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Triple Γ −Homomorphisms and Bi - Γ -Derivations on Jordan Γ −algebra

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Abstract

In this paper, we introduce the concept of Jordan Γ –algebra, special Jordan Γ –algebra and triple Γ –homomorphisms. We also introduce Bi - Γ –derivations and Annihilator of Jordan Γ −algebra. Finally, we study the triple Γ –homomorphisms and Bi - Γ –derivations on Jordan Γ −algebra.

Keywords: Derivation, Jordan algebra, Triple homomorphism, Centroid.

تشاكلات كاما الثلاثية واشتقاقات كاما الثنائية على جوردان كاما الجبرا

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قسم الرياضيات , كلية التربية, جامعة القادسية , القادسية , العراق

الخلاصة

في هذا البحث ،قدمنا مفهوم جوردان كاما الجبرا ،تشاكلات كاما الثلاثية واشتقاقات كاما الثنائية وكذلك درسنا اشتقاقات كاما الثلاثية واشتقاقات كاما الثنائية على جوردان كاما الجبرا .

1. Introduction

In [1], Jordon studied quantum mechanics, he introduced the concept of Jordon algebra. Recall that a Jordan algebra $\mathfrak{J}$ being an algebra over field $\mathcal{F}$ satisfies
(i) $u \circ v = v \circ u$ for all $u, v \in \mathfrak{J}$,
(ii) $(u^2 \circ v) \circ u = u^2 \circ (u \circ v), u^2 = u \circ u$ for all $u, v \in \mathfrak{J}$.
In [2,3], Albert developed a successful structure theory on all fields from zero property and renamed them Jordan algebras. It is called perfect if $\mathfrak{J} \circ \mathfrak{J} = \mathfrak{J}$, see [4].
In [5], Jacobson introduced the concept of Jordan module as follows: A Jordan module is a system that consists of a Jordan algebra $\mathfrak{J}$, vector space V and two compositions $a.b$ and $b.a$ for all a in V and b in $\mathfrak{J}$ where bi-linear check the following
1) $au = ua$,
2) $(ua)(b \circ c) + (ub)(c \circ a) + (uc)(a \circ b) = (u(b \circ c))a + (u(c \circ a))b + (u(a \circ b))c$,
3) $(((ua)b)c) + (((uc)b)a + u(a \circ c \circ b) = (ua)(b \circ c) + (ub)(c \circ a) + (uc)(a \circ b)$,
In [6], Hoque defined a centralizer as follows:

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An additive mapping $\boldsymbol{\delta}: \mathfrak{I} \to \mathfrak{I}$, is called a centralizer if
$\delta\,(x \circ y) = x\delta(y)$ for all $x, y \in \mathfrak{I}$.
In [7], Zhou introduced the concept of triple homomorphism as follows:
Let Ω and Ω_1 be Lie algebras on an abelian ring R . An R-linear mapping $\mathfrak{f} : \Omega \to \Omega_1$
It is named a triple homomorphism of Ω to Ω_1 if $\mathfrak{f}([x,[y,z]]) = [\mathfrak{f}(x),[\mathfrak{f}(y),\mathfrak{f}(z)]]$ for all $x, y, z \in \Omega$. Similarly, triple homomorphisms can be defined in Jordanian algebra.
In [8], Öztürk et.al. introduced the concept of bi−derivation on the Gamma ring as follows :
Let $D(.,.) : M \times M \to M$ be a bi-additive map which is called a bi-derivation if satisfy :
$D(x\alpha y, z) = D(x,z)\alpha y + x\alpha D(y,z)$
for all x, y, z ∈ M and $\alpha \in \Gamma$.
In [9], Rezaei and Davvaz introduced the concept of Γ−algebra as follows: Let V be a vector space on a field $\mathcal{F}$. V is said to be a Γ−algebra on $\mathcal{F}$ if there is a binary operation (product) on V ,. : $V \times \Gamma \times V \to V$ such that $(b, \alpha, c) \to a\,\alpha\, b, \alpha \in \Gamma, r \in \mathcal{F}$ which satisfies the following conditions :
1) $a\,\alpha\,(b + c) = a\,\alpha\, b + a\,\alpha\, c$, $(b + c)\alpha\, a = b\,\alpha\, a + c\,\alpha\, a$
2) $a\,(\alpha + \beta\,)\, b = a\,\alpha b + a\,\beta\, b$
3) $(ra)\alpha b = r(a\alpha b) = a\alpha(rb)$

The Γ−algebra V is commutative or abelian if $a\,\alpha\, b = b\,\alpha\, a, \forall\, a, b \in V, \alpha \in \Gamma$ and V is associative if $a\alpha(\,b\,\beta\,c\,) = (\,a\,\alpha b\,)\beta\,c, \forall a, b, c \in V, r \in \mathcal{F}, \alpha, \beta \in \Gamma$.
In this paper, we introduce the concept of Jordan Γ –algebra, triple Γ –homomorphism and Bi - Γ –derivation on Jordan Γ−algebra. We also give some results about its on Jordan Γ -algebra. For more details, see [10,11,12,13].
We will write triple Γ –homomorphism with the symbol (t. Γ−hom.) and JordanΓ−algebra with the symbol $\mathfrak{I}\,.\,\Gamma$−algebras
.
2. Triple *Γ*−homomorphism on Jordan *Γ*−algebra

In this section, we give a definition of Jordan Gamma algebra and study the concept of triple homomorphism on it.

