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Semi-Essentially Compressible Modules and Semi-Essentially Retractable Modules

Mohammed Baqer Hashim Al Hakeem*, Nuhad Salim Al-Mothafar
Department of Mathematics, college of Science, Baghdad University, Baghdad, Iraq

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Abstract:
Let R be a commutative ring with 1 and M be a left unitary $R-module$. In this paper, the generalizations for the notions of compressible module and retractable module are given.
An $R-module\ M$ is termed to be semi-essentially compressible if M can be embedded in every of a non-zero semi-essential submodules. An $R-module\ M$ is termed a semi-essentially retractable module, if $Hom_R(M, N) \neq 0$ for every non-zero semi-essentially submodule $K$of an $R-\ module\ M$. Some of their advantages characterizations and examples are given. We also study the relation between these classes and some other classes of modules.

Keywords: Compressible module, Retractable module, Essential compressible module, Essential retractable module, Semi- essential compressible, Semi- essential retractable.

مقاسات قابلة للانضغاط شبة جوهرية ومقاسات قابلة للسحب شبة جوهرية

محمد باقر هاشم الحكيم* ، نهاد سالم المظفر
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة
لتكن R حلقة إبداليه ذات عنصر محايد ولتكن M مقاس ايسري ذات عنصر محايد. بحثنا هذا هو إعمامات لمفهوم المقاسات القابلة للضغط والمقاسات القابلة للانسحاب. المقاس M يسمى مقاس قابل للانضغاط شبة جوهرية إذا اغمرت M في جميع المقاسات الجزئية شبة جوهري غير الصفري. يسمى المقاس M قابل للانسحاب شبة جوهري أذا كان $Hom_R(M, K) \neq 0$ لجميع المقاسات الجزئية K شبة جوهري غير الصفري. سنعطي في هذا البحث بعض الخواص والامثلة والعلاقة بين هذه المقاسات وأنواع من المقاسات الأخرى.

1.Introduction:

Let R be a commutative ring with 1 and M be a left unitary $R-module$. A non-zero submodule K is termed to be an essential submodule of M if $K \cap L \neq 0$ for every non-zero submodule L of M [1]. In [2], the authors introduced and studied the notion of semi-essential

*Email: mbhmsc2015110@gmail.com

submodules. Recall that a non-zero submodule N of an R −module M is termed to be a semi-essential submodule ($N \leq_{s.e} M$), if $N \cap P \neq 0$ for every non-zero prime submodule P of M. A submodule P of M is termed to be prime if P is a proper and whenever $rx \in P$ for every $r \in R, x \in M$ up to either $x \in P$ or $r \in [P:M]$, where $[P:M] = \{r \in R: rM \subseteq P\}$[3].

An $R - module\ M$ is termed to be compressible if M can be embedded in every of it is a non-zero submodule. In this work, the notion of a semi-essentially compressible module as a generalization of compressible $module$ is introduced and studied. Further, some of their advantages characterizations and examples are given. Also, we view that below definite terms semi-essentially compressible, semi-uniform , essentially compressible and compressible modules are equivalent. In[4], the author defined a semi-essentially compressible as an $R - module\ M$ is a semi-essentially compressible module if for every essential submodule $N$of M, he found a monomorphism $\theta : M \longrightarrow N^{(I)}$ for some set I. In this work, we give another definition for semi-essentially compressible, namely an R −module M is termed to be semi-essentially compressible module, if M can be embedded in every of it is nonzero semi-essentially submodules. An M is also called semi-essentially compressible**,** if one can find a monomorphism $f: M \longrightarrow N$ whenever N is a non-zero semi-essentially submodule of M. Moreover, we introduce the notion of a semi-essentially retractable module as a generalization of a retractable module . Finally, we study the relations between semi-essentially compressible modules and some of the other $modules$ as semi-essentially retractable.

