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On the First Natural Triangular Representations of the Symmetric Groups

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Abstract

The main objective of the research is to study the first natural triangular representation of the symmetric groups over a field K of characteristic $p \neq 2$ which deals with the partition $\lambda = (n-4,3,1)$ of the positive integer n. Furthermore, this work has proven that the $S(n-4,3,1)$ is a submodule of F_1. The $F_1 = KS_n(x_2x_4x_6x_7^2 - x_2x_3x_6x_7^2 - x_2x_4x_5x_7^2 + x_2x_3x_5x_7^2)$ can be only split when$p|(n-5)$.

Keywords: Exact sequence, Group algebra KS_n, KS_n −module, Symmetric group, Spechet module.

حول التمثيلات المثلثية الطبيعية الاولى للزمر التناظرية

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الخلاصة:

الهدف من هذا البحث هو دراسة تحليلية لمقاسات التمثيلات الطبيعية الاولى للزمر التناظرية ضمن الحقل K ذو المميز $p \neq 2$ والتي تتعامل مع التجزئة $\lambda = (n-4,3,1)$ للعدد الصحيح الموجب n . كما تم اثبات بان المقاس الجزئي M(n−4,3,1) من $F_1 = KS_n(x_2x_4x_6x_7^2 - x_2x_3x_6x_7^2 - x_2x_4x_5x_7^2 + x_2x_3x_5x_7^2)$ قابل للتجزئة عندما $p|(n-5)$ بالاضافة الى ان S(n−4,3,1) هو مقاس جزئي من F_1.

1. Preliminaries

Definition 1: Let S_n be the set of all permutations τ on the set $\{x_1, x_2, \dots, x_n\}$ and $K[x_1, x_2, \dots, x_n]$ be the ring of polynomials in $x_1, x_2, \dots, x_n$ with coefficients in K. Then each permutation $\tau \in S_n$ can be regarded as a bijective function from $K[x_1, x_2, \dots, x_n]$ onto $K[x_1, x_2, \dots, x_n]$ defined by$\left(f(x_1, x_2, \dots, x_n)\right) = f\left(\tau(x_1), \tau(x_2), \dots, \tau(x_n)\right)$ for all $f(x_1, x_2, \dots, x_n) \in K[x_1, x_2, \dots, x_n]$. Then KS_n forms a group algebra with respect to addition of functions, product of functions by scalars and composition of functions which is called the group algebra of the symmetric group S_n [1].

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Definition 2: Let n be a positive integer then the sequence $\lambda = (n_1, n_2, \dots, n_l)$ is called a partition of n if $n_1 \geq n_2 \geq \cdots \geq n_l > 0$ and $n_1 + n_2 + \cdots + n_l = n$. Then the set $D_\lambda = \{(i,j) | i = 1,2,\dots,l; 1 \leq j \leq n_I\}$ is called $\lambda-$diagram. In addition, any bijective function $t: D_\lambda \to \{x_1, x_2, \dots, x_n\}$ is called a λ–tableau. A λ–tableau may be thought as an array consisting of l row and n_1 columns of distinct variables $t\big((i,j)\big)$ where the variables occur in the first n_i positions of the i^{th} row and each variable $t((i,j))$ occurs in the i^{th} row and the j^{th} column ((i, j)-position) of the array. $t\big((i,j)\big)$ will be denoted by $t(i,j)$ for each $(i,j) \in D_\lambda$. The set of all λ-tableaux will be denoted by T_λ. i.e. $T_\lambda = \{t | t \text{ is a } \lambda - \text{tableau}\}$. Then the function $g: T_\lambda \to K[x_1, x_2, \dots, x_n]$ which is defined by $g(t) = \prod_{i=1}^{l} \prod_{j=1}^{n_i} \big(t(i,j)\big)^{i-1}, \forall\, t \in T_\lambda$ is called the row position monomial function of T_λ, and for each λ–tableau t, $g(t)$ is called the row position monomial of t. So $M(\lambda)$ is the cyclic KS_n −module generated by $g(t)$ over KS_n [2].

2. Introduction

It is well known that the purpose of representation theory is to discuss groups of endomorphism G of a vector space V and the relation between the building of abstract group G and the vector space. The concept of Specht polynomial was first initiated by Specht that proved how a given polynomial can be written as a linear combination of other polynomials which was the results of Specht study on representation theory of symmetric group. After that, he faced the problem when the symmetric group acts, in natural way, as a tableau. However, the result of permutation a standard tableau can be a nonstandard tableau and this nonstandard tableau can be written as a linear combination of Specht polynomials [3]. Al-Butahi [4] has been studied the third natural representation M(n-3,3) of the symmetric groups and proved that it is a split if and only if $p \nmid \frac{n(n-1)(n-2)}{6}$. In this work, the authors have been shown the first natural triangular representation of the symmetric groups over a field K of characteristic $p \neq 2$ and the variables defined over K are commuting linearly independent.

3. The First Natural Triangular Representation of S_n

In this section, some notations will be defined as follows:

1. Let $\sigma_1(n) = \sum_{j=1}^{m} x_i$
2. Let $\sigma_2(n) = \sum_{1 \leq i < j \leq n}^{m} x_i x_j$.
3. Let $\sigma_3(n) = \sum_{1 \leq i < j < k \leq n}^{m} x_i x_j x_k$.
4. Let $\sigma_4(n) = \sum_{1 \leq i < j \leq n} \sum_{\substack{k=1 \\ k \neq i,j}}^{n} x_i x_j x_k^2$.
5. Let $\sigma_5(n) = \sum_{1 \leq i < j \leq n} \sum_{\substack{l=1 \\ l \neq i,j,k}}^{n} x_i x_j x_k x_l^2$.
6. Let $c_l(n) = x_l \left(\sigma_2(n) - \sum_{\substack{1 \leq i < j \leq n \\ i,j \neq l}} x_l\, x_j x_l \right); l = 1,2,\dots,n$. Then $\sum_{i=1}^{n} c_i = \sigma_5(n)$.

7. Let $U_{ij}(n) = c_i(n) - c_j(n)$; $i,j = 1,2,\dots,n$.

We denote W to be the KS_n-modules generated by $c_1(n)$ over KS_n and W_0 to be the KS_n-submodule of W generated by $U_{12}(n)$ over KS_n. The set $B = \{c_i(n) |\, i = 1,2,\dots,n\}$ is a K-basis for $W = KS_n c_1(n)$ and $\dim_K W = n$.

Definition 3.1: The KS_n −module $M(n-(r+3), r+2, 1)$ defined by $M(n-(r+3), r+2, 1) = KS_n x_1 x_2 \dots x_{r+2} x_{r+3}^2$, is called the r^{th} –natural triangular representation of S_n over K, where $r \geq 0$ and $n \geq 2r+6$.

Lemma 3.2 [4]: The set $B(n-4,3,1)=\{x_ix_jx_kx_l^2 : 1\le i<j<k\le n,\ 1\le l\le n, l\ne i,j,k\}$ is a K-basis of $M(n-4,3,1)$ and $\dim_K M(n-4,3,1)=\binom{n}{3}(n-3); n\ge 7$.

Theorem 3.3: The set $B_0(n-4,3,1)=\{x_ix_jx_kx_l^2-x_1x_2x_3x_4^2 \mid 1\le i<j<k\le n, 1\le l\le n, l\ne i,j,k,(i,j,k,l)\ne(1,2,3,4)\}$ is a K-basis of $M_0(n-4,3,1)$, and $\dim_K M_0(n-4,3,1)=\binom{n}{3}(n-3)-1;\ n\ge 7$.

Proof: Since the KS_n-module $M_0(n-3,2,1)$ consists of all polynomials of the form $\sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k}^{n} k_{ijkl}\, x_ix_jx_kx_l^2$ with $\sum_{l=1, l\ne i,j,k}^{n}\sum_{1\le i<j<k\le n} k_{ijkl}=0$ and $k_{ijkl}\in K$. i. e. $M_0(n-3,2,1)=\{\sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k}^{n} k_{ijkl}\, x_ix_jx_kx_l^2 \mid \sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k}^{n} k_{ijkl}=0,\ k_{ijkl}\in K\}$. It is clear that $B_0(n-4,3,1)\subseteq M_0(n-4,3,1)$. To prove that $B_0(n-4,3,1)$ generates $M_0(n-4,3,1)$ over K. Let $x\in M_0(n-4,3,1)$.

$\Rightarrow x=\sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k}^{n} k_{ijkl}\, x_ix_jx_kx_l^2 ; \sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k}^{n} k_{ijkl}=0$ implies $x=\sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k}^{n} k_{ijkl}\,(x_ix_jx_kx_l^2-x_1x_2x_3x_4^2)$ with the term 1, 2, 3, and 4 excluded from the double summation since $k_{1234}\,(x_1x_2x_3x_4^2-x_1x_2x_3x_4^2)=0$. Thus, $B_0(n-4,3,1)$ generates $M_0(n-4,3,1)$ over K. $B_0(n-4,3,1)$ is linearly independent since if

$\sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k, (i,j,k,l)\ne(1,2,3,4)}^{n} k_{ijkl}\left(x_ix_jx_kx_l^2-x_1x_2x_3x_4^2\right)=0 \Rightarrow$

$\sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k}^{n} k_{ijkl}\, x_i\, x_jx_kx_l^2=0$, where $k_{1234}=-\sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k}^{n} k_{ijkl}$ with $\sum_{1\le i<j<k\le n}\sum_{l=1, l\ne i,j,k, (I,j,k,l)\ne(1,2,3,4)}^{n} k_{ijkl}=0$. Hence, $k_{ijkl}=0, \forall\, i,j,k,l;\ 1\le i<j<k\le n, 1\le l\le n, l\ne i,j,k$. Thus, $B_0(n-4,3,1)$ is a K-basis of $M_0(n-4,3,1)$, and

$\dim_K M_0(n-4,3,1)=\binom{n}{3}(n-3)-1=\frac{n(n-1)(n-2)(n-3)}{6}-1=\frac{n^4-6n^3+11n^2-6n-6}{6}$.

