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On gw −Prime Submodules

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Abstract

Our aim in this work is to investigate gw −prime submodules and prove some properties of them. We study the relations between gw −prime submodules of a given module and the extension of gw −prime submodules. The relations between gw −prime submodules of two given modules and the gw −prime submodules in the direct product of their quotient module are studied and investigated.

Keywords: Direct product of modules, quotient module, prime submodules, gw −prime submodules, extension of ideals and multiplicative systems.

حول المقاسات الجزئية الاولية من النمط – gw

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الخلاصة

الهدف من هذا العمل هو دراسة المقاسات الجزئية الاولية من النمط – gw حيث تمت البرهنة على بعض خصائصها وإعطاء بعض التصنيفات لها. و كذلك تم دراسة العلاقات بين المقاسات الجزئية الاولية من النمط – gw لمقاس معلوم وبين توسيع المقاسات الجزئية الاولية من النمط – gw و تمت كذلك التحقيق فى العلاقات بين المقاسات الجزئية الاولية من النمط – gw لمقاسين معلومين و المقاسات الجزئية الاولية من النمط – gw فى الجداء الديكارتى لتوسيعاتها.

1. Introduction and Preliminaries

In [1], P. Karimi Beiranvand and R. Beyranvand proved in some classes of modules that almost prime submodules and weakly prime submodules are equivalent. In [2], Rashid Abu-Dawwas, Khaldoun Al-Zoubi introduced the concept of graded weakly classical prime submodules and they gave some properties of these submodules. In [3], Hani A. Khashan and Ece Yetkin Celikel introduced a new type of weakly prime submodules which they called weakly S −prime submodules and gave many properties and characterizations of them in multiplication modules. In [4] Emel Aslankarayigit Ugurlu proved some results concerning S-prime and S-weakly prime submodules. In [5] Zehra Bilgin, Kursat Hakan Oral and Unsal Tekir introduced a new generalization of weakly prime submodules called gw −prime

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submodules and they studied the relation of this type of weakly prime submodules with valuation modules. In [6], Ali Sabah Sadiaq and Haibat K. Mohammadali introduced a new generalization of weakly prime submodules called weakly nearly prime submodules and they proved some basic properties of this new type of weakly prime submodules and gave some characterizations of them. Also, some other properties and some generalizations of weakly prime submodules are given in [7, 8, 9].

All the rings in this work are commutative with identity and all the modules are unital left R −modules. In this work, we try to extend some properties of some $gw-$ prime submodules to their quotient modules. Let M be an R −module. M is called a gw −prime module if whenever $abK = 0$, for $a, b \in R$ and K a submodule of M, then either $a^2K = 0$ or $b^2K = 0$ and a submodule N of M is called a gw −prime submodule if M/N is a gw −prime R −module [5]. If A is an ideal of R, then $S_R(A)$ is defined as $S_R(A) = \{r \in R: ra \in A$ for some $a \in R\backslash A\}$ [10] and if N is a submodule of M, then $S_M(N)$ is defined as $S_M(N) = \{r \in R: rx \in N$ for some $x \in M - N\}$ [11] and the nil radical $\sqrt{A}$, of A is $\sqrt{A} = \{r \in R: r^n \in A$ for some $n \in \mathbb{Z}_+\}$, which is an ideal of R [12]. Let $\emptyset \neq S \subseteq R$, then S is called a multiplicative system if $0 \notin S$ and $a, b \in S$ implies that $ab \in S$ [13]. If S is a multiplicatively system in R, then M_S (or $S^{-1}M$) is $M_S = \{\frac{m}{s}: m \in M, s \in S\}$ which is an R_S −module under the operations: $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$ and $\frac{x}{s}\frac{y}{t} = \frac{xy}{st}$, for $\frac{x}{s}, \frac{y}{t} \in M_S$, it is known as the quotient module (or the module of quotients), where $R_S = \{\frac{r}{s}: r \in R, s \in S\}$ (or $S^{-1}R$ [13]) is a commutative ring with identity $\frac{s}{s}$, for $s \in S$ under the operations $\frac{a}{s} + \frac{b}{t} = \frac{ta+sb}{st}$ and $\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$, for $\frac{a}{s}, \frac{b}{t} \in R_S$. If A is an ideal of R (N is a submodule of M), then A_S (resp. N_S) is an ideal of R_S (resp. a submodule of M_S) and A_S (resp. N_S) is known as the extension of A to R_S (N to M_S) [9], or A_S (resp. N_S) is called the extended ideal in R_S (resp. the extended submodule in M_S) and for a prime ideal P of R, $R \setminus P$ is a multiplicative system in R and $R_P = \{\frac{a}{p}: a \in R, p \notin P\}$ is a local ring and it is known as the localization of R at $R \setminus P$. If S is a multiplicative system in R and N is a submodule of M, then $(\frac{M}{N})_S \cong \frac{M_S}{N_S}$ [11, Corollary 3.4]. In particular, if P is a prime ideal of R, then $(\frac{M}{N})_P \cong \frac{M_P}{N_P}$.