Definition 2.1:-

A $\mathfrak{I}\,.\,\Gamma$−algebras $\mathfrak{I}$ of real vector space V with a bi−linear product map $:V \times \Gamma \times V \to V$, where $\psi(u,\alpha,v) = u\alpha v$ satisfying the commutative law and the $\mathfrak{I}.\Gamma$ −identity :

$u\alpha v = v\alpha u$ (commutative law),

$(u\lambda u\alpha v)\,\beta u = u\lambda u\alpha(v\beta u)$ (Jordan Γ-identity).

$u, v \in \mathfrak{I}, \lambda, \alpha, \beta \in \Gamma$

Definition 2.2:-

Let $\mathfrak{I}_1, \mathfrak{I}_2$ be Jordan Γ −algebras on over field $\mathcal{F}$. A linear map $\mathfrak{f} : \mathfrak{I}_1 \to \mathfrak{I}_2$ is called a triple Γ−homomorphism of $\mathfrak{I}_1$ to $\mathfrak{I}_2$ if $\mathfrak{f}(x \circ_\Gamma (y \circ_\Gamma z)) = \mathfrak{f}(x) \circ_\Gamma (\mathfrak{f}(y) \circ_\Gamma \mathfrak{f}(z)) \forall$ $x, y, z \in \mathfrak{I}_1$.
Suppose that $\mathfrak{f}$ is (t. Γ –hom.) from $\mathfrak{I}_1$ *to* $\mathfrak{I}_2$ where $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are $\mathfrak{I}\,.\,\Gamma$−algebras.
Define $A_{nn\mathfrak{f}}(\mathfrak{I}_2)$ as follows:
$A_{nn\mathfrak{f}}(\mathfrak{I}_2) = \{a \in \mathfrak{I}_2 \;/\; a \circ_\Gamma \mathfrak{f}(x) = 0, for\ all\ x \in \mathfrak{I}_1\}$, where $x \circ_\Gamma y = \frac{1}{2}(x\alpha y + y\alpha x)$

Lemma 2.3:-

Suppose $\mathfrak{I}_1$ is a perfect Jordan Γ−algebra over $\mathcal{F}$ and $\mathfrak{f}$ is (t. Γ–hom.) from $\mathfrak{I}_1$ *to* $\mathfrak{I}_2$ that satisfies $A_{nnf}(\mathfrak{I}_2) = \{0\}$, where $\mathfrak{I}_2$ be arbitrary $\mathfrak{I}.\Gamma$−algebra over $\mathcal{F}$. There is $\mathcal{F}$ –linear map $\psi_f : \mathfrak{I}_1 \to \mathfrak{I}_2$ such that for all $x \in \mathfrak{I}_1$ with x $= \sum_{i\in I}(x_{1i} \circ_\Gamma x_{2i})$ where $x_{1i}, x_{2i} \in \mathfrak{I}_1$, $\psi_f(\mathrm{x}) = \sum_{i\in I}(\mathfrak{f}(x_{1i}) \circ_\Gamma \mathfrak{f}(x_{2i}))$.

Proof

It is sufficient to prove that $\sum_{i\in I}(\mathfrak{f}(x_{1i}) \circ_\Gamma \mathfrak{f}(x_{2i}))$ is independent of the expression of x.
Assume that x $= \sum_{i\in I}(x_{1i} \circ_\Gamma x_{2i}) = \sum_{j\in H}(x_{1j} \circ_\Gamma x_{2j})$
Let $\Omega = \sum_{i\in I}(\mathfrak{f}(x_{1i}) \circ_\Gamma \mathfrak{f}(x_{2i}))$, $'\Omega = \sum_{j\in H}(\mathfrak{f}(x_{1j}) \circ_\Gamma \mathfrak{f}(x_{2j}))$.
For any $z \in \mathfrak{I}_1$, we have

$$\mathfrak{f}(z) \circ_\Gamma (\Omega - '\Omega) = \mathfrak{f}(z) \circ_\Gamma (\sum_{i\in I}(\mathfrak{f}(x_{1i}) \circ_\Gamma \mathfrak{f}(x_{2i})) - \sum_{j\in H}(\mathfrak{f}(x_{1j}) \circ_\Gamma \mathfrak{f}(x_{2j}))).$$

$= \mathfrak{f}(z) \circ_\Gamma (\sum_{i\in I}(\mathfrak{f}(x_{1i}) \circ_\Gamma \mathfrak{f}(x_{2i})) - \mathfrak{f}(z) \circ_\Gamma \sum_{j\in H}(\mathfrak{f}(x_{1j}) \circ_\Gamma \mathfrak{f}(x_{2j})))$
$=\sum_{i\in I}(\mathfrak{f}(z) \circ_\Gamma (\mathfrak{f}(x_{1i}) \circ_\Gamma \mathfrak{f}(x_{2i})) - \sum_{j\in H}(\mathfrak{f}(z) \circ_\Gamma (\mathfrak{f}(x_{1j}) \circ_\Gamma \mathfrak{f}(x_{2j})$

$$= \sum_{i\in I}\mathfrak{f}(z \circ_\Gamma (x_{1i} \circ_\Gamma x_{2i}) - \sum_{j\in H}\mathfrak{f}(z \circ_\Gamma (x_{1j} \circ_\Gamma x_{2j}))$$

$$= \mathfrak{f}(z \circ_\Gamma (\sum_{i\in I}(x_{1i} \circ_\Gamma x_{2i})) - \mathfrak{f}(z \circ_\Gamma (\sum_{j\in H}(x_{1j} \circ_\Gamma x_{2j})).$$

$$= \mathfrak{f}(z \circ_\Gamma x) - \mathfrak{f}(z \circ_\Gamma x) = 0.$$

Thus, we have
$\Omega - '\Omega = 0$, *i.e.* $\Omega = '\Omega$. The proof is completed.