Theorem 3.4: $W=KS_nc_1(n)$ and $M(n-1,1)$ are isomorphic over KS_n.

Proof: Let $\varphi: M(n-1,1)\to W$ be defined as follows: $\varphi(x_i)=c_i(n);\ i=1,2,\dots,n$.
Then if $\tau=(x_ix_j)\in S_n$ we get that $\varphi(\tau x_i)=\varphi(x_j)=c_j(n)$ and since $\tau c_i(n)=c_j(n)$, then $\varphi(\tau x_i)=\tau\varphi(x_i)$. Therefore, φ is a KS_n homomorphism. For any $y\in W$ we have $y=\sum_{i=1}^{n}k_ic_i(n)$. Thus there exists $w=\sum_{i=1}^{n}k_ix_i\in M(n-1,1)$, such that: $\varphi(w)=\varphi(\sum_{i=1}^{n}k_ix_I)=\sum_{i=1}^{n}\varphi(k_ix_i)=\sum_{i=1}^{n}k_i\varphi(x_i)=\sum_{i=1}^{n}k_ic_i(n)=w$. Hence, φ is an epimorphism.
$\Rightarrow \dim_K \ker\varphi=\dim_K M(n-1,1)-\dim_K W=0 \Rightarrow \ker\varphi=0$.
Hence, φ is monomorphism. Thus, φ is a KS_n–isomorphism. Therefore, $M(n-1,1)$ and W are isomorphic over KS_n.

Theorem 3.5: $W_0=KS_nu_{12}(n)$ is irreducible submodule when p does not divide n.

Proof: From Theorem (3.4), we have a KS_n–homomorphism $\varphi: M(n-1,1)\to W$, such that $\varphi(x_i)=c_i(n);\ i=1,2,\dots,n$. Since $M_0(n-1,1)=KS_n(x_2-x_1)\subset M(n-1,1)$, thus $\varphi(x_i-x_1)=\varphi(x_i)-\varphi(x_1)=c_i(n)-c_1(n)=u_{i1}(n)\in W_0$.
Let $\psi=\varphi|_{M_0(n-1,1)}$. Then $\psi: M_0(n-1,1)\to W_0$, such that $\psi(x_i-x_1)=u_{i1}(n);\ i=1,2,\dots,n$ is a KS_n –homomorphism. Also, for all $u_{ij}\in W_0$, there exists $x_i-x_j\in M_0(n-$

1,1), such that $\psi(x_i - x_j) = \psi(x_i - x_1 + x_1 - x_j) = \psi(x_i - x_1) - \psi(x_j - x_1) = u_{i1}(n) - u_{j1}(n) = c_i(n) - c_1(n) - c_j(n) + c_1(n) = c_i(n) - c_j(n) = u_{ij}(n)$.
Since ψ is an epimorphism, $\dim_K M_0(n-1,1) = n-1$ and $\dim_K W_0 = n-1$. Then we get $\dim_K \ker\psi = \dim_K M_0(n-1,1) - \dim_K W_0 = 0$ i.e. $\ker\psi = 0$. Hence, ψ is monomorphism, which implies that ψ is a KS_n −isomorphism. Thus $M_0(n-1,1)$ and W_0 are isomorphic over KS_n. By [Peel:1979][5] we have $M_0(n-1,1)$ is irreducible submodule when $p \nmid n$. Hence, $W_0 = KS_n u_{12}(n)$ is irreducible submodule when $p \nmid n$.

Proposition 3.6: The submodule $W_0 = KS_n u_{12}(n)$ has the following composition series when p divides n.

$$0 \subset K\sigma_5 \subset W_0$$

Proof: Since W= $KS_n c_1(n)$ and $\sigma_5(n)=\sum_{i=1}^{n} c_i(n)$, then the sum of the coefficient is equal to n and $\sigma_5(n) \in W$ which implies that $K\sigma_5 \subset W$. Then $K\sigma_5(n) \subset W_0$ and $\frac{W_0}{K\sigma_5}$ is an irreducible module over KS_n when p divides n. Therefore, W_0 has the following composition series $0 \subset K\sigma_5 \subset W_0$.

Proposition 3.7: If $p \nmid n$, then $W(n) = W_0(n) \oplus K\sigma_5(n)$.

Proof: By implementing Theorem (3.5) we have $W_0 \cong M_0(n-1,1)$, and irreducible submodule over KS_n when $p \nmid n$ and $\sigma_5(n) \notin W_0(n)$ when $p \nmid n$ since the sum of the coefficients of the $c_i(n)$ in $\sigma_5(n)$ is n. Since $K\sigma_5(n)$ is irreducible submodule over KS_n. Hence, $W_0(n) \cap K\sigma_5(n) = 0$, $K\sigma_5(n) \subset W(n)$ and $W_0(n) \subset W(n)$. But, $\dim_K W_0(n) + \dim_K K\sigma_5(n) = n - 1 + 1 = n = \dim_K W(n)$. Hence $W(n) = W_0(n) \oplus K\sigma_5(n)$ when $p \nmid n$.

Proposition 3.8: If p does not divide n, then W has the following two composition series
$0 \subset W_0(n) \subset W(n)$ and $0 \subset K\sigma_5(n) \subset W(n)$.

Proof: Since $p \nmid n$, then by means of Proposition (3.7) we have
W=$W_0(n) \oplus K\sigma_5$, and by using Theorem (3.5) we have $W_0(n) \cong M_0(n-1,1)$ and $W_0(n)$ is irreducible submodule when $p \nmid n$. Hence $\frac{W}{K\sigma_5(n)} = \frac{W_0(n) \oplus K\sigma_5(n)}{K\sigma_5(n)} \cong W_0(n)$. Thus $\frac{W}{K\sigma_5(n)}$ is irreducible module when $p \nmid n$. Moreover $\frac{W}{W_0(n)} = \frac{W_0(n) \oplus K\sigma_5(n)}{W_0(n)} \cong K\sigma_5(n)$. Hence $\frac{W}{W_0(n)}$ is irreducible module over KS_n. So we get the following two composite series $0 \subset W_0(n) \subset W$ and $0 \subset K\sigma_5(n) \subset W$.

Theorem 3.9: The following sequence of KS_n −modules is exact

$$0 \to \operatorname{Ker} d \overset{i}{\to} M(n-4,3,1) \overset{d}{\to} M(n-3,3) \to 0 \tag{1}$$

over a field K with $p \neq 2$.

Proof Let d: $M(n-4,3,1) \to M(n-3,3)$ be a map defined in terms of the partial operators by $d(x_i x_j x_k x_l^2) = \sum_{q=1}^{n} \frac{\partial^2}{\partial x_q^2} (x_i x_j x_k x_l^2)$. It is clear that the map d is KS_n–homomorphism. Moreover, it is onto map since $\forall \sum_{1\le i<j<k\le n} k_{ijk}\, x_i x_j x_k \in M(n-3,3)$, $\exists\ \frac{1}{2}\left(\sum_{1\le i<j<k\le n} k_{ijk}\, x_i x_j x_k x_l^2\right) \in M(n-4,3,1)$ for some l ($l \neq I, j, k$) such that
$d(\frac{1}{2}(\sum_{1\le i<j\le n} k_{ijk}\, x_i x_j x_k x_l^2)) = \sum_{1\le i<j\le n} k_{ijk}\, x_i x_j x_k$.
Since the inclusion map i is 1-1 and $\operatorname{Im} i = \ker d$. Hence, the sequence (1) is exact.

Theorem 3.10: If $p \neq 2$, then sequence (1) is split if and only if $p \nmid (n-3)$.

Proof: Assume p does not divide (n-3). We can define a function $\varphi: M(n-3,3) \to M(n-4,3,1)$ by $\varphi(x_i x_j x_k) = \frac{1}{2(n-3)} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} x_i x_j x_k x_l^2$ which is a KS_n-homomorphism. Since for any $\tau \in S_n$ then $\varphi\left(\tau(x_i x_j x_k)\right) = \varphi\left(\tau(x_i)\tau(x_j)\tau(x_k)\right) = \frac{1}{2(n-3)} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} \tau(x_i)\tau(x_j)\tau(x_k) x_{l_1}^2$ where $\tau(x_i) = x_{i_1}, \tau(x_j) = x_{j_1}, \tau(x_k) = x_{k_1}$.

$\Longrightarrow \varphi(\tau(x_i x_j x_k)) = \frac{1}{2(n-3)} \tau(x_i x_j x_k x_l^2) = \tau\left(\frac{1}{2(n-3)} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} x_i x_j x_k x_l^2\right) = \tau\,\varphi(x_i x_j x_k)$, and $d\,\varphi(x_i x_j x_k) = d\left(\frac{1}{2(n-3)} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} x_i x_j x_k x_l^2\right) = \frac{1}{2(n-3)} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} d(x_i x_j x_k x_l^2) = \frac{1}{2(n-3)}\left(2(n-3) x_i x_j x_k\right) = x_i x_j x_k$. Then $d\varphi = I$ on $M(n-3,3)$. Hence, the sequence (1) is split. Thus, $M(n-4,3,1) = L \oplus \ker d$, where $L = \varphi(M(n-3,3))$.

Now assume that the sequence (1) is split. Then there exist a KS_n-homomorphism $\psi: M(n-3,3) \longrightarrow M(n-4,3,1)$ such that $d\,\psi = I$ on $M(n-3,3)$, i.e. $d\,\psi(x_i x_j x_k) = x_i x_j x_k$. Then ψ has the form $\psi(x_{i_1} x_{j_1} x_{k_1}) = \sum_{1\le i<j<k\le n} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} k_{ijkl} x_i x_j x_k x_l^2, 1 \le i_1 < j_1 < k_1 \le n$.