2. Some Properties of Quotient Modules

We prove some properties of quotient modules on which we depend to drive the results of next sections. We start with the following result.

Proposition 2.1. Let N be a submodule of M. For $r \in R$ and $s \in S$ we have $\frac{r}{s}N_S = (rN)_S$. In particular, $\frac{r}{s}M_S = (rM)_S$.

Proof. First, to prove that $\frac{r}{s}N_S \subseteq (rN)_S$, it is enough to prove that $\frac{r}{s}.\frac{x}{t} \in (rN)_S$ for all $\frac{x}{t} \in N_S$, so let $\frac{x}{t} \in N_S$, where $x \in M$ and $t \in S$, then $qx \in N$ for some $q \in S$. Now, we have $\frac{r}{s}.\frac{x}{t} = \frac{r}{s}.\frac{q}{q}.\frac{x}{t} = \frac{rqx}{sqt} \in (rN)_S$, so that $\frac{r}{s}N_S \subseteq (rN)_S$. Next, we prove that $(rN)_S \subseteq \frac{r}{s}N_S$, that is to show that $\frac{rx}{s} \in \frac{r}{s}M_S$ for all $x \in N$, so let $x \in N$, then we have $\frac{rx}{s} = \frac{s}{s}.\frac{rx}{s} = \frac{r}{s}.\frac{sx}{s} \in \frac{r}{s}M_S$, so that $(rN)_S \subseteq \frac{r}{s}N_S$. Therefore $\frac{r}{s}N_S = (rN)_S$. For the second part, since M is a submodule of M, if we put $N = M$ in the above we get $\frac{r}{s}M_S = (rM)_S$.

As a corollary to the above result, we give the following.

Corollary 2.2. Let N a submodule of M. For $r \in R$ and $p \notin P$ we have $\frac{r}{p}N_P = (rN)_P$. In particular, $\frac{r}{p}M_P = (rM)_P$.

Proof. As $R\backslash P$ is a multiplicative system in R, by putting $S = R\backslash P$ and $N = M$ in Proposition 2.1, we get the result.

Proposition 2.3. Let N be a proper submodule of M. Then, $(N:M)_S \subseteq (N_S:M_S)$. In particular, $(N:M)_P \subseteq (N_P:M_P)$.

Proof. Let $\frac{r}{p} \in (N:M)_S$, for $r \in R$ and $p \in S$, then $qr \in (N:M)$, for some $q \in S$, so that $qrM \subseteq N$. If $\frac{x}{t} \in M_S$ is any element, where $x \in M$ and $t \in S$, then we get $qrx \in N$. Now, we have $\frac{r}{p}.\frac{x}{t} = \frac{q}{q}.\frac{rx}{pt} = \frac{qrx}{qpt} \in N_S$, so that $\frac{r}{p}M_S \subseteq N_S$, that means $\frac{r}{p} \in (N_S:M_S)$. Therefore $(N:M)_S \subseteq (N_S:M_S)$. By putting $S = R\backslash P$, we get $(N:M)_P \subseteq (N_P:M_P)$.