Theorem 2.4:-

Assume that $\mathfrak{I}_1$ is a perfect $\mathfrak{I}$.Γ–algebra on $\mathcal{F}$ and $\mathfrak{f}$ is (t. Γ –hom.) from $\mathfrak{I}_1$to $\mathfrak{I}_2$ that satisfies $A_{nnf}(\mathfrak{I}_2) = \{0\}$. where $\mathfrak{I}_2$ be arbitrary $\mathfrak{I}$.Γ–algebra on $\mathcal{F}$. Then $\mathfrak{f}(x\alpha x) = \mathfrak{f}(\mathrm{x})\alpha\mathfrak{f}(x)$ or $\mathfrak{f}(x\alpha x) = -\mathfrak{f}(x)\alpha\mathfrak{f}(x)$ *for all* $x \in \mathfrak{I}_1$, $\alpha \in \Gamma$.

Proof

For any, $y \in \mathfrak{I}_1$, we have
$\mathfrak{f}((x\alpha x) \circ_\Gamma y) \circ_\Gamma \mathfrak{f}(x) = \psi_\mathfrak{f}((x\alpha x) \circ_\Gamma y) \circ_\Gamma x) = \psi_\mathfrak{f}(x\alpha x) \circ_\Gamma (y \circ_\Gamma x)) =$
$\mathfrak{f}(x\alpha x) \circ_\Gamma \mathfrak{f}(y \circ_\Gamma x)$
On the other hand,
$\mathfrak{f}(x\alpha x) \circ_\Gamma y) \circ_\Gamma \mathfrak{f}(x) = ((\mathfrak{f}(\mathrm{x}) \circ_\Gamma \mathfrak{f}(\mathrm{x})) \circ_\Gamma \mathfrak{f}(\mathrm{y})) \circ_\Gamma \mathfrak{f}(\mathrm{x}) = (\mathfrak{f}(\mathrm{x}) \circ_\Gamma \mathfrak{f}(\mathrm{x})) \circ_\Gamma (\mathfrak{f}(\mathrm{y}) \circ_\Gamma \mathfrak{f}(\mathrm{x}))$.
then, we have

$$\mathfrak{f}(x\alpha x) \circ_\Gamma \mathfrak{f}(y \circ_\Gamma x) = (\mathfrak{f}(x) \circ_\Gamma \mathfrak{f}(x)) \circ_\Gamma (\mathfrak{f}(y) \circ_\Gamma \mathfrak{f}(x)).$$

Especially, take $x = y$, then we have
$\mathfrak{f}(x\alpha x)\alpha\mathfrak{f}(x\alpha x) = (\mathfrak{f}(\mathrm{x}) \circ_\Gamma \mathfrak{f}(x))\alpha(\mathfrak{f}(x) \circ_\Gamma \mathfrak{f}(x))$
Which implies that
$\mathfrak{f}(x\alpha x) = \mathfrak{f}(x)\alpha\mathfrak{f}(x)$ or $\mathfrak{f}(x\alpha x) = -\mathfrak{f}(x)\alpha\mathfrak{f}(x)$.

Corollary 2.5:-

Suppose that $\mathfrak{I}_1$ be perfect $\mathfrak{I}.\Gamma$–algebra on $\mathcal{F}$ and $\mathfrak{f}$ is (t. Γ –hom.) $\mathfrak{I}_1$ to $\mathfrak{I}_2$ that satisfies $A_{nnf}(\mathfrak{I}_2) = \{0\}$ where $\mathfrak{I}_2$ be an arbitrary $\mathfrak{I}$. Γ–algebra on $\mathcal{F}$. Hence $\mathfrak{f}$ is a homo. if and only if $\mathfrak{f}(x\alpha x) = \mathfrak{f}(x)\alpha\mathfrak{f}(x)$, $\forall x \in \mathfrak{I}_1$, $\alpha \in \Gamma$when char $\mathcal{F} \neq 2$

Proof:

Assume that f is a $\Gamma -$ hom. Hence, for any $x, y \in \mathfrak{I}_1$, we have
$\mathfrak{f}(x \circ_\Gamma y) = \mathfrak{f}(x) \circ_\Gamma \mathfrak{f}(y)$.
Especially, take $x = y$, we have

$$\mathfrak{f}(x\alpha x) = \mathfrak{f}(x) \circ_{\Gamma} \mathfrak{f}(x) = \mathfrak{f}(x)\alpha\mathfrak{f}(x)$$

Now, suppose that

$\mathfrak{f}(x\alpha x) = \mathfrak{f}(x)\alpha\mathfrak{f}(x)$, for all $x \in \mathfrak{I}_1$, $\alpha \in \Gamma$.