Therefore, we get $d\psi(x_{i_1} x_{j_1} x_{k_1}) = d\left(\sum_{1\le i<j<k\le n} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} k_{ijkl} x_i x_j x_k x_l^2\right) = \sum_{1\le i<j<k\le n}\left(2 \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} k_{ijkl}\right) x_i x_j x_k = x_{i_1} x_{j_1} x_{k_1}$

which implies that $2\left(\sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} k_{ijkl}\right) = \begin{cases} 0, \text{if } (i,j,k) \neq (i_1,j_1,k_1) \\ 1, \text{if } (i,j,k) = (i_1,j_1,k_1) \end{cases}$.

Moreover, if $\tau = (x_r x_s) \in S_n; 1 \le r < s \le n$ such that $\tau(x_{i_1} x_{j_1} x_{k_1}) = x_{i_1} x_{j_1} x_{k_1}$. Then $\psi(\tau(x_{i_1} x_{j_1} x_{k_1})) = \psi(x_{i_1} x_{j_1} x_{k_1}) = \tau\,\psi(x_{i_1} x_{j_1} x_{k_1}) \Longrightarrow \psi(x_{i_1} x_{j_1} x_{k_1}) - \tau\,\psi(x_{i_1} x_{j_1} x_{k_1}) = 0$

$\Longrightarrow \sum_{1\le i<j<k\le n} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} k_{ijkl} x_i x_j x_k x_l^2 - \tau\left(\sum_{1\le i<j<k\le n} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} k_{ijkl} x_i x_j x_k x_l^2\right) = 0$

$\Longrightarrow \sum_{1\le i<j<k\le n} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} (k_{ijkl} x_i x_j x_k x_l^2 - k_{ijkl}\,\tau(x_i x_j x_k x_l^2)) = 0 \Longrightarrow$

$\sum_{\substack{r<j<k\le n \\ j\neq s}} \sum_{\substack{l=1 \\ l\neq r,s,j,k}}^{n} (k_{rjkl} - k_{sjkl}) x_r x_j x_k x_l^2 + \sum_{s<j<k\le n} \sum_{\substack{l=1 \\ l\neq r,s,j,k}}^{n} (k_{sjkl} - k_{rjkl}) x_s x_j x_k x_l^2 + \sum_{\substack{1\le i<r<k\le n \\ k\neq s}} \sum_{\substack{l=1 \\ l\neq i,r,s,k}}^{n} (k_{irkl} - k_{iskl}) x_i x_r x_k x_l^2 + \sum_{\substack{1\le i<s<k\le n \\ i\neq r}} \sum_{\substack{l=1 \\ l\neq i,r,s,k}}^{n} (k_{iskl} - k_{irkl}) x_i x_s x_k x_l^2 + \sum_{1\le i<j<r} \sum_{\substack{l=1 \\ l\neq i,j,r,s}}^{n} (k_{ijrl} - k_{ijsl}) x_i x_j x_r x_l^2 + \sum_{\substack{1\le i<j<s \\ i,j\neq r}} \sum_{\substack{l=1 \\ l\neq i,j,r,s}}^{n} (k_{ijsl} - k_{ijrl}) x_i x_j x_s x_l^2 + \sum_{\substack{1\le i<j<k\le n \\ i,j,k\neq r,s}} (k_{ijkr} - k_{ijks}) x_i x_j x_k x_r^2 + \sum_{\substack{1\le i<j<k\le n \\ i,j,k\neq r,s}} (k_{ijks} - k_{ijkr}) x_i x_j x_k x_s^2 + \sum_{k=s+1}^{n} \sum_{\substack{l=1 \\ l\neq r,s,k}}^{n} (k_{rskl} - k_{srkl}) x_r x_s x_k x_l^2 + \sum_{j=r+1}^{s-1} \sum_{\substack{l=1 \\ l\neq r,j,s}}^{n} (k_{rjsl} - k_{sjrl}) x_r x_j x_s x_l^2 + \sum_{s<j<k\le n} (k_{sjkr} - k_{rjks}) x_s x_j x_k x_r^2 + \sum_{\substack{r<j<k\le n \\ j,k\neq s}} (k_{rjks} - k_{sjkr}) x_r x_j x_k x_s^2 + \sum_{i=1}^{r-1} \sum_{\substack{l=1 \\ l\neq i,r,s}}^{n} (k_{irsl} - k_{isrl}) x_i x_r x_s x_l^2 + \sum_{i=1}^{r-1} \sum_{\substack{k=r+1 \\ k\neq s}}^{n} (k_{irks} - k_{iskr}) x_i x_r x_k x_s^2 + \sum_{1\le i<j<r} (k_{ijrs} - k_{ijsr}) x_i x_j x_r x_s^2 + \sum_{i=1}^{s-1} \sum_{k=s+1}^{n} (k_{iskr} - k_{irks}) x_i x_s x_k x_r^2 + \sum_{\substack{1\le i,j<s \\ i,j\neq r}} (k_{ijsr} - k_{ijrs}) x_i x_j x_s x_r^2 = 0.$

$\Longrightarrow \sum_{\substack{r<j<k\le n \\ j\neq s}}^{n} \sum_{\substack{l=1 \\ l\neq r,s,j,k}}^{n} (k_{rjkl} - k_{sjkl})(x_r x_j x_k x_l^2 - x_s x_j x_k x_l^2) + \sum_{\substack{1\le i<r<k\le n \\ k\neq s}} \sum_{\substack{l=1 \\ l\neq i,r,s,k}}^{n} (k_{irkl} - k_{iskl})(x_i x_r x_k x_l^2 - x_i x_s x_k x_l^2) + \sum_{1\le i<j<r} \sum_{\substack{l=1 \\ l\neq i,j,r,s}}^{n} (k_{ijrl} - k_{ijsl})(x_i x_j x_r x_l^2 - x_i x_j x_s x_l^2) +$

$\sum_{\substack{1\le i<j<k\le n\\ i,j,k\neq r,s}}(k_{ijkr}-k_{ijks})(x_ix_j\,x_kx_r^2-x_ix_j\,x_kx_s^2)+\sum_{k=s+1}^{n}\sum_{\substack{l=1\\ l\neq r,s,k}}^{n}(k_{rskl}-k_{srkl})x_rx_sx_k\,x_l^2+\sum_{i=r+1}^{s-1}\sum_{\substack{l=1\\ l\neq r,j,s}}^{n}(k_{rjsl}-k_{sjrl})x_rx_jx_s\,x_l^2+\sum_{\substack{r<j<k\le n\\ j,k\neq s}}(k_{rjks}-k_{sjkr})(x_rx_j\,x_kx_s^2-x_sx_j\,x_kx_r^2)+\sum_{i=1}^{r-1}\sum_{\substack{k=r+1\\ k\neq s}}^{n}(k_{irks}-k_{iskr})(x_ix_rx_k\,x_s^2-x_ix_sx_k\,x_r^2)+\sum_{i=1}^{r-1}\sum_{\substack{l=1\\ l\neq i,r,s}}^{n}(k_{irsl}-k_{isrl})x_ix_rx_s\,x_l^2+\sum_{1\le i<j<r}(k_{ijrs}-k_{ijsr})(x_ix_j\,x_rx_s^2-x_ix_j\,x_sx_r^2)=0.$

Then by equalling the coefficient of the above equation we get for any $r,s;1\le r<s\le n$ that $k_{rjkl}=k_{iskl}=k_{sjkl}=k_{irkl}=k_{ijrl}=k_{ijsl}=k_{rskl}=k_{srkl}=k_{rjsl}=k_{sjrl}=k_{rjks}=k_{sjkr}=k_{irks}=k_{iskr}=k_{irsl}=k_{ijsr}=k$ for any i,j,k,l. But, we have $\sum_{\substack{l=1\\ l\neq i,j,k}}^{n}k_{ijkl}=0$ if $(i,j,k)\neq(i_1,j_1,k_1)$, thus $\sum_{\substack{l=1\\ l\neq i,j,k}}^{n}k=0$ which implies that (n-3) k=0 $\Longrightarrow$ $p|(n-3)$ or $k=0$.

From other side, we get for any $r,s;1\le r<s\le n$ that $k_{ijkr}=k_{ijks}=k_1$. But we have $\sum_{\substack{l=1\\ l\neq i,j,k}}^{n}k_{ijkl}=1$ when $(i,j,k)=(i_1,j_1,k_1)$ which implies that $\sum_{\substack{l=1\\ l\neq i,j,k}}^{n}k_{ijkl}=\sum_{\substack{l=1\\ l\neq i,j,k}}^{n}k_1=1$. i.e. $(n-3)k_1=1\Longrightarrow p\nmid(n-3)$ and $k_1\neq0$. Hence, we get that $p\nmid(n-3)$, $k_1\neq0$ and $k=0$. i.e. if the sequence (1) is split, then $p\nmid(n-3)$.

Proposition 3.11: $S(n-4,3,1)$ is a proper submodule of ker d.

Proof: Since $S(n-4,3,1)=KS_n\Delta(x_1,x_2,x_3)\Delta(x_4,x_5)\Delta(x_6,x_7)$.
Let $y=\Delta(x_1,x_2,x_3)\Delta(x_4,x_5)\Delta(x_6,x_7)=(x_3-x_1)(x_3-x_2)(x_2-x_1)(x_5-x_4)(x_7-x_6)$. Then $y\in$ kerd. But the dimension of kerd over K of the KS_nhomomorphism $d: M(n-4,3,1)\to M(n-3,3)$ is $\frac{n(n-1)(n-2)(n-4)}{6}$ and $\dim_K S(n-4,3,1)=\frac{n(n-1)(n-3)(n-6)}{8}<\frac{n(n-1)(n-2)(n-4)}{6}$.
Hence $S(n-4,3,1)$ is a proper submodule of ker d.