Proposition 2.4. Let N be a proper submodule of M such that $S_M(N) \cap S = \emptyset$, then $(N_S:M_S) \subseteq (N:M)_S$. In particular, if $S_M(N) \subseteq P$, then $(N_P:M_P) \subseteq (N:M)_P$.

Proof. Suppose that $\frac{r}{p} \in (N_S:M_S)$, where $r \in R$ and $p \in S$, then $\frac{r}{p}M_S \subseteq N_S$. To show $rM \subseteq N$. Let $x \in M$ be any element, then $\frac{x}{p} \in M_S$ and thus $\frac{rx}{pp} = \frac{r}{p}.\frac{x}{p} \in N_S$, then $srx \in N$ for some $s \in S$. If $rx \notin N$, then $s \in S_M(N)$, so that $s \in S_M(N) \cap S$, this gives that $S_M(N) \cap S \neq \emptyset$ which is a contradiction, so that $rx \in N$. Hence, $rM \subseteq N$, that means, $r \in (N:M)$ and thus we get $\frac{r}{p} \in (N:M)_S$, which gives that $(N_S:M_S) \subseteq (N:M)_S$. We get the result by putting $S = R\backslash P$ and from the fact that, $S_M(N) \cap S = \emptyset$ if and only if $S_M(N) \subseteq P$.

Corollary 2.5. Let N be a proper submodule of M such that $S_M(N) \cap S = \emptyset$, then $(N:M)_S = (N_S:M_S)$. In particular, if $S_M(N) \subseteq P$, then $(N:M)_P = (N_P:M_P)$.

If M, T are $R-$modules and $f: M \to T$ is a homomorphism, then $f_S: M_S \to T_S$, defined by $f_S\left(\frac{m}{s}\right) = \frac{f(m)}{s}$, for $m \in M$ and $s \in S$, is a homomorphism, which is known as the induced homomorphism on M_S and putting $S = R\backslash P$, the above homomorphism $f_P: M_P \to T_P$ given by $f_P\left(\frac{m}{p}\right) = \frac{f(m)}{p}$, for $m \in M$ and $p \notin P$.

Proposition 2.6. Let M, T be $R-$modules. If $f: M \to T$ is a module homomorphism and N is a submodule of T, then $(f^{-1}(N))_S = f_S^{-1}(N_S)$. In particular, $(f^{-1}(N))_P = f_P^{-1}(N_P)$.

Proof: In [14, Theorem 2.3 and Theorem 2.5], it is proved that: For an $R-$module M and a multiplicative system S in R and an endomorphism $f: M \to M$, $\ker f_S = (kerf)_S$ and $Imf_S = (Im\, f)_S$. Now, we prove the same results but for any module homomorphism $f: M \to T$, and for any two modules M, T and the technique of the proofs are the same as in [14, Theorem 2.3 and Theorem 2.5].

Proposition 2.7. Let M, T be $R-$modules and $f: M \to T$ is a homomorphism. Then we have $\ker f_S = (\ker f)_S$, where the homomorphism $f_S: M_S \to T_S$ defined by $f_S\left(\frac{m}{s}\right) = \frac{f(m)}{s}$, for $m \in M$ and $s \in S$. In particular, $\ker f_P = (\ker f)_P$.

Proof. The proof follows by the same technique as in [14, Theorem 2.3].

Proposition 2.8. Let M be an $R-$module and $f: M \to T$ be a homomorphism. If S is a multiplicative system in R and N is a submodule of M, then $f_S(N_S) = (f(N))_S$. In particular, $f_P(N_P) = (f(N))_P$.

Proof. The proof follows by the same technique as in [14, Theorem 2.5].

3. Some properties and characterizations of gw −prime submodules

In this section we prove some properties of $gw-$prime submodules and give some characterizations of them.
First we prove that the extension of a $gw-$prime module is $gw-$prime.

Proposition 3.1. Let M be a $gw-$prime module, then M_S is a $gw-$prime R_S-module.