Replace x by $x + y$ where $\in \mathfrak{I}_1$. So we have

$$\mathfrak{f}((x+y)\alpha(x+y)) = \mathfrak{f}(x+y) \circ_{\Gamma} \mathfrak{f}(x+y),$$

$$\mathfrak{f}(x\alpha x) + 2\mathfrak{f}(x \circ_{\Gamma} y) + \mathfrak{f}(y\alpha y) = \mathfrak{f}(x) \circ_{\Gamma} \mathfrak{f}(y) + 2\mathfrak{f}(x) \circ_{\Gamma} \mathfrak{f}(y) + \mathfrak{f}(y) \circ_{\Gamma} \mathfrak{f}(y),$$

Since char $F \neq 2$, we have

$$\mathfrak{f}(x \circ_{\Gamma} y) = \mathfrak{f}(x) \circ_{\Gamma} \mathfrak{f}(y),$$

Therefore, $\mathfrak{f}$ is a hom.

Definition 2.6:-

Assume that $\mathfrak{I}_1$, $\mathfrak{I}_2$ are $\mathfrak{I}.\Gamma$–algebra on $\mathcal{F}$ and $\mathfrak{f}:\mathfrak{I}_1 \to \mathfrak{I}_2$ is (t. Γ –hom.) then $\mathfrak{f}$ is called a special (t. Γ –hom.). if $\mathfrak{f}(\mathfrak{I}_1^{''}) = 0$, where $\mathfrak{I}_1^{''}$ denotes all elements of type $(x \circ_{\Gamma} y) \circ_{\Gamma} z$. $x, y, z \in \mathfrak{I}_1$.

It is obvious that every (t. Γ –hom.) $\mathfrak{f}:\mathfrak{I}_1 \to \mathfrak{I}_2$ satisfies $(\mathfrak{I}_1^{''}) \subseteq \mathfrak{I}_2^{''}$. Thus, we have (t. Γ –hom.) $\mathfrak{f}^{'}:\mathfrak{I}_1^{''} \to \mathfrak{I}_2^{''}$ by restricting $\mathfrak{f}$ to $\mathfrak{I}_1^{''}$.

Theorem 2.7:-

Let $\mathfrak{I}_1$, $\mathfrak{I}_2$ be $\mathfrak{I}.\Gamma$–algebra on $\mathcal{F}$.

1) Up to a special (t. Γ –hom.) any (t. Γ –hom.) $\mathfrak{f}$ from $\mathfrak{I}_1$ to $\mathfrak{I}_2$ can be extended from unique (t. Γ –hom.) of $\mathfrak{I}_1^{''}$ to $\mathfrak{I}_2^{''}$.

2) If $\mathfrak{I}_1$ is perfect, then there is no non-zero distinct (t. Γ –hom.) of $\mathfrak{I}_1$ to $\mathfrak{I}_2$ where $\mathfrak{I}_2$ is an arbitrary $\mathfrak{I}.\Gamma$–algebra.

Proof

1– Assume that $\mathfrak{f}_1$, $\mathfrak{f}_2$ are two (t. Γ –hom.) from $\mathfrak{f}_1$ *to* $\mathfrak{f}_2$ such that $\mathfrak{f}_1^{'} = \mathfrak{f}_2^{'}$ .let $\mathfrak{f} = \mathfrak{f}_1 - \mathfrak{f}_2$ then we have

$$\mathfrak{f}(\mathfrak{I}_1^{''}) = (\mathfrak{f}_1 - \mathfrak{f}_2)(\mathfrak{I}_1^{''}) = (\mathfrak{f}_1^{'} - \mathfrak{f}_2^{'})(\mathfrak{I}_1^{''}) = 0$$

This means that $\mathfrak{f}$ is a special (t. Γ –hom.).

2– Assume that $\mathfrak{f}$ is a special (t. Γ –hom.) from $\mathfrak{I}_1$ to $\mathfrak{I}_2$ i.e. , $f(\mathfrak{I}_1^{''}) = 0$, where $\mathfrak{I}_1$ is perfect , then we get $\mathfrak{f}(\mathfrak{I}_1) = f(\mathfrak{I}_1^{''}) = 0$.

This implies that $\mathfrak{f}$ is zero homomophism.

3. Bi$-\Gamma$ $-$derivation on Jordan Γ –algebra

In this section, we shall study the concept of Bi$-\Gamma$ $-$derivation on Jordan Γ –algebra. First, we shall introduce the definition of $\mathfrak{I}.\Gamma$ –module as follows:

Definition 3.1: -

A $\mathfrak{I}.\Gamma$ $-$module be a system Consists of a $\mathfrak{I}.\Gamma$ $-$algebra $\mathfrak{I}$, vector space V and two compositions $a . b$ and $b . a$ for a in V *and* b in $\mathfrak{I}$ such that they are bi-linear and the following conditions are satisfied

1) $a\alpha u = u\alpha a$,

2) $(u\alpha a)\beta(b \circ_{\Gamma} c) + (u\alpha b)\beta(c \circ_{\Gamma} a) + (u\alpha c)\beta(a \circ_{\Gamma} b) = (u\alpha(b \circ_{\Gamma} c))\beta a + (u\alpha(c \circ_{\Gamma} a))\beta b + (u\alpha(a \circ_{\Gamma} b))\beta c$,

3) $(((u\alpha a)\beta b)\lambda c) + (((u\alpha c)\beta b)\lambda a + u\alpha(a \circ_{\Gamma} c \circ_{\Gamma} b) = (u\alpha a)\beta(b \circ_{\Gamma} c) + (u\alpha b)\beta(c \circ_{\Gamma} a) + (u\alpha c)\beta(a \circ_{\Gamma} b)$.

