\alpha \cap \beta) \) by Lemma (3.16). Thus \( I_T(\alpha \cap \beta) = I_T(0), I_F(\alpha \cap \beta) = I_F(0) \), and \( \alpha \cap \beta \in K(1) \), hence \( K(1) \) is \((\rho - \text{SA})\).

**Proposition 3.35.** Let \( I \) be an IFS then \( I \) is \((\text{IFS } -\bar{\rho} - I)\) if and only if \( I^c \) is \((\text{CIF } -\bar{\rho} - I)\).

**Proof:** Let \( I \) be \((\text{IFS } -\bar{\rho} - I)\), we obtain \( I_T(0) \geq I_T(\alpha), I_F(0) \leq I_F(\alpha) \). Thus \( I_T(\alpha) = 1 - I_T(\bar{\alpha}) \geq 1 - I_T(0) = I_T(0) \), \( I_F(\alpha) = 1 - I_F(\bar{\alpha}) \leq 1 - I_F(0) = I_F(\bar{\alpha}) \). Now, \( I_T(\alpha \cap \beta) = 1 - I_T(\bar{\alpha \cap \beta}) \leq 1 - \min \{ I_T(\alpha), I_T(\beta) \} \) \( = \max \{ 1 - I_T(\alpha), 1 - I_T(\beta) \} \) \( = \max \{ I_T(\alpha), I_T(\beta) \} \), \( I_F(\alpha \cap \beta) = 1 - I_F(\bar{\alpha \cap \beta}) \geq 1 - \max \{ I_F(\alpha), I_F(\beta) \} \) \( = \min \{ 1 - I_F(\alpha), 1 - I_F(\beta) \} \) \( = \min \{ I_F(\alpha), I_F(\beta) \} \). Hence \( I^c \) is \((\text{CIF } -\bar{\rho} - I)\).

Conversely, let \( I^c \) be \((\text{CIF } -\bar{\rho} - I)\), then \( I_T(0) \leq I_T(\alpha), I_F(0) \geq I_F(\alpha) \), \( I_T(0) = 1 - I_T(0) \) \( \geq 1 - I_T(\alpha) = I_T(\bar{\alpha}) \). \( I_F(0) = 1 - I_F(0) \leq 1 - I_F(\alpha) = I_F(\bar{\alpha}) \), and from the following \( I_T(\alpha \cap \beta) \leq \max \{ I_T(\alpha), I_T(\beta) \}, I_F(\alpha \cap \beta) \geq \min \{ I_F(\alpha), I_F(\beta) \} \), for any \( \alpha, \beta \in U \). We obtain \( I_T(\alpha \cap \beta) = 1 - I_T(\bar{\alpha \cap \beta}) \geq 1 - \max \{ I_T(\alpha), I_T(\beta) \} \) \( = \min \{ I_T(\alpha), I_T(\beta) \} \), \( I_F(\alpha \cap \beta) = 1 - I_F(\bar{\alpha \cap \beta}) \leq 1 - \min \{ I_F(\alpha), I_F(\beta) \} \) \( = \max \{ 1 - I_F(\alpha), 1 - I_F(\beta) \} \) \( = \max \{ I_F(\alpha), I_F(\beta) \} \). Hence, \( I \) is \((\text{IFS } -\bar{\rho} - I)\).

**Corollary 3.36.** Let \( I^c \) be is \((\text{CIF } -\bar{\rho} - I)\). Then,
1- \( I_1 \) is \( \rho - \text{subalgebra} \).
2- \( I^c \) is \((\text{CIF } -\rho - \text{SA})\)
3- \( I_1 \) is \( \bar{\rho} - \text{ideal} \).

**Proof 1:** From Proposition (3.35) and Corollary (3.29)-1, we get the proof. We can also get the proof from Proposition (3.35) and Corollary (3.29)-2. Further, we get the proof from Proposition (3.35) and Proposition (3.30).
4. CONCLUSION

This paper investigates and discusses several new ideals in intuitionistic fuzzy $\rho$ -algebra. This study will be helpful in the future if we apply the neutrosophic fuzzy sets theory to consider new conceptions in the neutrosophic fuzzy $\rho$ -algebra.

REFERENCES