Proof. Let $\frac{a}{s}.\frac{b}{t}\overline{N} = 0$, where $a, b \in R, s, t \in S$ and $\overline{N}$ a submodule of M_S. By [13, Theorem 2.2], there exists a submodule N of M such that $\overline{N} = N_S$ and $S_M(N) \cap S = \emptyset$. So that we get $\frac{ab}{st}N_S = \frac{a}{s}.\frac{b}{t}\overline{N} = 0$. By Proposition 2.1, we get $(abN)_S = \frac{ab}{st}N_S = 0$. Hence, we get $qabN = 0$ for some $q \in S$, since M is a $gw-$prime module, so we have $(qa)^2N = 0$ or $b^2N = 0$. If $(qa)^2N = 0$, then $(\frac{a}{s})^2\overline{N} = \frac{a}{s}\frac{a}{s}N_S = \frac{q}{q}\frac{a}{s}\frac{q}{q}\frac{a}{s}N_S = \frac{(qa)^2}{(qs)^2}N_S = ((qa)^2N)_S = 0$ and if $b^2N = 0$, the by the same technique as in above, we get $(\frac{b}{t})^2\overline{N} = 0$. Hence, we get M_S is a $gw-$prime R_S-module.

Corollary 3.2. Let M be a $gw-$prime module, then M_P is a $gw-$prime R_P-module.

Proof. Put $S = R\backslash P$ in Proposition 3.1.

Next, we give a necessary and sufficient condition for the extension module to be $gw-$prime.

Proposition 3.3. Let $S_M(0) \cap S = \emptyset$. Then, M_S is a $gw-$prime R_S-module if and only if for each $m \in M$ and $a, b \in R$ with $abm = 0$, then $a^2m = 0$ or $b^2m = 0$.

Proof. Let M_S be a $gw-$prime R_S-module and $m \in M, a, b \in R$ such that $abm = 0$. Now $< m >$ is a submodule of M and to show $ab < m >= 0$. Thus, $(< m >)_S$ is a submodule of M_S. As $S \neq \emptyset$, let $s \in S$ be an element, then $\frac{a}{s}, \frac{b}{s} \in R_S$. Now we show that $\frac{a}{s}.\frac{b}{s}(< m >)_S = 0$. Let $x \in \frac{a}{s}.\frac{b}{s}(< m >)_S$, then $x = \frac{a}{s}.\frac{b}{s}.\frac{r}{t}\frac{y}{u}$, where $r \in R, t \in S$ and $\frac{y}{u} \in (< m >)_S$, then $vy \in < m >$ for some $v \in S$, this gives $vy = cm$, for some $c \in R$. Now $= \frac{a}{s}.\frac{b}{s}.\frac{r}{t}\frac{y}{u} = x = \frac{a}{s}.\frac{b}{s}.\frac{r}{t}.\frac{v}{v}.\frac{y}{u} = \frac{abrvy}{sstvu} = \frac{abrcm}{sstvu} = \frac{rcabm}{sstvu} = \frac{0}{sstvu} = 0$. Hence, $\frac{a}{s}.\frac{b}{s}(< m >)_S = 0$. As M_S is a $gw-$prime R_S-module, we get $(\frac{a}{s})^2(< m >)_S = 0$ or $(\frac{b}{s})^2(< m >)_S = 0$, this gives that $\frac{a^2}{s^2}(< m >)_S = 0$ or $\frac{b^2}{s^2}(< m >)_S = 0$. If $\frac{a^2}{s^2}(< m >)_S = 0$, then $(a^2 < m >)_S = 0$, and as $a^2m \in a^2 < m >$, so that $\frac{a^2m}{s} \in (a^2 < m >)_S = 0$, which gives that $pa^2m = 0$ for some $p \in S$. If $a^2m \neq 0$, then $p \in S_M(0)$. Hence, $S_M(0) \cap S \neq \emptyset$, which is a contradiction, so that $a^2m = 0$ and if $\frac{b^2}{s^2}(< m >)_S = 0$, by the same technique we get $b^2m = 0$.