Definition 3.2:-

Let $\Im$ be $\Im$ Γ–algebra on field$\mathcal{F}$ where Ḿ a $\Im$ – Γ– module, which is a subset S of $\Im$, the collection A_{nn}Ḿ$(S) = \{v \in$ Ḿ$: S\ \alpha\ v = 0\}$

Now, we shall introduce the definition of Bi– Γ– derivation on Jordan Γ −algebra as follows

Definition 3.3:-

A bi– Γ– derivation from a $\Im$. Γ–algebra $\Im$ on Γ–moduleḾ being abilinearly mapping $\psi : \Im \times \Im \to$ Ḿ such that the following conditions are satisfied

$(i)\ \psi\ (x\ \circ_\Gamma\ y, \alpha, z) = x\ \alpha\ \psi\ (y, z) + y\ \alpha\psi\ (x, z),$

$(ii)\ \psi\ (x, \alpha, y\ \circ_\Gamma\ z) = y\ \alpha\ \psi(x, z) + z\ \alpha\ \psi(x, y).$

If Ψ satisfies

$\psi\ (x, \alpha, y) = \psi\ (y, \alpha, x), \forall x, y \in \Im, \alpha \in \Gamma$

then ψ is called a symmetric bi– Γ–derivation.

If Ψ satisfies

$\psi\ (x, \alpha, y) = -\psi(y, \alpha, x),\ \forall x, y \in \Im, \alpha \in \Gamma,$ then ψ is called a skew-symmetric bi– Γ–derivation.

Definition 3.4:-

An associatively $\Gamma -$ algebra B with the multiplication defined by $x\ \circ_\Gamma\ y = \frac{1}{2}(x\alpha y + y\alpha x)$ makes up a $\Im$. Γ–algebra, which is referred to B^+. This type from Jordan algebras is called the distinct $\Im$. Γ–algebra.

Lemma3.5:-

Let B anassociative Γ–algebra where B^+acorresponding distinct $\Im$. Γ–algebra where $\mathrm{u} \circ \mathrm{v} = \frac{1}{2}(\mathrm{u\alpha v} + \mathrm{v\alpha u})$, for all $\mathrm{u, v} \in \mathrm{B}$, $\alpha, \beta \in \Gamma$.

Suppose that $: \mathrm{B} \times \Gamma \times \mathrm{B} \to \mathrm{B}$, is bi$-\ \Gamma$ −Derivation on B then ψ is bi$-\ \Gamma$ −Derivation on B^+.

Proof : $\forall$ u , v , z $\in$ B^+

$\psi(\mathrm{uov}, \beta, \mathrm{z}) = \psi(\frac{1}{2}(\mathrm{u\alpha v} + \mathrm{v\alpha u}), \beta, \mathrm{z})$

$\psi(\mathrm{uov}, \beta, \mathrm{z}) = \psi(\frac{1}{2}\mathrm{u\alpha v} + \frac{1}{2}\mathrm{v\alpha u}, \beta, \mathrm{z})$

$= \frac{1}{2}\psi(\mathrm{u\alpha v}, \beta, \mathrm{z}) + \frac{1}{2}\psi(\mathrm{v\alpha u}, \beta, \mathrm{z})$

$= \frac{1}{2}(\psi(\mathrm{u}, \beta, \mathrm{z})\alpha\mathrm{v} + \mathrm{u}\alpha\psi(\mathrm{v}, \beta, \mathrm{z})) + \frac{1}{2}(\psi(\mathrm{v}, \beta, \mathrm{z})\alpha\mathrm{u} + \mathrm{v}\alpha\psi(\mathrm{u}, \beta, \mathrm{z}))$

$= \frac{1}{2}(\psi(\mathrm{u}, \beta, \mathrm{z})\alpha\mathrm{v} + \mathrm{v}\alpha\psi(\mathrm{u}, \beta, \mathrm{z})) + \frac{1}{2}(\mathrm{u}\alpha\psi(\mathrm{v}, \beta, \mathrm{z}) + \psi(\mathrm{v}, \beta, \mathrm{z})\alpha\mathrm{u}$

$= \psi(\mathrm{u}, \beta, \mathrm{z})\ \circ_\Gamma\ \mathrm{v} + \mathrm{u} \circ_\Gamma\ \psi(\mathrm{v}, \beta, \mathrm{z})$

Similarly, we have

$\psi(u, \beta, v \circ_\Gamma\ z) = \psi(u, \beta, v) \circ_\Gamma\ z + v \circ_\Gamma \psi(u, \beta, z).$

Corollary 3.6:-

Let B be an acommutative anassociative Γ–algebra where ψ asymmetric bi$-$ Γ–derivation on B satisfies

$\psi(\mathrm{u\alpha v}, \beta, \mathrm{z}) = \mathrm{z}\alpha\ \psi\ (\mathrm{u}, \beta, \mathrm{v}), \forall\ \mathrm{u}, \mathrm{v}, \mathrm{z} \in\ \mathrm{B}\ , \alpha, \beta \in \Gamma$

, then

$\psi(\mathrm{u} \circ_\Gamma\ \mathrm{v}, \beta, \mathrm{z}) = \mathrm{z} \circ_\Gamma\ \psi\ (\mathrm{u}, \beta, \mathrm{v})\ , \forall \mathrm{u}, \mathrm{v}, \mathrm{z} \in \mathrm{B}^+, \alpha, \beta \in \Gamma$