2. Semi-Essentially Compressible Modules:

An $R - moduleM$ is termed to be a compressible $module,$ if M can be embedded in every of it is a non-zero submodule of M. Equivalently, M is compressible if one can find a monomorphism $f: M \longrightarrow N$ whenever $0 \neq N \leq M$ [5].
A ring R is termed compressible if the $R - module$ is compressible[5]. That is R can be embedded in any of it is non-zero ideal.

Definition (2.1): An $R - module\ M$ is termed to be a semi-essentially compressible module if M can be embedded in every nonzero semi-essential submodule. Equivalently, M is a semi-essentially compressible**,** if one can find a monomorphism $f: M \longrightarrow N$ whenever N is a non-zero semi-essential submodule of M.
A ring R is termed to be a semi-essentially compressible, if R is a semi-essentially compressible as R −module.

Remarks and Examples(2.2):
1. $Z\ as\ Z - module$ is a semi-essentially compressible module, because $nZ \leq_{s.e} Z, \forall n \in N$ and there exists a monomorphism $f: Z \longrightarrow nZ$.
2. $Q\ as\ Z - module$ is not a semi-essentially compressible module, because there is no monomorphism from Q to nZ , where $nZ \leq_{s.e} Q$.
3. It is obvious that every compressible module is a semi-essentially compressible $module$, but the converse is not true in general for example $Z_6\ as\ Z - module$ is semi-essentially compressible $module$ because Z_6 is the only semi-essential submodule of Z_6 , but it is
4. Every simple $R - module$ is semi-essentially compressible module, however, the converse is not true, because $Z\ as\ Z - module$ is a semi-essentially compressible $module$ but it is not simple.
5. $Z_4\ as\ Z - module$ is not semi-essentially compressible. Because Z_4 can not be embedded in $\langle\bar{2}\rangle$ and $\langle\bar{2}\rangle \leq_{s.e} Z_4$.

6. A homomorphic image of a semi-essentially compressible $module$ needs not be semi-essentially compressible in general. For example, $Z\ as\ Z-module$ is a semi-essentially compressible module and $\frac{z}{4z} \simeq z_4$ is not a semi-essentially compressible module by (5).
7. If M is a semi-simple module, then M is a semi-essentially compressible module, since M is the only semi-essential submodule of M, see Example (2) (2) [2].

Recall that a non-zero $R-module\ M$ is termed to be semi-uniform, if every non-zero submodule of an $R-module$ is semi-essential, see Definition(1) [2].

Proposition(2.3): If M is a semi- uniform $R-module$, then M is a compressible $module$ if and only if M is semi-essentially compressible $module$.

Proof: Suppose that M is semi-essentially compressible $module$. Let N be a non-zero submodule of M because M is semi- uniform, then N is semi-essential submodule in M. But M is semi-essentially compressible $module$, then M can be embedded in $N$for every $0 \neq N \leq_{s.e} M$. Thus, $M$is compressible. Conversely, it is obvious, by Remarks and Examples (2.2) (3).

Proposition(2.4): If M is a semi-essentially compressible $R-module$ in which all submodule of M contains a non-zero semi-essential submodule of M, then M is a compressible $module$.

Proof: Let $0 \neq N \leq M$. By assumption found a semi-essential submodule $0 \neq K \leq N$, then $K \leq_{s.e} M$. Because M is semi-essentially compressible, then there exists a monomorphism $f: M \longrightarrow K$ and $i: K \longrightarrow N$ is an inclusion monomorphism . Hence, $i \circ f: M \longrightarrow N$ is a monomorphism. Therefore, M is compressible.

Proposition(2.5): A $module\ M$ is a semi-essentially compressible module if and only if M can be embedded in R_x for every $0 \neq x \in M\ and\ R_x \leq_{s.e} M$.

Proof: The first direction is obvious by Definition (2.1). Conversely, let $0 \neq x \in N \leq_{s.e} M$, then $0 \neq R_x \leq_{s.e} N$ Corollary(6)[2]. Consider $i: R_x \longrightarrow N$ is the inclusion homomorphism by assumption R_x is semi-essentially submodule in M, there exists a monomorphism $f: M \longrightarrow R_x$, then $i \circ f: M \longrightarrow N$ is a monomorphism . Therefore, M is semi-essentially compressible.