Next, assume that whenever $abm = 0$, where $a, b \in R$ and $m \in M$, then we have $a^2m = 0$ or $b^2m = 0$, then by [5, Proposition 2.2], we get M is a $gw-$prime $R-$module and by Proposition 3.1, we get M_S is a $gw-$prime R_S-module.
Now, we give the following corollary.

Corollary 3.4. If $S_M(0) \subseteq P$, then M_P is a $gw-$prime R_P-module if and only if for each $m \in M$ and $a, b \in R$ the equation $abm = 0$ implies $a^2m = 0$ or $b^2m = 0$.

Proof. By taking $S = R\backslash P$ in Proposition 3.3, and since $S_M(0) \cap S = \emptyset$ if and only if $S_M(0) \subseteq P$, so that the result follows directly.

Now, we give a condition under which the quotient submodule of a $gw-$prime submodule is a $gw-$prime submodule.

Proposition 3.5. Let N be a submodule of M such that $S_M(N) \cap S = \emptyset$. Then N is a $gw-$prime submodule of M if and only if N_S is a $gw-$prime R_S-submodule of M_S.

Proof. Let N is a $gw-$prime submodule of M, then by [5, Proposition 2.5], we get N_S is a $gw-$prime R_S-submodule of M_S. Conversely, let N_S be a $gw-$prime R_S-submodule of M_S. To show N is a $gw-$prime submodule of M. Let $abm \in N$, where $a, b \in R$ and $m \in M$. As $S \neq \emptyset$, take $s \in S$, then $\frac{a}{s}.\frac{b}{s}.\frac{m}{s} = \frac{abm}{sss} \in N_S$ and as N_S is a $gw-$prime R_S-submodule of M_S, by [5, Corollary 2.3], we get $(\frac{a}{s})^2\frac{m}{s} \in N_S$ or $(\frac{b}{s})^2\frac{m}{s} \in N_S$. Let $(\frac{a}{s})^2\frac{m}{s} \in N_S$, then $\frac{a^2m}{s^2s} = (\frac{a}{s})^2\frac{m}{s} \in N_S$, so that $ta^2m \in N$ for some $t \in S$ and if $a^2m \notin N$, then $t \in S_M(N)$, so that $t \in S_M(N) \cap S$, this gives $S_M(N) \cap S \neq \emptyset$ which is a contradiction. Hence, $a^2m \in N$ and if $(\frac{b}{s})^2\frac{m}{s} \in N_S$, then by the same way we get $b^2m \in N$. Hence, by [5, Corollary 2.3], we get N is a $gw-$prime submodule of M.

Corollary 3.6. Let N be a submodule of M such that $S_M(N) \subseteq P$. Then, N_P is a $gw-$prime R_P-submodule of M_P if and only if for each $m \in M$ and $a, b \in R$ the equation $abm \in N$ implies $a^2m \in N$ or $b^2m \in N$.

Proof. By taking $S = R\backslash P$ in Proposition 3.5, and since $S_M(N) \cap S = \emptyset$ if and only if $S_M(N) \subseteq P$, so that the result follows directly.

Corollary 3.7. If $\overline{N}$ is a $gw-$prime submodule of M_S, then $\overline{N} = N_S$ for some $gw-$prime submodule N of M for which $S \cap S_M(N) = \emptyset$.

Proof. By [15, Theorem 2.2], $\overline{N} = N_S$ for some submodule N of M with $S_M(N) \cap S = \emptyset$. As N_S is a $gw-$prime submodule of M_S, by Proposition 3.5, we get N is a $gw-$prime submodule of M.

Corollary 3.8. Let N, K be submodules of M such that $K \subseteq N$. If S is a multiplicative system in R such that $S \cap S_M(N) = \emptyset$ and N_S is a $gw-$prime submodule of M_S, then $\frac{N}{K}$ is a $gw-$prime submodule of $\frac{M}{K}$.

Proof. As $S\cap S_M(N)=\emptyset$ and N_S is a gw −prime submodule of M_S, by Proposition 3.5, we get N is a gw −prime submodule of M and by [5, Proposition 2.4], we get $\frac{N}{K}$ is a gw −prime submodule of $\frac{M}{K}$.