Proof : For all $\mathrm{u}, \mathrm{v}, \mathrm{z}\ \in\ \mathrm{B}^+$ we have

$\psi(\mathrm{u} \circ_{\Gamma} \mathrm{v}, \beta, \mathrm{z}) = \psi\left(\frac{1}{2}(\mathrm{u}\alpha\mathrm{v} + \mathrm{v}\alpha\mathrm{u}), \beta, \mathrm{z}\right)$
$= \psi\left(\frac{1}{2}\mathrm{u}\alpha\mathrm{v} + \frac{1}{2}\mathrm{v}\alpha\mathrm{u}, \beta, \mathrm{z}\right)$
$= \frac{1}{2}\psi(\mathrm{u}\alpha\mathrm{v}, \beta, \mathrm{z}) + \frac{1}{2}\psi(\mathrm{v}\alpha\mathrm{u}, \beta, \mathrm{z})$
$= \frac{1}{2}\mathrm{z}\alpha\,\psi(\mathrm{u}, \beta, \mathrm{v}) + \frac{1}{2}\mathrm{z}\alpha\,\psi(\mathrm{v}, \beta, \mathrm{u})$
$= \frac{1}{2}\mathrm{z}\alpha\psi(\mathrm{u}, \beta, \mathrm{v}) + \frac{1}{2}\psi(\mathrm{u}, \beta, \mathrm{v})\alpha\mathrm{z}$
$= \mathrm{z} \circ_{\Gamma} \psi(\mathrm{u}, \beta, \mathrm{v})$.

Definition 3.7:-

A $\mathfrak{I}$ −module Ḿ of $\mathfrak{I}.\Gamma$ −algebra $\mathfrak{I}$ who be referring to $Cent$(Ḿ) as a linearly mapping ∶ $\mathfrak{I} \to$ Ḿ, refer $Cent$(Ḿ) if for all $x, y \in \mathfrak{I}$,
$\delta(x \circ_{\Gamma} y) = x\alpha\,\delta(y)$.

Theorem 3.8:-

Let $\mathfrak{I}$ be perfect $\mathfrak{I}$. Γ–algebra on $\mathcal{F}$where Ḿ is $\mathfrak{I}$– Γ −module checking A_{nn}Ḿ$(\mathfrak{I}) = \{0\}$. Let ψ:$\mathfrak{I} \times \Gamma \times \mathfrak{I} \to$ Ḿ be asymmetric bi−Γ −derivation and γ belong on $cent$(Ḿ). Thus, the equivalence of the following equations is achieved.

$$\psi(\mathrm{w}, \alpha, \mathrm{u} \circ_{\Gamma} \mathrm{v}) = \mathrm{w}\alpha\psi(\mathrm{u}, \mathrm{v}) \,.\, \forall \mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathfrak{I} \;\ldots\ldots\ldots. \text{①}$$
$$\mathrm{z}\alpha\delta(\mathrm{x} \circ_{\Gamma} \mathrm{y}) = \mathrm{x}\alpha\delta(\mathrm{y} \circ_{\Gamma} \mathrm{z}) + \mathrm{y}\alpha\delta(\mathrm{x} \circ_{\Gamma} \mathrm{z}) \,.\, \forall \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathfrak{I} \ldots\ldots\ldots\ldots \text{②}$$

Proof:

Suppose that ψ:$\mathfrak{I} \times \Gamma \times \mathfrak{I} \to$ Ḿ is asymmetric bi−Γ −derivation checking ①. Defining
δ ∶ $\mathfrak{I} \to$ Ḿ as a linearly mapping by
$\delta(\mathrm{x} \circ_{\Gamma} \mathrm{y}) = \psi(x, \alpha, y)$, $\forall \mathrm{x}, \mathrm{y} \in \mathfrak{I}$
Suppose $\sum_i x_i \circ_{\Gamma} y_i = 0$ then we have
$0 = \psi(u, \alpha, \sum_i x_i \circ_{\Gamma} y_i) = \sum_i \psi(u, x_i \circ_{\Gamma} y_i)$
$= \sum_i u\alpha\psi(x_i \circ_{\Gamma} y_i) = u\alpha(\sum_i \psi(x_i, y_i))$
Since A_{nn}Ḿ$(\mathfrak{I}) = \{0\}$, $\sum_I \psi(x_i, \alpha, y_i) = 0$
Hence δ is well–define
$\forall\, u, v \in \mathfrak{I}$, suppose $v = \sum_i x_i \circ_{\Gamma} y_i$ then we have
$\psi(u, \alpha, v) = \psi(u, \sum_i x_i \circ_{\Gamma} y_i)$
$= \sum_i \psi(u, \alpha, x_i \circ_{\Gamma} y_i) = \sum_i u\alpha\psi(x_i \circ_{\Gamma} y_i)$
$= \sum_i u\alpha\delta(x_i \circ_{\Gamma} y_i) = u\alpha(\sum_i \delta(x_i \circ_{\Gamma} y_i)$
$= u\alpha\delta(\sum_i x_i \circ_{\Gamma} y_i) = u\alpha\delta(v)$.
Then we have $\delta(x \circ_{\Gamma} y) = \psi(x, \alpha, y) = x\alpha\gamma(y)$.
This implies that $\delta \in cent(M)$.
For all $x, y, z \in \mathfrak{I}$.
$\mathrm{z}\alpha\delta(\mathrm{x} \circ_{\Gamma} \mathrm{y}) = \mathrm{z}\alpha\psi(\mathrm{x}, \alpha, \mathrm{y}) = \psi(\mathrm{x} \circ_{\Gamma} \mathrm{y}, \alpha, \mathrm{z})$
$= \mathrm{x}\alpha\psi(\mathrm{y}, \alpha, \mathrm{z}) + \mathrm{y}\alpha\psi(\mathrm{x}, \alpha, \mathrm{z})$
$= \mathrm{x}\alpha\delta(y \circ_{\Gamma} z) + y\alpha\delta(x \circ_{\Gamma} z)$
This means that δ satisfies ②
Now, we assume that δ belongs to $cent$(Ḿ) and satisfying ②. And define ∶ $\mathfrak{I} \times \Gamma \times \mathfrak{I} \to$ M is being a bi−linearly mapping by $\psi(x, \alpha, y) = \delta(x \circ_{\Gamma} y)$, for all $x, y, z \in \mathfrak{I}$.
$\psi(\mathrm{x} \circ_{\Gamma} \mathrm{y}, \alpha, \mathrm{z}) = \delta((\mathrm{x} \circ_{\Gamma} \mathrm{y}) \circ_{\Gamma} \mathrm{z}) = \delta(\mathrm{z} \circ_{\Gamma} (\mathrm{x} \circ_{\Gamma} \mathrm{y})) = \mathrm{z}\alpha\delta(\mathrm{x} \circ_{\Gamma} \mathrm{y})$
$= \mathrm{x}\alpha\delta(\mathrm{y} \circ_{\Gamma} \mathrm{z}) + \mathrm{y}\alpha\delta(\mathrm{x} \circ_{\Gamma} \mathrm{z}) = \mathrm{x}\alpha\psi(\mathrm{y}, \mathrm{z}) + \mathrm{y}\alpha\psi(\mathrm{x} \circ_{\Gamma} \mathrm{z})$
Hence, ψ is symmetric bi−Γ −derivation satisfying ①