Corollary 3.12: The following sequence of KS_n-modules

$$0\to \text{Ker d}\xrightarrow{i}M_0(n-4,3,1)\xrightarrow{\bar d}M_0(n-3,3)\to 0 \qquad (2)$$

is exact over a field K with $p\neq2$.

Proof: Since $M_0(n-4,3,1)\subset M(n-4,3,1)$ and the K-basis of $M_0(n-4,3,1)$ is $\{x_ix_jx_kx_l^2-x_1x_2x_3x_4^2|\ 1\le i<j<k\le n,1\le l\le n,l\neq i,j,k,(i,j,k,l)\neq(1,2,3,4)\}$, thus $d(x_ix_jx_kx_l^2-x_1x_2x_3x_4^2)=2x_ix_jx_k-2x_1x_2x_3\in M_0(n-3,3)$. Hence, $d|M_0(n-4,3,1): M_0(n-4,3,1)\to M_0(n-3,3)$. Let $\bar d=d|M_0(n-4,3,1)$, then $\bar d: M_0(n-4,3,1)\to M_0(n-3,3)$ such that $\bar d(x_ix_jx_kx_l^2-x_1x_2x_3x_4^2)=2x_ix_jx_k-2x_1x_2x_3$. Then $\bar d$ is onto map since $\forall\,\alpha(x_ix_jx_k-x_1x_2x_3)\in M_0(n-3,3),\exists\frac{\alpha}{2}(x_ix_jx_kx_l^2-x_1x_2x_3x_4^2)\in M_0(n-4,3,1)$ such that $\bar d(\frac{\alpha}{2}(x_ix_jx_kx_l^2-x_1x_2x_3x_4^2))=\alpha(x_ix_jx_k-x_1x_2x_3)$; $\alpha\in K$. Thus the following sequence

$$0\to \text{Ker}\,\bar d\xrightarrow{i}M_0(n-4,3,1)\xrightarrow{\bar d}M_0(n-3,3)\to 0$$

is exact sequence since the inclusion map is one-to-one and Ker $\bar d$ = Imi. Since $\bar d=d|M_0(n-4,3,1)$, then Ker $\bar d\subset$ Ker d. But $\dim_K\text{Ker}\,\bar d=\dim_K M_0(n-4,3,1)-\dim_K M_0(n-3,3)=\dim_K M(n-4,3,1)-\dim_K M(n-3,3)=\dim_K\text{Ker d}$ which implies that Ker $\bar d$ = Ker d. Thus, we get the following sequence

$0\to \text{Ker d}\xrightarrow{i}M_0(n-4,3,1)\xrightarrow{\bar d}M_0(n-3,3)\to 0$ is exact.

Corollary 3.13: The sequence (2) is split if and only if $p \nmid (n-3)$ over a field K with $p \neq 2$.
Proof: Assume $p \nmid (n-3)$. By utilizing Theorem (3.10) we have a KS_n-homomorphism $\varphi: M(n-3,3) \to M(n-4,3,1)$ such that $\varphi(x_i x_j x_k) = \frac{1}{2(n-3)} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} x_i x_j x_k x_l^2$. Then $\varphi(x_i x_j x_k - x_1 x_2 x_3) = \varphi(x_i x_j x_k) - \varphi(x_1 x_2 x_3) = \frac{1}{2(n-3)} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} x_i x_j x_k x_l^2 - \frac{1}{2(n-3)} \sum_{\substack{l=1 \\ l\neq 1,2,3}}^{n} x_1 x_2 x_3 x_l^2 = \frac{1}{2(n-3)} (\sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} x_i x_j x_k x_l^2 - \sum_{\substack{l=1 \\ l\neq 1,2,3}}^{n} x_1 x_2 x_3 x_l^2) \in M_0(n-4,3,1)$. Let $\overline{\varphi} = \varphi|M_0(n-4,3,1)$, then $\overline{\varphi}: M_0(n-3,3) \to M_0(n-4,3,1)$ which is a KS_n-homomorphism such that $\bar{d}\overline{\varphi}(x_i x_j x_k - x_1 x_2 x_3) = \bar{d}\left(\frac{1}{2(n-3)}\left(\sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} x_i x_j x_k x_l^2 - \sum_{\substack{l=1 \\ l\neq 1,2,3}}^{n} x_1 x_2 x_3 x_l^2\right)\right) = \frac{1}{2(n-3)} (\bar{d}(\sum_{\substack{l=1 \\ l\neq I,j,k}}^{n} x_i x_j x_k x_l^2 - \sum_{\substack{l=1 \\ l\neq 1,2,3}}^{n} x_1 x_2 x_3 x_l^2)) = \frac{1}{2(n-3)} (2(n-3) x_i x_j x_k - 2(n-3) x_1 x_2 x_3) = x_i x_j x_k - x_1 x_2 x_3$. Hence, $\bar{d}\overline{\varphi} = I$ on $M_0(n-3,3)$. Thus, the sequence (2) is split and$M_0(n-4,3,1) \cong \text{Ker } d \oplus L_0; L_0 \cong M_0(n-3,3)$. Now, assume the sequence (2) is split. Then there exists a KS_n-homomorphism $\overline{\psi} = \psi|M_0(n-3,3)$ where ψ as it is define in Theorem (3.9) such that $\bar{d}\overline{\psi} = I$. Thus $x_i x_j x_k - x_1 x_2 x_3 = \bar{d}\overline{\psi}(x_i x_j x_k - x_1 x_2 x_3) = d\psi(x_i x_j x_k - x_1 x_2 x_3) = d\psi(x_i x_j x_k) - d\psi(x_1 x_2 x_3) = d(\sum_{\substack{l=1 \\ l\neq I,j,k}}^{n} x_i x_j x_k x_l^2) - d(\sum_{\substack{l=1 \\ l\neq 1,2,3}}^{n} x_1 x_2 x_3 x_l^2) = 2(n-3)k x_i x_j x_k - 2(n-3)k_1 x_1 x_2 x_3$. By equaling the coefficients we get that $2(n-3)k = 1$ and $2(n-3)k_1 = 1$ which implies that $p \nmid (n-3)$.

Theorem 3.14: The following sequence

$$0 \to M_0(n-4,3,1) \xrightarrow{i} M(n-4,3,1) \xrightarrow{f} K \to 0 \qquad (3)$$

is split if and only if $p \nmid \frac{n(n-1)(n-2)(n-3)}{6}$.

Proof: It is clear that the inclusion map is one-to-one and for any $k \in K$ we have $f\left(\sum_{1\leq i<j<k\leq n} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} k_{ijkl} x_i x_j x_k x_l^2\right) = \sum_{1\leq i<j<k\leq n} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} k_{ijkl} = k$ is onto map. Moreover,$\ker f = Imi$, thus the sequence (3) is exact.
If $p \nmid \frac{n(n-1)(n-2)(n-3)}{6}$ we can define a function $h: K \to M(n-4,3,1)$ by $h(k) = \frac{6k\sigma_5(n)}{n(n-1)(n-2)(n-3)}$, where h is a KS_n-homomorphism since $\sum_{\tau\in S_n} r\tau\, h(k) = \sum_{\tau\in S_n} r\tau \left(\frac{6k\sigma_5(n)}{n(n-1)(n-2)(n-3)}\right) = \sum_{\tau\in S_n} \frac{6r\tau k\sigma_5(n)}{n(n-1)(n-2)(n-3)}$ - $\sum_{\tau\in S_n} \frac{6rk\sigma_5(n)}{n(n-1)(n-2)(n-3)} = \sum_{\tau\in S_n} r\, h(k) = h(\sum_{\tau\in S_n} r\, k) = h(\sum_{\tau\in S_n} r\tau(k))$, $\tau(k) = k$ and $\tau\sigma_5(n) = \sigma_5(n)$, then $h(\tau k) = \tau h(k)$. Moreover, $h(k) = f\left(\frac{6k\sigma_5(n)}{n(n-1)(n-2)(n-3)}\right) = \frac{6kf(\sigma_5(n))}{n(n-1)(n-2)(n-3)} = \frac{6k}{n(n-1)(n-2)(n-3)} \frac{n(n-1)(n-2)(n-3)}{6} = k$. Hence, $fh = I$ on K, thus the sequence (3) is split.
Now assume the sequence (3) is split. Then there exist a KS_n-homomorphism $g: K \to M(n-4,3,1)$ such that $fg = I$ on K.
Let $g(1) = \sum_{1\leq i<j<k\leq n} \sum_{\substack{l=1 \\ l\neq i,j,k}}^{n} k_{ijkl} x_i x_j x_k x_l^2$, then $g(1) = g(\tau(1)) = \tau g(1)$; $\tau = (x_r x_s)$, $1 \leq r < s \leq n$, thus $g(1) - \tau g(1) = 0$. i.e. $\sum_{s<j<k\leq n} \sum_{\substack{l=1 \\ l\neq r,s,j,k}}^{n} (k_{rjkl} - k_{sjkl}) x_r x_j x_k x_l^2 +$