Corollary(2.6): A semi-essentially compressible $module\ M$ is compressible if every cyclic submodule of M is semi-essential in M .

Proof: The result is obviously obtained by Proposition 2.5.

Proposition(2.7): A semi-essential submodule of a semi- essentially compressible module is also a semi- essentially compressible module.

Proof: Let $0 \neq K \leq_{s.e} M$ and M be a semi-essentially compressible module. Let $0 \neq L \leq_{s.e} K \leq_{s.e} M$, by Proposition(1.5)[7] $L \leq_{s.e} M$. Because M is semi- essentially compressible, so there exists a monomorphism $f: M \longrightarrow L$ and $i: K \longrightarrow M$ is the inclusion monomorphism ,then $f \circ i: K \longrightarrow L$ be a monomorphism. Therefore, Kcan be embedded in L.

Proposition(2.8): Let M be a fully prime $R-module$, then M is a semi- essentially compressible module if and only if M is an essentially compressible module, where an $R-module\, M$ is termed to be fully prime if every proper submodule of M is a prime submodule[9].

Proof: Because M is fully prime, then by proposition(2.1)[7] every semi-essential submodule is an essential submodule , hence every essential compressible is semi-essentially compressible. Conversely, see Remarks and Examples (2.2) (3).

Remark(2.9): Let L be any R −submodule of M containing a submodule N of M such that $\frac{L}{N}$ is a semi-essentially of $\frac{M}{N}$, then L is a semi-essentially submodule of M.

Proof: The same proof of Proposition(1.4) [9].

Proposition(2.10): Let M be a semi- essentially compressible R −module and N be a submodule of M such that N contains in the inverse image of R −monomorphism of N, then $\frac{M}{N}$ is a semi-essentially compressible R −module.

Proof: Let $\frac{L}{N}$ be a semi-essential submodule of $\frac{M}{N}$, then by Remark(2.9) L is a semi-essential submodule of M and M be a semi-essentially compressible $R-module$. Thus, there exists a monomorphism $h: M \longrightarrow L$. Now define $\Psi: \frac{M}{N} \longrightarrow \frac{L}{N}$, by $\Psi(m+N) = h(m) + N$, $for\ all\ m \in M$. Because a homomorphism $h: M \longrightarrow L$ is well-define, then $\Psi: \frac{M}{N} \longrightarrow \frac{L}{N}$ is well- define. Let $\Psi(m_1 + N) = \Psi(m_2 + N)$, then $h(m_1) + N = h(m_2) + N$, thus $h(m_1) - h(m_2) \in N \leq Ker\ h$, then $h(m_1) = h(m_2)$, but h is a monomorphism , then $m_1 + N = m_2 + N$. Therefore, $\frac{M}{N}$ is a semi- essential compressible $R-module$.

Proposition(2.11): If M_1 and M_2 are isomorphic $R-modules$, then $M_1$is a semi-essentially compressible if and only if $M_2$is a semi-essentially compressible.

Proof: Suppose that $M_1$is a semi-essentially compressible and let $\phi: M_1 \longrightarrow M_2$be an isomorphism. Let $0 \neq N \leq_{s.e} M_2$, then $0 \neq \phi^{-1}(N) \leq_{s.e} M_1$. Put $K = \phi^{-1}(N)$, so $\alpha: M_1 \longrightarrow K$ is a monomorphism , let $g = \phi \restriction g$, then $g: K \longrightarrow M_2$ is a monomorphism. $g(K) = \phi(\phi^{-1}(N)) = N$, hence we have a composition . Let $\hbar = g \circ \alpha \circ \phi^{-1}$, hence $\hbar: M_1 \longrightarrow N$ is a monomorphism. Therefore, $M_2$is semi-essentially compressible $module$.
Recall that an $R-module\ M$ subisomorphism to an $R-module\ M'$, if there exists R −monomorphism $\phi: M \longrightarrow M'$ and $: M' \longrightarrow M$. In this case, we say that $R-module\, M$ and $M'$are subisomorphic[10].