In [5, Proposition 2.4] it is proved that, if N is a gw −prime submodule of M, then $\sqrt{(N:M)}$ is a prime ideal of R. Now we prove that if the quotient of N is a gw −prime submodule, then $\sqrt{(N:M)}$ is a prime ideal of R.

Proposition 3.9. Let N be a proper submodule of M and $S_M(N)\cap S=\emptyset=S_R(\sqrt{(N:M)}\cap S$. If N_S is a gw −prime R_S −submodule of M_S, then $\sqrt{(N:M)}$ is a prime ideal of R. In particular, if $S_M(N)\subseteq P$ and $S_R(\sqrt{(N:M)}\subseteq P$, then $\sqrt{(N:M)}$ is a prime ideal of R.

Proof. As $(N:M)$ is an ideal of R, by [12, Proposition 2.1.1], we have $\sqrt{(N:M)_S}=(\sqrt{(N:M)})_S$ and as $S_M(N)\cap S=\emptyset$, by Theorem 2.5, we get $(N:M)_S=(N_S:M_S)$, so that $\sqrt{(N_S:M_S)}=\sqrt{(N:M)_S}=(\sqrt{(N:M)})_S$. Since, N_S is a gw −prime R_S −submodule of M_S, so by [5, Proposition 2.4 (i)], we get $\sqrt{(N_S:M_S)}$ is a prime ideal of R_S, so that $(\sqrt{(N:M)})_S$ is a prime ideal of R_S. To show $\sqrt{(N:M)}$ is a prime ideal of R. If $\sqrt{(N:M)}=R$, then we get $1\in\sqrt{(N:M)}$, so that $1=1^n\in(N:M)$ for some $n\in\mathbb{Z}_+$, so that $1.M\subseteq N$ and hence $M\subseteq N$, so that $N=M$ which is a contradiction (since N is proper), so that $\sqrt{(N:M)}\neq R$. Next, let $ab\in\sqrt{(N:M)}$, where $a,b\in R$. As $S\neq\emptyset$, let $s\in S$, then $\frac{a}{s}.\frac{b}{s}=\frac{ab}{ss}\in(\sqrt{(N:M)})_S$, so that $\frac{a}{s}\in(\sqrt{(N:M)})_S$, or $\frac{b}{s}\in(\sqrt{(N:M)})_S$. If $\frac{a}{s}\in(\sqrt{(N:M)})_S$, then $ta\in\sqrt{(N:M)}$, for some $t\in S$. If $a\notin\sqrt{(N:M)}$, then $t\in S_R(\sqrt{(N:M)}$, so that $S_R(\sqrt{(N:M)}\cap S$, this gives that $S_R(\sqrt{(N:M)}\cap S\neq\emptyset$ which is a contradiction. Hence, $a\in\sqrt{(N:M)}$ and if $\frac{b}{s}\in(\sqrt{(N:M)})_S$, then by the same technique as in the above we get $b\in\sqrt{(N:M)}$, so that $\sqrt{(N:M)}$ is a prime ideal of R. By putting $S=R\backslash P$ and from the fact that $S_M(N)\cap S=\emptyset=S_R(\sqrt{(N:M)}\cap S$ if and only if $S_M(N)\subseteq P$ and $S_R(\sqrt{(N:M)}\subseteq P$.

Next, we prove that if the quotient module of a given submodule is a gw −prime submodule, then the epimorphic inverse image of a submodule is a gw −prime submodule.

Proposition 3.10. Let M,T be R −modules and $f:M\to T$ is a module homomorphism. If N is a submodule of T and N_S is a gw −prime submodule of T_S, then $f^{-1}(N)$ is a gw −prime submodule of M.

Proof. As N_S is a gw −prime submodule of T_S, and since, by [5, Proposition 2.4], we get $f_S^{-1}(N_S)$ is a gw −prime submodule of M_S and since $(f^{-1}(N))_S=f_S^{-1}(N_S)$ by Proposition 2.6, so we get $(f^{-1}(N))_S$ is a gw −prime submodule of M_S.