$\sum_{\substack{1\le i<r<k\le n\\ k\ne s}}\sum_{\substack{l=1\\ l\ne i,r,s,k}}^{n}(k_{irkl}-k_{iskl})x_ix_rx_k\,x_l^2+\sum_{1\le i<j<r}\sum_{\substack{l=1\\ l\ne i,j,r,s}}^{n}(k_{ijrl}-k_{ijsl})x_ix_jx_r\,x_l^2+$
$\sum_{\substack{1\le i<j<k\le n\\ i,j,k\ne r,s}}(k_{ijkr}-k_{ijks})x_i\,x_j\,x_kx_r^2+\sum_{k=s+1}^{n}\sum_{\substack{l=1\\ l\ne r,s,k}}^{n}(k_{rskl}-k_{srkl})\,x_rx_sx_k\,x_l^2+$
$\sum_{j=r+1}^{s-1}\sum_{\substack{l=1\\ l\ne r,j,s}}^{n}(k_{rjsl}-k_{sjrl})x_rx_jx_s\,x_l^2+\sum_{\substack{r<j<k\le n\\ j,k\ne s}}(k_{rjks}-k_{sjkr})\,x_rx_j\,x_kx_s^2+$
$\sum_{i=1}^{r-1}\sum_{\substack{l=1\\ l\ne i,r,s}}^{n}(k_{irsl}-k_{isrl})\,x_ix_rx_s\,x_l^2+\sum_{i=1}^{r-1}\sum_{\substack{k=r+1\\ k\ne s}}^{n}(k_{irks}-k_{iskr})x_ix_rx_k\,x_s^2\;+$
$\sum_{1\le i<j<r}(k_{ijrs}-k_{ijsr})\,x_ix_j\,x_rx_s^2=0.$
By equalling the coefficients, one can obtain for any $r,s;\ 1\le r<s\le n$ and any i,j,k,l that $k_{rjkl}=k_{iskl}=k_{sjkl}=k_{irkl}=k_{ijrl}=k_{ijsl}=k_{rskl}=k_{srkl}=k_{rjsl}=k_{sjrl}=k_{rjks}=k_{sjkr}=k_{irks}=k_{iskr}=k_{irsl}=k_{ijsr}=k$. Then $g(1)=\sum_{1\le i<j<k\le n}\sum_{\substack{l=1\\ l\ne i,j,k}}^{n}k_{ijkl}x_ix_jx_kx_l^2=k\sigma_5(n)$.
Since $fg=I$, then we have
$1=fg(1)=f\big(k\sigma_5(n)\big)=kf\left(\sum_{1\le i<j<k\le n}\sum_{\substack{l=1\\ l\ne i,j,k}}^{n}x_ix_jx_kx_l^2\right)=k\,\frac{n(n-1)(n-2)(n-3)}{6}$ which implies that $p\nmid\frac{n(n-1)(n-2)(n-3)}{6}$.

Corollary 3.15: $M_0(n-4,3,1)$ is not a direct summand of $M(n-4,3,1)$ when p divides $\frac{n(n-1)(n-2)(n-3)}{6}$.

Proof: Assume $M_0(n-4,3,1)$ is a direct summand of $M(n-4,3,1)$ when p divides $\frac{n(n-1)(n-2)(n-3)}{6}$. Then there exists a KS_n-submodule F of the KS_n-module $M(n-4,3,1)$ s.t. $M(n-4,3,1)=M_0(n-4,3,1)\oplus F$, which implies that the sequence (2) is split and this is contradiction. Hence, $M_0(n-4,3,1)$ is not a direct summand of $M(n-4,3,1)$ when p divides $\frac{n(n-1)(n-2)(n-3)}{6}$.

Theorem 3.16: If $p\ne 2,3$ and $p|(n-1)$, then we have a series of submodules of M(n-4,3,1) as in figure (1) in the appendix.

Proof: If $p\ne 2,3$ and $p|(n-1)$, then $p\nmid(n-3)$. Thus, by Corollary (3.12) one can reach to obtain $M_0(n-4,3,1)\cong \mathrm{Ker}\,d\oplus L_0;\ L_0\cong M_0(n-3,3)$. Since $W=KS_nc_1(n);\ c_1(n)=\sum_{1\le i<j<k\le n}x_ix_jx_kx_1^2$, then the sum of coefficients is $\frac{(n-1)(n-2)(n-3)}{6}$ which implies that $c_1(n)\in M_0(n-4,3,1)$ since $p|(n-1)$. Moreover, $\bar{d}(c_1(n))=2\sum_{1\le i<j<k\le n}x_ix_jx_k\ne 0$. Hence,$c_1(n)\notin \ker\bar{d}$. i.e. $W\cap\ker\bar{d}=0$ thus $W\subset L_0$. Also since $p|(n-1)$, then $p\nmid n$ and by Proposition (3.8) W has the following two composition series
1) $0\subset W_0(n)\subset W(n)$.
2) $0\subset K\sigma_5(n)\subset W(n)$.
Moreover, we have $S(n-4,3,1)\subset\ker\bar{d}$, then $S(n-4,3,1)\oplus L_0\subset\ker\bar{d}\oplus L_0=M_0(n-4,3,1)$. Therefore, we get the proof.

Theorem 3.17: If $p\ne 2,3$ and $p|(n-3)$, then we have a series of submodules of M(n-4,3,1) as in Figure (2) in the appendix.

Proof: Since $W = KS_n c_1(n)$; $c_1(n) = \sum_{1\le i<j<k\le n} x_i x_j x_k x_1^2$, then the sum of coefficients $\frac{n(n-1)(n-2)(n-3)}{6} = 0 \pmod p$ which implies that $W \subset M_0(n-4,3,1)$. But we have $S(n-4,3,1) \subset \ker\bar{d} \subset M_0(n-4,3,1)$ and $\bar{d}(c_1(n)) = 2\sum_{1\le i<j<k\le n} x_i x_j x_k \neq 0$.
Hence $W \cap \ker\bar{d} = 0$. Moreover, since $p \neq 2,3$ and $p|(n-3)$, then $p \nmid n$. Thus, we obtain the proof.

Corollary 3.18: If $p \neq 2,3$ and $p|(n-1)$ then one can achieve a series of submodules of M(n-4,3,1), where $F_2 = KS_n(x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 - x_{r_1}x_{s_1}x_{t_1}x_m^2 + x_{r_1}x_{s_2}x_{t_1}x_m^2)$ and $F_3 = KS_n(x_r x_s x_t x_n^2 - x_r x_s x_t x_m^2)$, where $n \neq m$. As shown in figure (3) in the appendix.

Proof: Using definition of F_2 and F_3, we obtain that $F_2, F_3 \subset M_0(n-4,3,1)$. Moreover, F_2, $F_3 \subset \ker\bar{d}$. Since $x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 - x_{r_1}x_{s_1}x_{t_1}x_m^2 + x_{r_1}x_{s_2}x_{t_1}x_m^2 = (x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_1}x_{t_1}x_m^2) - (x_{r_1}x_{s_2}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_m^2) \in F_3$, then $F_2 \subset F_3$. Thus if $p \neq 2,3$ and $p|(n-1)$, by using Theorem (3.17) one can get a series of submodules of M(n-4,3,1) as shown in Figure (3) in the appendix.

Corollary 3.19: If $p \neq 2,3$ and $p|(n-1)$, then one can obtain a series of submodules of M(n-4,3,1), where $K_2 = KS_n\big(x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 + x_{r_2}x_{s_1}x_{t_2}x_n^2 - x_{r_2}x_{s_2}x_{t_2}x_n^2\big), K_3 = KS_n(x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 + x_{r_2}x_{s_2}x_{t_1}x_n^2 - x_{r_2}x_{s_1}x_{t_1}x_n^2 + x_{r_2}x_{s_1}x_{t_2}x_n^2 - x_{r_2}x_{s_2}x_{t_2}x_n^2 + x_{r_1}x_{s_2}x_{t_2}x_n^2 - x_{r_1}x_{s_1}x_{t_2}x_n^2)$ and $K_4 = KS_n(x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 + x_{r_2}x_{s_1}x_{t_2}x_n^2 - x_{r_2}x_{s_2}x_{t_2}x_n^2 + x_{r_2}x_{s_1}x_{t_1}x_m^2 - x_{r_2}x_{s_2}x_{t_1}x_m^2 + x_{r_3}x_{s_1}x_{t_2}x_m^2 - x_{r_3}x_{s_2}x_{t_2}x_m^2)$. As it is displayed in Figure (4) in the appendix.

Proof: By utilizing definition of K_2, K_3 and K_4 we get $K_2, K_3, K_4 \subset M_0(n-4,3,1)$. Moreover, we have $K_3, K_4 \subset K_2$ since $x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 + x_{r_2}x_{s_2}x_{t_1}x_n^2 - x_{r_2}x_{s_1}x_{t_1}x_n^2 + x_{r_2}x_{s_1}x_{t_2}x_n^2 - x_{r_2}x_{s_2}x_{t_2}x_n^2 + x_{r_1}x_{s_2}x_{t_2}x_n^2 - x_{r_1}x_{s_1}x_{t_2}x_n^2 = \big(x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 + x_{r_2}x_{s_1}x_{t_2}x_n^2 - x_{r_2}x_{s_2}x_{t_2}x_n^2\big) + (x_{r_1}x_{s_2}x_{t_2}x_n^2 - x_{r_1}x_{s_1}x_{t_2}x_n^2 + x_{r_2}x_{s_2}x_{t_1}x_n^2 - x_{r_2}x_{s_1}x_{t_1}x_n^2) \in K_2$, and $x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 + x_{r_2}x_{s_1}x_{t_2}x_n^2 - x_{r_2}x_{s_2}x_{t_2}x_n^2 + x_{r_2}x_{s_1}x_{t_1}x_m^2 - x_{r_2}x_{s_2}x_{t_1}x_m^2 + x_{r_3}x_{s_1}x_{t_2}x_m^2 - x_{r_3}x_{s_2}x_{t_2}x_m^2 = (x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 + x_{r_2}x_{s_1}x_{t_2}x_n^2 - x_{r_2}x_{s_2}x_{t_2}x_n^2) + (x_{r_2}x_{s_1}x_{t_1}x_m^2 - x_{r_2}x_{s_2}x_{t_1}x_m^2 + x_{r_3}x_{s_1}x_{t_2}x_m^2 - x_{r_3}x_{s_2}x_{t_2}x_m^2) \in K_2$. Moreover, $\bar{d}\big(x_{r_1}x_{s_1}x_{t_1}x_n^2 - x_{r_1}x_{s_2}x_{t_1}x_n^2 + x_{r_2}x_{s_1}x_{t_2}x_n^2 - x_{r_2}x_{s_2}x_{t_2}x_n^2\big) = 2(x_{r_1}x_{s_1}x_{t_1} - x_{r_1}x_{s_2}x_{t_1} + x_{r_2}x_{s_1}x_{t_2} - x_{r_2}x_{s_2}x_{t_2}) \neq 0$, thus $K_2 \cap \ker\bar{d} = 0$ which implies that $K_2 \subset L_0$. From definition of the submodule K_2 and the submodule W, one can obtain that $K_2 \cap W = 0$. Then if $p \neq 2,3$ and $p|(n-1)$ we achieve the proof.
Corollary 3.20: If $p \neq 2$, then the following sequence of KS_n-modules