Proposition(2.12): The following statements are equivalent for $R-module\ M$:
(1)M is a semi-essentially compressible $module$.
(2)M is subisomorphic to a semi-essentially compressible $module$.
(3)M contains a semi-essentially compressible submodule N such that there exists a monomorphism $\varphi: M \longrightarrow N$.

Proof: (1) ⟹ (2) because M is a semi-essentially compressible , then there exists a monomorphism $f: M \longrightarrow N$, for all $N \leq_{s.e} M$ and there exists $i: N \longrightarrow M$; i is an inclusion monomorphism , then by Proposition(2.7) M is subisomorphic to a semi-essentially compressible $module$.

(2) $\Longrightarrow$ (3) Suppose that M is subisomorphic of a semi-essentially compressible $module\ B$ and one can find a monomorphisms $\psi: M \longrightarrow B\ and\ \phi: B \longrightarrow M$. Let $L = \phi(B) \leq_{s.e} M$, then by Proposition(2.7) L is a semi-essentially compressible submodule of M. Thus, $\phi \circ \psi: M \longrightarrow L$ is a monomorphism. Therefore, M contains a semi-essentially compressible submodule L.
(3) $\Longrightarrow$ (1) Let L be any semi-essential submodule of M, then $L \cap N \leq_{s.e} N$ so by(3) there exists a monomorphism $f: N \longrightarrow L \cap N$ and there exists a monomorphism $\varphi: M \longrightarrow N$ also $i: L \cap N \longrightarrow L$, is the inclusion monomorphism , thus $i \circ f \circ \varphi: M \longrightarrow L$ is a monomorphism. Therefore, M is semi-essentially compressible.

Remark(2.13): The direct sum of a semi-essentially compressible $module$ needs not to be semi-essentially compressible. Consider the following example let $Z_4 \simeq Z_2 \oplus Z_2$ as Z-module . Z_2 is a semi-essentially compressible module, but Z_4 is not a semi-essentially compressible module.

Proposition(2.14): Let $M = M_1 \oplus M_2$ be an $R-$module such that $ann_R M_1 \oplus ann_R M_2 = R$. If M_1 and M_2 are semi-essentially compressible $modules$, then M is semi-essentially compressible.

Proof: Let $0 \neq N \leq_{s.e} M$. Then by Proposition (2.5)[11] . $N = K_1 \oplus K_2$ for some $0 \neq K_1 \leq M_1 \leq M$ and $0 \neq K_2 \leq M_2 \leq M$, so by[7] K_1 and K_2 semi-essential. But $M_1$and M_2 semi-essentially compressible, so $\exists$ a monomorphisms $f: M_1 \longrightarrow K_1$and $g: M_2 \longrightarrow K_2$. Define $\hbar: M \longrightarrow N$ by $\hbar(a,b) = (f(a), g(b))$, it can be readily to show that $\hbar$is a monomorphism. Therefore, M is a semi-essentially compressible.

3. **Semi-Essentially Retractable Modules:**
An $R - module\ M$ is termed to be a retractable if $Hom(M,N) \neq 0$for every non-zero submodule N of M. [8]
A ring R is termed to be retractable if the $R - module\ R$ is retractable. [8]

Definition (3.1): An $R - module\ M$ is termed to be a semi-essentially retractable module, if $Hom(M,N) \neq 0$ for every nonzero semi-essentially submodule N of M.

An $R - module\ M$ is termed to be essentially retractable if $Hom_R(M, N) \neq 0$ for every non-zero essential submodule N of M.
A ring R is termed to be essentially retractable if the R-module R is essentially retractable. That is $Hom_R(R, I) \neq 0$ for every nonzero small ideal I of a ring R[12]

Remarks and Examples(3.2):
1. $Z_n\ as\ Z - module$ is a semi- essentially retractable module for all $n \in Z^+$.
2. $Q\ as\ Z - module$ is not a semi- essential retractable $module$, because $Hom(Q,Z) = 0$ and $Z \leq_{s.e} Q$
3 .$Z\ as\ Z - module$ is a semi-essentially retractable $module$, because, $nZ \leq_{s.e} Z$ for all $n \in N$, then there exists a homomorphism $f: Z \longrightarrow nZ$.