Proposition 3.11. Let $f:M\to T$ be an epimorphism. If N is a submodule of T such that $f^{-1}(N)$ is a gw −prime submodule of M, then N_S is a gw −prime submodule of T_S.
Proof. First, we prove that N is a gw −prime submodule of T. If $x\in\ker f$, then $f(x)=0\in N$, so that we get $x\in f^{-1}(N)$, so that $\ker f\subseteq f^{-1}(N)$ and as f is an epimorphism, by [5, Proposition 2.4], we get that $ff^{-1}(N)$ is a gw −prime submodule of T. Let $x\in N$, then $x=f(m)$ for some $m\in M$, then $m=f^{-1}(x)\in f^{-1}(N)$, and then $x=f(m)=ff^{-1}(x)\in$

$ff^{-1}(N)$, so that $N \subseteq ff^{-1}(N)$. Conversely suppose that $x \in ff^{-1}(N)$, then $x = f(m)$ for some $m \in f^{-1}(N)$ which gives that $f(m) \in N$ and hence, $x \in N$, so that $ff^{-1}(N) \subseteq N$. Hence, $ff^{-1}(N) = N$, so that N is a gw −prime submodule of T and then by [5, Proposition 2.5], we get N_S is a gw −prime submodule of T_S.
From Proposition 3.10 and Proposition 3.11 we conclude the following result.

Corollary 3.12. Let M, T be R −modules and $f: M \to T$ be an epimorphism. If S is a multiplicative system in R and N is a submodule of T such that N_S is a gw −prime submodule of T_S, then $f^{-1}(N)$ is a gw −prime submodule of M.

Proof. The proof follows by Proposition 3.10 and Proposition 3.11.

Proposition 3.13. Let N be a submodule of M and $f: M \to T$ be an epimorphism such that $\ker f \subseteq N$. If $S \cap S_M(T) = \emptyset$ and N_S is a gw −prime submodule of M_S, then $f(N)$ is a gw −prime submodule of T.

Proof. The homomorphism $f_S: M_S \to T_S$ is defined by $f_S\left(\frac{m}{s}\right) = \frac{f(m)}{s}$ for $m \in M$ and $s \in S$, and by Proposition 2.7, we have $\ker f_S = (\ker f)_S$. Thus we get $\ker f_S = (\ker f)_S \subseteq N_S$. To show f_S is onto. Let $\frac{t}{s} \in T_S$, where $t \in T$ and $s \in S$. As f is onto, we have $t = f(m)$ for some $m \in M$, then we get $\frac{m}{s} \in M_S$ and $f_S\left(\frac{m}{s}\right) = \frac{f(m)}{s} = \frac{t}{s}$, so that f_S is an epimorphism, so by [5, Proposition 2.4], we get $f_S(N_S)$ is a gw −prime submodule of T_S and by Proposition 2.8, we have $f_S(N_S) = (f(N))_S$, so $(f(N))_S$ is a gw −prime submodule of T_S. As $S \cap S_M(T) = \emptyset$, by Proposition 2.5, we get $f(N)$ is a gw −prime submodule of T.

4. The gw −Prime Submodules in Direct Product of Modules

In this section we determine the relations between gw −prime submodule of two modules and the gw −prime submodules in the direct product of their quotient modules.

Proposition 4.1. Let M and G be R −modules and N, L be proper submodules of M and G, respectively. If S is a multiplicative system in R such that $S \cap S_M(0) = \emptyset = S \cap S_G(0)$ and $S \cap S_M(N) = \emptyset = S \cap S_M(L)$, then
(i) $N_S \times G_S$ is a gw −prime submodule of $M_S \times G_S$ if and only if N is a gw −prime submodule of M.
(ii) If $N_S \times L_S$ is a gw −prime submodule of $M_S \times G_S$, then N is a gw −prime submodule of M and L is a gw −prime submodule of G.