Corollary 3.9:-

If $\Im$ is perfect $\Im.\Gamma$ −algebra and checks that $A_{nn}(\Im) = \{0\}$. Hence, all symmetric bi−Γ −derivation ψ on $\Im$ satisfies ①is of the from $\psi(\mathrm{u},\alpha,\mathrm{v}) = \delta(\mathrm{u} \circ_{\Gamma} \mathrm{v})$, where $\delta \in \mathrm{cent}(\Im)$ such ② satisfies.

Remark 3.10:-

Assume that $\Im$ a $\Im.\Gamma$–algebra, clear that $\psi: \Im \times \Gamma \times \Im \to A_{nn}(\Im)$ $with\, \psi(\Im,\alpha,\Im') = 0$ be a symmetric bi−Γ–derivation over $\Im$ and call it trivial bi−derivation .

Let ψ :$\Im \times \Gamma \times \Im \to \Im$ be an optional symmetric bi−Γ −derivation for all $u, v \in \Im, z \in A_{nn}(\Im)$, we have got

$0 = \psi(z \circ_{\Gamma} u,\alpha,v) = z \circ_{\Gamma} \psi(u,\alpha,v) + u \circ_{\Gamma} \psi(z \circ_{\Gamma} v) = u \circ_{\Gamma} \psi(z,\alpha,v)$. Hence, $\psi(z(\Im),\Im) \subseteq A_{nn}(\Im)$.

As a result, setting $\overline{\Im} = \Im / A_{nn}(\Im)$, then we can define a symmetric bi−Γ −derivation $\bar{\psi} : \overline{\Im} \times \bar{\Gamma} \times \overline{\Im} \to \overline{\Im}$

By

$\bar{\psi}(\bar{u},\bar{\alpha},\bar{v}) = \overline{\psi(u,\alpha,v)}$ where

$\bar{u} = u + A_{nn}(\Im) \in \overline{\Im}$ for $\in \Im$.

Lemma 3.11:-

Let $\Im$ be a $\Im.\Gamma$−algebra be trivial bi−Γ −derivation over $\Im$. The map $\psi \to \bar{\psi}$ be 1−1 map of symmetric bi−Γ–derivation satisfying ① over $\Im$ of symmetric bi−Γ–derivation checking ①over $\overline{\Im}$.

Proof

Assume ψ_1, ψ_2 be symmetric bi−Γ −derivation on $\Im$ such that $\overline{\psi_1} = \overline{\psi_2}$, then $\psi = \psi_1 - \psi_2$ is asymmetric bi−Γ −derivation on $\Im$. since $\overline{\psi_1} = \overline{\psi_2}$ then

$\overline{\psi_1}(\overline{\Im}) = \overline{\psi_2}(\overline{\Im})$

Subsequently

$\overline{\psi_1(\Im) - \psi_2(\Im)} = \overline{\psi_1(\Im)} - \overline{\psi_2(\Im)} = \overline{\psi_1}(\overline{\Im}) - \overline{\psi_2}(\overline{\Im}) = 0,$

Which implies

$\psi_1(\Im) - \psi_2(\Im) \in A_{nn}(\Im)$

$i.e.\, \psi(\Im,\alpha,\Im) \subseteq A_{nn}(\Im)$ moreover, we have

$$\psi(\Im,\alpha,\Im') = \psi(\Im,\alpha,\Im o \Im) = \Im o \psi(\Im,\alpha,\Im) = 0$$

So , ψ is a trivial bi−Γ −derivation on $\Im$.

Definition 3.12

let $\Im$ be a $\Im.\Gamma$−algebra . A special bi−Γ–derivation be a symmetric bi−Γ–derivation $\psi: \Im \times \Im \to \Im$ such that

1) $\psi(\Im',\Im') = 0$

2) $\psi(\Im,\Im') \subseteq A_{nn_{\Im}}(\Im')$.