4. Every semi-essentially compressible $module$ is a semi-essential retractable $module$, but the converse is not true in general. For example, $Z_4\ as\ Z-module$ is a semi-essential retractable $module$, however, it is not semi-essentially compressible $module$.
5. Every semi-essentially retractable module is an essentially retractable $module$
6. Every retractable $module$ is a semi-essentially retractable $module$.
7. Every semi-simple $R-module$ is a semi-essentially retractable because it is retractable.
8. Every compressible $module$ is a semi-essential retractable $module$, however, the converse is not true for example $Z_{12}\ \ as\ Z-module$ is a semi-essential retractable module, then there exists a honomorphsim $f: Z_{12} \rightarrow (\bar{3})$ by $f(\bar{x}) = 3\bar{x}$,which is not monomorphism.

Proposition(3.3): If M is a semi- uniform $R-module$, then M is a retractable $module$ if and only if M is semi-essentially retractable $module$.

Proof: Suppose that M is semi-essentially retractable $module$. Let N be a non-zero submodule of M because M is semi-uniform, then N is semi-essential submodule in M. But M is semi-essentially retractable module. Thus,$Hom(M, N) \neq 0$. Therefore, $M$is retractable. Conversely, it is obvious by Remarks and Examples(3.2) (5).

Proposition(3.4): Let M be a fully prime $R-module$, then M is a semi- essentially retractable $module$ if and only if M is an essentially retractable $module$.

Proof: Because M is fully prime , then by Proposition(2.1)[7] every semi-essential submodule is essential submodule , hence every essential retractable is semi-essentially retractable. Conversely, see Remarks and Examples(3.2) (5).

Proposition(3.5): A semi-essential submodule of a semi- essentially retractable $module$ is also a semi-essentially retractable module.

Proof: Let $0 \neq K \leq_{s.e} M$. Let $0 \neq L \leq_{s.e} K \leq_{s.e} M$, by Proposition(1.5) [7] $L \leq_{s.e} M$, M is a semi-essentially retractable, then there exists a homomorphism $f: M \longrightarrow L$ and $i: K \longrightarrow M$ is the inclusion homomorphism. Therefore, $Hom(K, L) \neq 0$.

Proposition(3.6): If M is a semi-essentially retractable $R-module$ in which every submodule of M contains a non-zero semi-essentially submodule of M, then M is a retractable $module$.

Proof: Let $0 \neq N \leq M$ and $K \leq N$, $K \leq_{s.e} M$. Because M is a semi-essentially retractable $module$, then there exists a homomorphism $f: M \longrightarrow K$ and $i: K \longrightarrow N$ is inclusion homomorphism. Hence, $i \circ f: M \longrightarrow N$ is a homomorphism. Therefore, M is retractable.

Proposition(3.7): If M_1 and M_2 be isomorphic$R-$modules, then $M_1$is a semi-essentially retractable module if and only if M_2 is a semi-essentially retractable $module$.

Proof: Suppose that M_1 is a semi-essentially retractable $module$ and $\phi: M_1 \longrightarrow M_2$ is an isomorphism . Let $N \leq_{s.e} M_2$, then $\phi^{-1}(N) \leq_{s.e} M_1$ see Proposition(13)[2], thus $Hom(M_1, K) \neq 0$; $K = \phi^{-1}(N) \leq M_1$,hence there exists $f: M_1 \longrightarrow K$. put $= \phi\mid_K$, so $g: K \longrightarrow M_2$, then $g(K) = \phi\big(\phi^{-1}(N)\big) = N$, then $g \circ f \circ \phi^{-1}: M_2 \longrightarrow N$. Therefore, $Hom(M, N) \neq 0$.

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