Proof. Since, N is a proper submodule of M and L is a proper submodule of G and $S \cap S_M(0) = \emptyset = S \cap S_G(0)$, so it can be shown that N_S is a proper submodule of M_S and L_S is a proper submodule of G_S.
(i) Since, N_S is a proper submodule of M_S, and $S \cap S_M(N) = \emptyset$, so by [5, Proposition 2.6 (i)] and Proposition 3.5, we have $N_S \times G_S$ is a gw −prime submodule of $M_S \times G_S$ if and only if N_S is a gw −prime submodule of M if and only if N is a gw −prime submodule of M.
(ii) Let $N_S \times L_S$ be a gw −prime submodule of $M_S \times G_S$. As, N_S, L_S are proper submodule of M_S and L_S respectively, by [5, Proposition 2.6 (ii)] we get N_S is a gw −prime submodule of M_S and L_S is a gw −prime submodule of G_S and since, $S \cap S_M(N) = \emptyset = S \cap S_M(L)$, so that by Proposition 3.5, we get N is a gw −prime submodule of M and L is a gw −prime submodule of G.

Corollary 4.2. Let N be a proper submodule of M and L be a proper submodule of G such that $S \cap S_M(N) = \emptyset = T \cap S_G(L)$. If $N_S \times L_T$ is a gw −prime submodule of $M_S \times G_T$, then N is a gw −prime submodule of M and L is a gw −prime submodule of G.

Proof. Let $N_S \times L_T$ be a gw −prime submodule of $M_S \times G_T$. As $S \neq \emptyset$, take $s \in S$. If $N_S = M_S$, then for any $m \in M$, we have $\frac{m}{s} \in M_S = N_S$, so that $tm \in N$ for some $t \in S$. If $m \notin N$, then $t \in S_M(N)$, so that $S \cap S_M(N) \neq \emptyset$ which is a contradiction. Hence, $m \in N$ and that $M \subseteq M$ and hence, $N = M$ which is a contradiction, so that $N_S \neq M_S$, that means N_S is a proper submodule of M_S. Similar we get L_T is a proper submodule of G_T. Next, since we have $N_S \times L_T$ is a gw −prime submodule of $M_S \times G_T$, so by [5, Proposition 2.8], we get that N_S is a gw −prime submodule of M_S and L_T is a gw −prime submodule of G_T and as $S \cap S_M(N) = \emptyset = T \cap S_G(L)$, by Proposition 3.5, we get N is a gw −prime submodule of M and L is a gw −prime submodule of G.

Proposition 4.3. If N is a proper submodule of M such that $S \cap S_M(N) = \emptyset$, then $N_S \times G_T$ is a gw −prime submodule of $M_S \times G_T$ if and only if N is a gw −prime submodule of M.

Proof. Let $N_S \times G_T$ be a gw −prime submodule of $M_S \times G_T$. Since $S \cap S_M(N) = \emptyset$ and N is a proper submodule of M, so by the same technique as in the proof of Proposition 4.2, we get N_S is a proper submodule of M_S and hence, by [5, Proposition 2.8], we get N_S is a gw −prime submodule of M_S and as $S \cap S_M(N) = \emptyset$, by Proposition 3.5, we get N is a gw −prime submodule of M. Conversely, suppose that N is a gw −prime submodule of M, then by Proposition 3.5, we get N_S is a proper submodule of M_S and by [5, Proposition 2.8], we get $N_S \times G_T$ is a gw −prime submodule of $M_S \times G_T$.

Conclusions

(1)The extension of gw −prime submodules are gw −prime and the converse is true when the intersection of multiplicative systems does not meet the set of those elements which are not prime to the given submodule.
(2) If the quotient of a submodule N of an R −module M is a gw −prime submodule of an R −module, then $\sqrt{(N:M)}$ is a prime ideal of R.
(3) If the quotient module of a given submodule is a gw −prime submodule, then the epimorphic inverse image of a submodule is a gw −prime submodule.
(4)If N is a gw −prime submodule of an R −module M and L is a gw −prime submodule of an R −module G, then the direct product of their extensions is a gw −prime submodule.

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