$$0 \to \mathrm{Ker}\,\bar{d}_1 \xrightarrow{i} F_1 \xrightarrow{\bar{d}_1} F \to 0 \tag{4}$$

is split if and only if $p \nmid (n-5)$, where $F = KS_n(x_2x_4x_6 - x_2x_3x_6 - x_2x_4x_5 + x_2x_3x_5)$ and $F_1 = KS_n(x_2x_4x_6x_7^2 - x_2x_3x_6x_7^2 - x_2x_4x_5x_7^2 + x_2x_3x_5x_7^2)$.
Proof: Using definition of F_1 we get that $F_1 \subset M_0(n-4,3,1)$. Since $\bar{d}: M_0(n-4,3,1) \to M_0(n-3,3)$ is onto map and
$\bar{d}(x_2x_4x_6x_7^2 - x_2x_3x_6x_7^2 - x_2x_4x_5x_7^2 + x_2x_3x_5x_7^2) = 2(x_2x_4x_6 - x_2x_3x_6 - x_2x_4x_5 + x_2x_3x_5)$, where $(x_2x_4x_6 - x_2x_3x_6 - x_2x_4x_5 + x_2x_3x_5)$ is a generator of F, thus $\bar{d}|F_1 : F_1 \to F$ is onto map. Let $\bar{d}_1 = \bar{d}|F_1$ then $\bar{d}_1$ is onto map. It is clear that the inclusion map i is one-to-one and $\mathrm{Ker}\,\bar{d}_1 = \mathrm{Im}\,i$. Hence, the sequence (4) is exact. Now, assume that $p \nmid (n-5)$. Let $\psi: F \to F_1$ such that $\psi(x_r x_v x_t - x_r x_s x_t - x_r x_v x_l + x_r x_s x_l) =$

$\frac{1}{2(n-5)}\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}\left(x_rx_vx_tx_k^2-x_rx_sx_tx_k^2-x_rx_vx_lx_k^2+x_rx_sx_lx_k^2\right)$, then for any $\tau\in S_n$ we get $\psi\left(\tau(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)\right)=\psi\left(\tau(x_r)\tau(x_s)\tau(x_t)-\tau(x_r)\tau(x_v)\tau(x_t)-\tau(x_r)\tau(x_s)\tau(x_l)+\tau(x_r)\tau(x_v)\tau(x_l)\right)=\psi\left(x_{r_1}x_{s_1}x_{t_1}-x_{r_1}x_{v_1}x_{t_1}-x_{r_1}x_{s_1}x_{l_1}+x_{r_1}x_{v_1}x_{l_1}\right)=\frac{1}{2(n-5)}\sum_{\substack{k=1\\k_1\neq r_1,s_1,v_1,t_1,l_1}}^{n}(x_{r_1}x_{s_1}x_{t_1}x_{k_1}^2-x_{r_1}x_{v_1}x_{t_1}x_{k_1}^2-x_{r_1}x_{s_1}x_{l_1}x_{k_1}^2+x_{r_1}x_{v_1}x_{l_1}x_{k_1}^2)$, where $\tau(x_r)=x_{r_1},\tau(x_s)=x_{s_1},\tau(x_t)=x_{t_1}$, $\tau(x_v)=x_{v_1}$ and $\tau(x_l)=x_{l_1}$. Hence, $\psi(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)=\tau\,\psi(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)$. Thus ψ is a KS_n-homomorphism. Moreover, we have

$\bar{d}_1\,\psi(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)=$
$\bar{d}_1\left(\frac{1}{2(n-5)}\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}\left(x_rx_sx_tx_k^2-x_rx_vx_tx_k^2-x_rx_sx_lx_k^2+x_rx_vx_lx_k^2\right)\right)=$
$\frac{1}{2(n-5)}\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}2(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)=\frac{1}{2(n-5)}(2(n-5)(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)=(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)$. Hence, $\bar{d}_1\psi=I$ on F and the sequence (4) is split when $p\nmid(n-5)$ and $F_1\cong \operatorname{Ker}\bar{d}_1\oplus F$.

Now if the sequence (4) is split. Then there exists a KS_n-homomorphism $\phi: F\to F_1$ such that $d_1\phi=I$ on F .i.e. $d_1\phi(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)=(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)$. Then ϕ has the form

$\phi\left(x_{r_1}x_{s_1}x_{t_1}-x_{r_1}x_{v_1}x_{t_1}-x_{r_1}x_{s_1}x_{l_1}+x_{r_1}x_{v_1}x_{l_1}\right)=\sum_{1\le r<s,v<t,1\le n}\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}k_{rsvtlk}(x_rx_sx_tx_k^2-x_rx_vx_tx_k^2-x_rx_sx_lx_k^2+x_rx_vx_lx_k^2)$; $1\le r_1<s_1,v_1<t_1,l_1\le n$.

Since $d_1\phi=I$. Hence, $d_1\phi\left(x_{r_1}x_{s_1}x_{t_1}-x_{r_1}x_{v_1}x_{t_1}-x_{r_1}x_{s_1}x_{l_1}+x_{r_1}x_{v_1}x_{l_1}\right)=d_1\left(\sum_{l\le r<s,v<t,l\le n}\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}k_{rsvtlk}(x_rx_sx_tx_k^2-x_rx_vx_tx_k^2-x_rx_sx_lx_k^2+x_rx_vx_lx_k^2)\right)=\sum_{1\le r<s,v<t,1\le n}\left(2\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}k_{rsvtlk}\right)(x_rx_sx_t-x_rx_vx_t-x_rx_sx_l+x_rx_vx_l)=\left(x_{r_1}x_{s_1}x_{t_1}-x_{r_1}x_{v_1}x_{t_1}-x_{r_1}x_{s_1}x_{l_1}+x_{r_1}x_{v_1}x_{l_1}\right)$.

Thus, we obtain

$$2\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}k_{rsvtlk}=\begin{cases}1, & \text{if } (r,s,v,t,l)=(r_1,s_1,v_1,t_1,l_1)\\ 0, & \text{if } (r,s,v,t,l)\neq(r_1,s_1,v_1,t_1,l_1)\end{cases}$$

For $\tau=(x_ix_j)\in S_n$, $1\le i<j\le n$ such that $\tau\left(x_{r_1}x_{s_1}x_{t_1}-x_{r_1}x_{v_1}x_{t_1}-x_{r_1}x_{s_1}x_{l_1}+x_{r_1}x_{v_1}x_{l_1}\right)=\left(x_{r_1}x_{s_1}x_{t_1}-x_{r_1}x_{v_1}x_{t_1}-x_{r_1}x_{s_1}x_{l_1}+x_{r_1}x_{v_1}x_{l_1}\right)$ which implies that $\phi\left(x_{r_1}x_{s_1}x_{t_1}-x_{r_1}x_{v_1}x_{t_1}-x_{r_1}x_{s_1}x_{l_1}+x_{r_1}x_{v_1}x_{l_1}\right)=\phi\left(\tau\left(x_{r_1}x_{s_1}x_{t_1}-x_{r_1}x_{v_1}x_{t_1}-x_{r_1}x_{s_1}x_{l_1}+x_{r_1}x_{v_1}x_{l_1}\right)\right)$. By equalling the coefficients of the above equation and for any $\tau=(x_ix_j)\in S_n$, $1\le i<j\le n$ we obtain

$k_{isvtlk}=k_{jsvtlk}=k_{rivtlk}=k_{rjvtlk}=k_{rsitlk}=k_{rsjtlk}=k_{rsvilk}=k_{rsvjlk}=k_{rsvtik}=k_{rsvtjk}=k_{isvtlj}=k_{jsvtli}=k_{rivtlj}=k_{rjvtli}=k_{rsitlj}=k_{rsjtli}=k_{rsvilj}=k_{rsvjli}=k_{rsvtij}=k_{rsvtji}=k_{ijvtlk}=k_{jivtlk}=k_{isjtlk}=k_{jsitlk}=k_{isvjlk}=k_{jsvilk}=k_{isvtjk}=k_{jsvtik}=k_{rijtlk}=k_{rjitlk}=k_{rivjlk}=k_{rjvilk}=k_{rivtjk}=k_{rjvtik}=k_{rsijlk}=k_{rsjilk}=k_{rsitjk}=k_{rsjtik}=b$ for any r,s,v,t,l,k. But we have $\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}k_{rsvtlk}=0$ when $(r,s,v,t,l)\neq(r_1,s_1,v_1,t_1,l_1)$. Hence, $\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}k_{rsvtlk}=\sum_{\substack{k=1\\k\neq r,s,v,l,t}}^{n}b=0$ which implies that $b(n-5)=0\implies b=0$ or $p\,|(n-5)$. From other side, we get for any $i,j;1\le i<j\le n$ that $k_{rsvtli}=k_{rsvtlj}=b_1$. But we have

$\sum_{\substack{k=1 \\ k\neq r,s,v,l,t}}^{n} k_{rsvtlk} = 1$ when $(r,s,v,t,l) = (r_1,s_1,v_1,t_1,l_1) \Rightarrow \sum_{\substack{k=1 \\ k\neq r,s,v,l,t}}^{n} k_{rsvtlk} = \sum_{\substack{k=1 \\ k\neq r,s,v,l,t}}^{n} b_1 = 1 \Rightarrow (n-5)b_1 = 1 \Rightarrow b_1 \neq 0$ and $p \nmid (n-5)$. Hence, we obtain that $b = 0, b_1 \neq 0$ and $p \nmid (n-5)$. Thus, if the sequence (4) is split then $p \nmid (n-5)$.