Every symmetric bi −Γ–derivation

$\psi: \Im \times \Im \to \Im$ satisfying

$$\psi(u, x \circ_{\Gamma} y) = x \circ_{\Gamma} \psi(u,y) + y \circ_{\Gamma} \psi(u,x) \in \Im' \quad \ldots\ldots\ldots (3)$$

For all $u, x, y \in \Im$.

Thus, we have a symmetric bi−Γ–derivation

$\psi' : \Im' \times \Gamma \times \Im' \to \Im'$ by restricting ψ over $\Im' \times \Gamma \times \Im'$

Lemma 3.13

Let $\mathfrak{I}$ is $\mathfrak{I}.\Gamma-$algebra where satisfies $A_{nn}(\mathfrak{I}) = \{0\}$

1) Up to a specific bi$-\Gamma-$derivation , all symmetric bi$-\Gamma-$derivation ψ satisfying ① on $\mathfrak{I}$ is an extension from a single symmetry bi$-\Gamma-$ derivation checking (1) over $\mathfrak{I}'$.
2) if $\mathfrak{I}'$ is perfect, then $\mathfrak{I}$ has no non zero distinct bi$-\Gamma-$derivation.

Proof

1) Let ψ_1, ψ_2 be bi$-\Gamma-$ derivation on $\mathfrak{I}$ satisfy

$\psi_1' = \psi_2'$. Set $\psi = \psi_1 - \psi_2$,then

$$\psi(\mathfrak{I}', \alpha, \mathfrak{I}') = (\psi_1 - \psi_2)(\mathfrak{I}', \alpha, \mathfrak{I}') = (\psi_1' - \psi_2')(\mathfrak{I}', \alpha, \mathfrak{I}') = 0$$

Take $u, y \in \mathfrak{I}'$ in (3), then we have

$y \circ_\Gamma \psi(u, \alpha, x) = 0, \forall x \in \mathfrak{I}, y, u \in \mathfrak{I}'$

I.e. $\psi(\mathfrak{I}, \alpha, \mathfrak{I}') \subseteq A_{nn_{\mathfrak{I}}}(\mathfrak{I}')$.

Hence, ψ is a distinct bi$-$derivation over $\mathfrak{I}$

2) Suppose that ψ is a distinct bi$-\Gamma-$derivation over $\mathfrak{I}$. From proof (1), one can get $\psi(\mathfrak{I}, \alpha, \mathfrak{I}') \subseteq A_{nn_{\mathfrak{I}}}(\mathfrak{I}')$.

Since $\mathfrak{I}$ is perfect and satisfies $A_{nn_{\mathfrak{I}}}(\mathfrak{I}) = \{0\}$, can be $\psi(\mathfrak{I}, \alpha, \mathfrak{I}') = 0$.

Suppose that $\psi \neq 0$, $x_1, x_2 \in \mathfrak{I}$ such that

$\psi(x_1, x_2) = Z_{12} \neq 0$.

Since $A_{nn}(\mathfrak{I}) = \{0\}$, we can see $x_3 \in \mathfrak{I}$ such that

$x_3 \circ_\Gamma Z_{12} = Z \neq 0$.

Let $\psi(x_1, x_3) = Z_{13}$, $\psi(x_2, x_3) = Z_{23}$.

Then we have

$0 = \psi(x_1 \circ_\Gamma x_3, \alpha, x_2) = x_1 \circ_\Gamma \psi(x_3, \alpha, x_2) + x_3 \circ_\Gamma \psi(x_1, \alpha, x_2) =$
$x_1 \circ_\Gamma Z_{23} + Z \ldots\ldots\ldots\ldots\ldots (1)$

$0 = \psi(x_1 \circ_\Gamma x_2, \alpha, x_3) = x_1 \circ_\Gamma \psi(x_2, \alpha, x_3) + x_2 \circ_\Gamma \psi(x_1, \alpha, x_3) =$
$x_1 \circ_\Gamma Z_{23} + x_2 \circ_\Gamma Z_{13} \ldots\ldots (2)$

$0 = \psi(x_2 \circ_\Gamma x_3, \alpha, x_1) = x_2 \circ_\Gamma \psi(x_3, \alpha, x_1) + x_3 \circ_\Gamma \psi(x_2, \alpha, x_1) =$
$x_2 \circ_\Gamma Z_{13} + Z \ldots\ldots\ldots\ldots (3)$

From (1), we get

$x_1 \circ_\Gamma Z_{23} + Z = 0 \Rightarrow x_1 \circ_\Gamma Z_{23} = -Z$

From (2), we obtained

$x_1 \circ_\Gamma Z_{23} + x_2 \circ_\Gamma Z_{13} = 0 \Rightarrow x_1 \circ_\Gamma Z_{23} = -x_2 \circ_\Gamma Z_{13}$

From (3)

$x_2 \circ_\Gamma Z_{13} + Z = 0 \Rightarrow x_2 \circ_\Gamma Z_{13} = -Z$

We deduce that

$Z = -x_1 \circ_\Gamma Z_{23} = x_2 \circ_\Gamma Z_{13} = -Z$,

Which is a contradiction. Therefore, $\psi = 0$.

Conclusions:

The concept of Jordan Γ –algebra, special Jordan Γ –algebra and triple Γ –homomorphisms are introduced and studied. In addition, Bi - Γ –derivations and Annihilator of Jordan Γ-algebra are introduced and discussed. Finally, many results of these concepts are obtained.

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