Proposition 3.21: $S(n-4,3,1)$ is a KS_n-submodule of F_1.
Proof: Since $S(n-4,3,1) = KS_n\Delta(x_1,x_2,x_3)\Delta(x_4,x_5)\Delta(x_6,x_7)$.
Let $y = \Delta(x_1,x_2,x_3)\Delta(x_4,x_5)\Delta(x_6,x_7) = (x_3-x_1)(x_3-x_2)(x_2-x_1)(x_5-x_4)(x_7-x_6) = x_2x_5x_7x_3^2 - x_2x_4x_7x_3^2 + x_1x_4x_7x_3^2 - x_1x_5x_7x_3^2 + x_1x_5x_7x_2^2 - x_1x_4x_7x_2^2 + x_3x_4x_7x_2^2 - x_3x_5x_7x_2^2 + x_3x_5x_7x_1^2 - x_2x_5x_7x_1^2 + x_2x_4x_7x_1^2 - x_3x_4x_7x_1^2 - x_2x_5x_6x_3^2 + x_2x_4x_6x_3^2 - x_1x_4x_6x_3^2 + x_1x_5x_6x_3^2 - x_1x_5x_6x_2^2 + x_1x_4x_6x_2^2 - x_3x_4x_6x_2^2 + x_3x_5x_6x_2^2 + x_2x_5x_6x_1^2 - x_3x_5x_6x_1^2 + x_3x_4x_6x_1^2 - x_2x_4x_6x_1^2 = (x_2x_5x_7x_3^2 - x_2x_4x_7x_3^2 - x_2x_5x_6x_3^2 + x_2x_4x_6x_3^2) + (x_1x_4x_7x_3^2 - x_1x_5x_7x_3^2 - x_1x_4x_6x_3^2 + x_1x_5x_6x_3^2) + (x_1x_5x_7x_2^2 - x_1x_4x_7x_2^2 - x_1x_5x_6x_2^2 + x_1x_4x_6x_2^2) + (x_3x_4x_7x_2^2 - x_2x_5x_7x_2^2 - x_3x_4x_6x_2^2 + x_2x_5x_6x_2^2) + (x_3x_5x_7x_1^2 - x_3x_4x_7x_1^2 - x_3x_5x_6x_1^2 + x_3x_4x_6x_1^2) + (x_2x_4x_7x_1^2 - x_2x_5x_7x_1^2 - x_2x_4x_6x_1^2 + x_2x_5x_6x_1^2) \in F_1$.
Hence, $S(n-4,3,1) \subset F_1$.

7. Conclusions and discussion

This work presents the definition of the r^{th}–natural triangular representation of $S_n$overK, where $r \geq 0$ and $n \geq 2r+6$ that deals with when $r = 1$. The authors have been proved the following:

1. Each $M(n-4,3,1)$ and $M_0(n-4,3,1)$ are split when $p \nmid (n-3)$.
2. $M_0(n-4,3,1)$ is not a direct summand of $M(n-4,3,1)$ when $p \nmid \frac{n(n-1)(n-2)}{6}$.
3. $F_1 = KS_n(x_2x_4x_6x_7^2 - x_2x_3x_6x_7^2 - x_2x_4x_5x_7^2 + x_2x_3x_5x_7^2)$ is split when $p \nmid (n-3)$ and $p \neq 2$.
4. $S(n-4,3,1)$ is a KS_n-submodule of F_1.

Based on the good results that are achieved in the present work, it has encouraged the forthcoming work to focus on developing and starting to work on the second triangular representation $M(n,,6,3,2,1)$ on symmetric group. This will be a key issue in our subsequent works.

8. Disclosure and conflict of interest

The authors declare that they have no conflicts of interest.

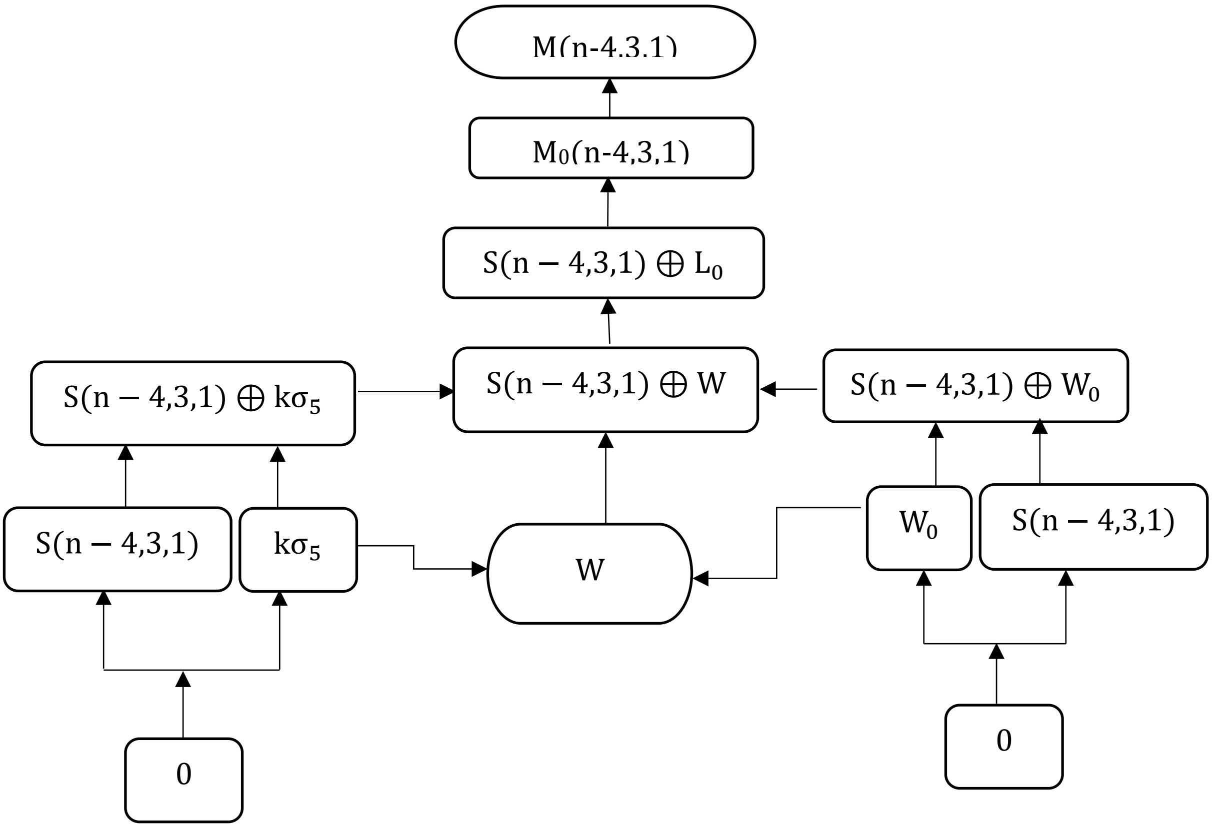

Figure 1: Schematic diagram of submodules of M(n-4,3,1) when $p|(n-1)$ of theorem (3.16)

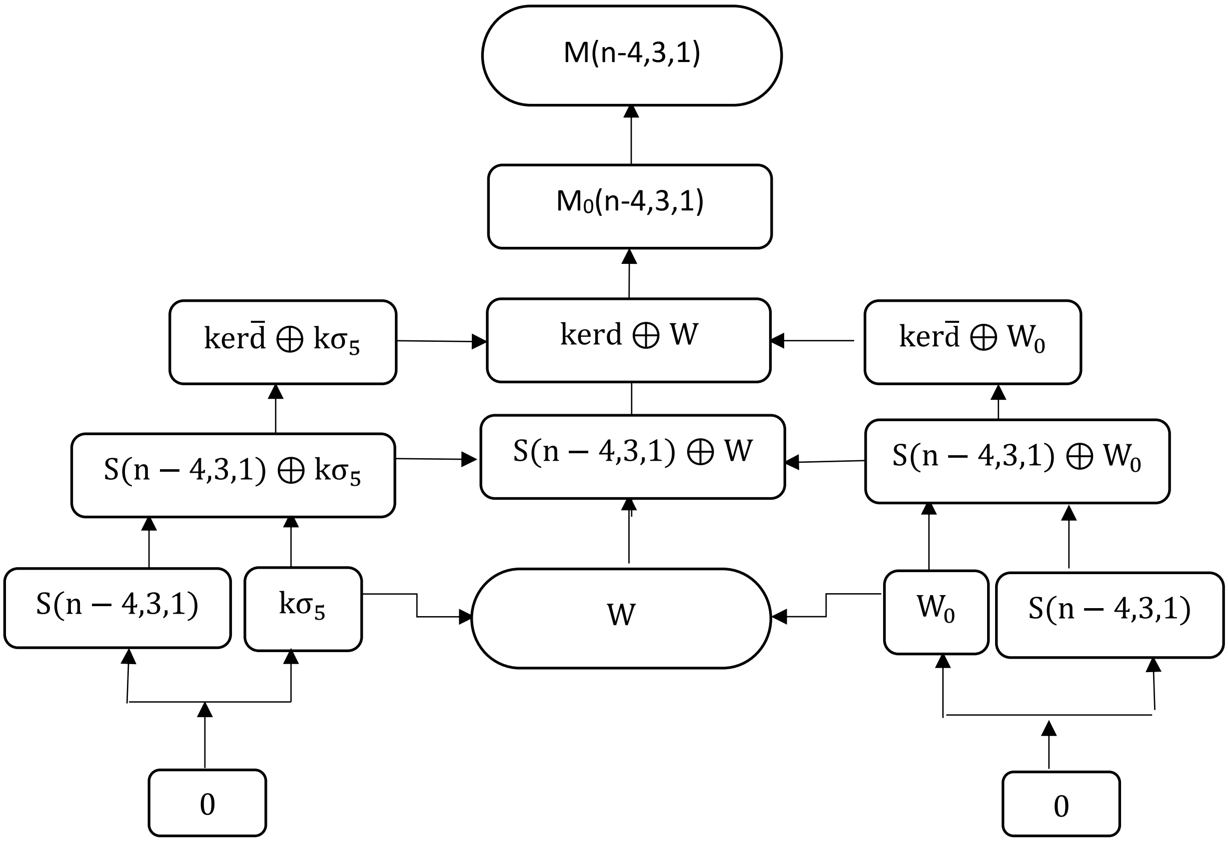

Figure 2- Schematic diagram of submodules of M(n-4,3,1) when $p|(n-3)$ of theorem (3.17)

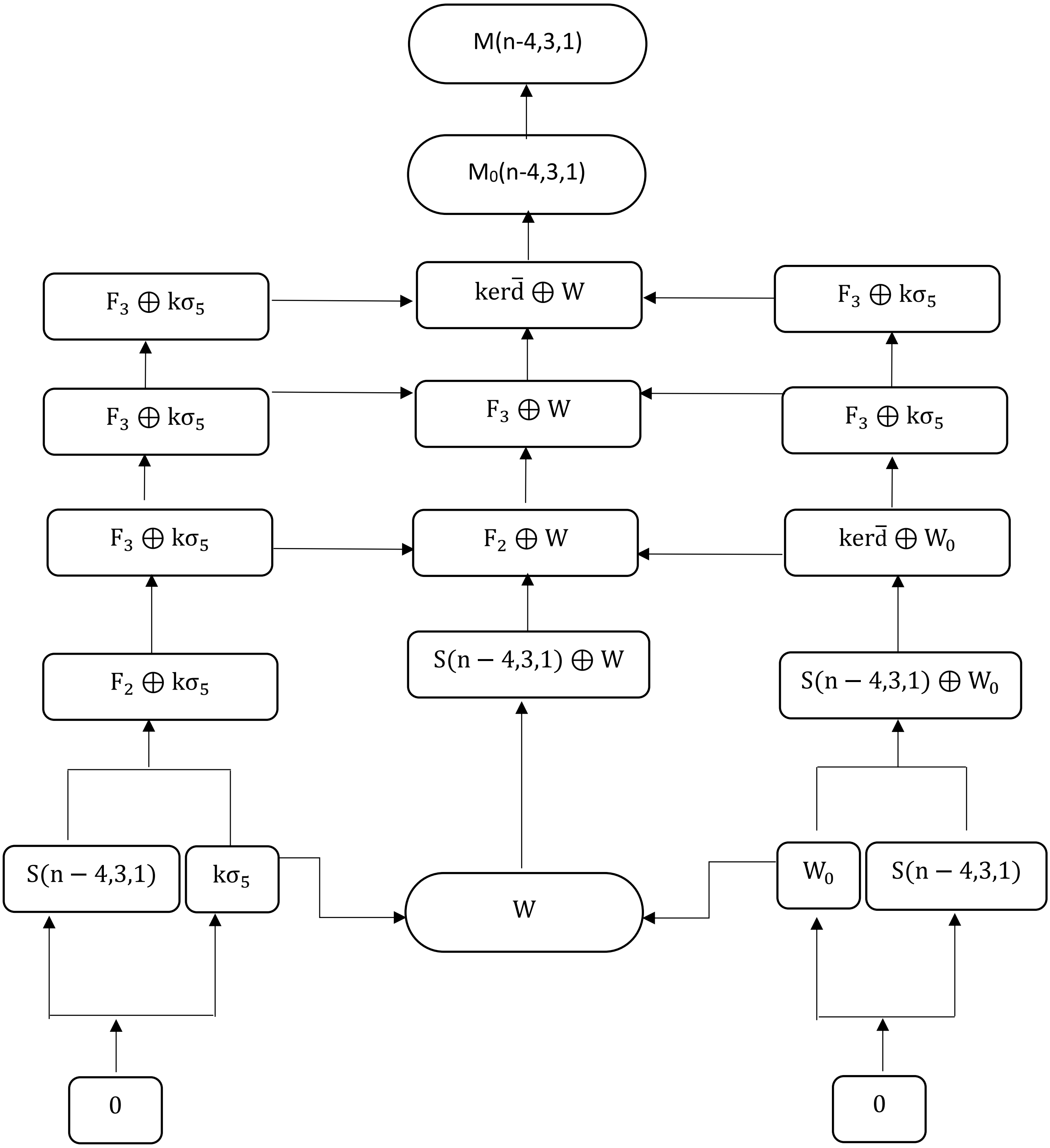

Figure 3: Schematic diagram of submodules of M(n-4,3,1) when $p|(n-1)$ of corollary (3.18)

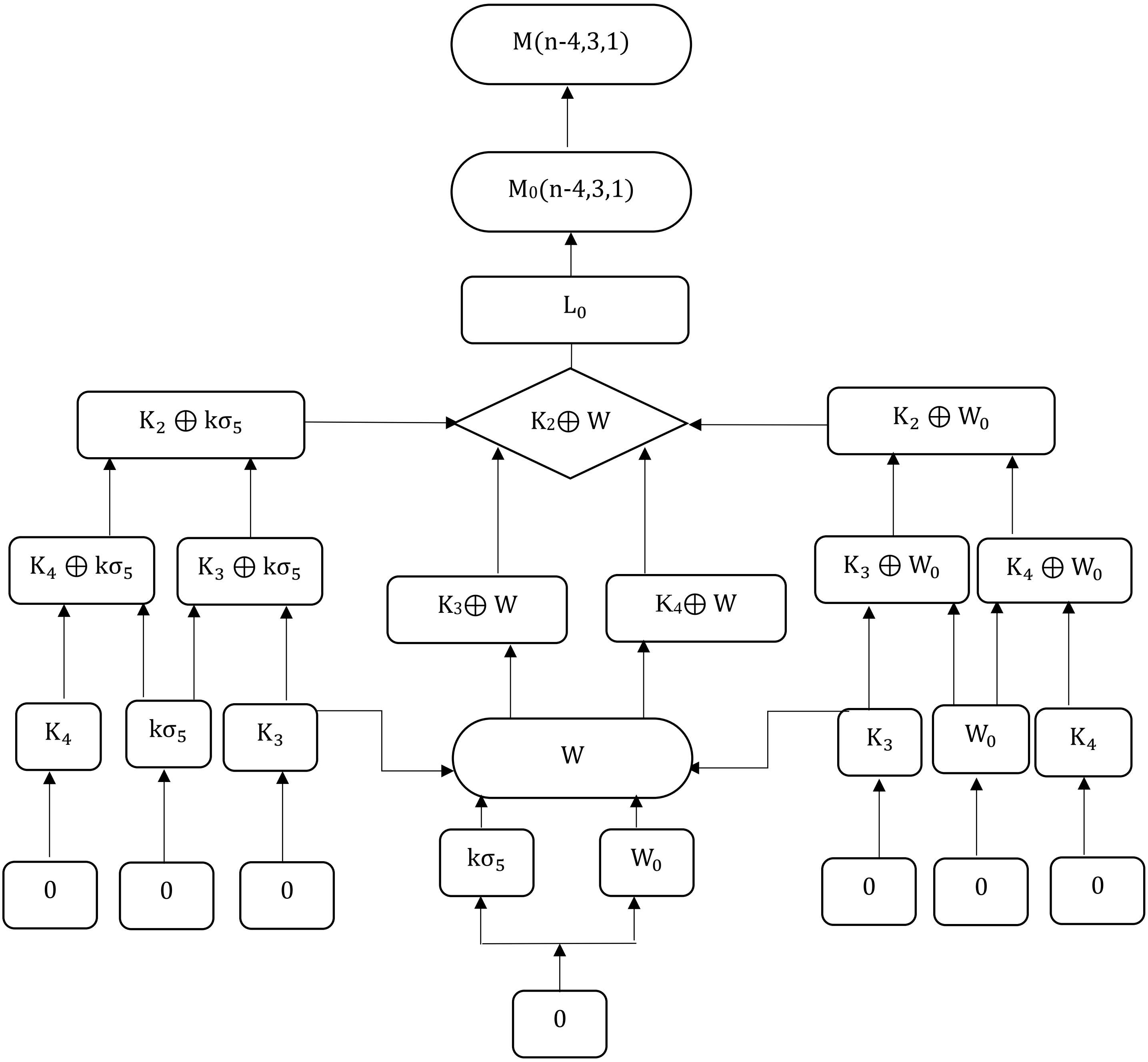

Figure 4: Schematic diagram of submodules of M(n-4,3,1) when $p|(n-1)$ of corollary (3.19)

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