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Characteristic Zero Resolution (Lascoux Resolution) of Weyl Module in the Case of the Skew- Partition (11, 7, 5)/ (1, 1, 1)

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Abstract:
In this paper, the terms of Lascoux and boundary maps for the skew-partition (11,7,5) / (1,1,1) are found by using the Jacobi-Trudi matrix of partition. Further, Lascoux resolution is studied by using a mapping Cone without depending on the characteristic-free resolution of the Weyl module for the same skew-partition.

Keywords: Resolution, Lascoux resolution, Weyl module, Characteristic free, mapping Cone ,skew-shape.

تحلل المميز الصفري (تحلل لاسكو) لمقاس وايل في حالة شبه التجزئة (11, 7, 5) / (1, 1, 1)

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الخلاصة
في هذا البحث تم ايجاد عناصر المميز الصفري (لاسكو) والدوال الحدودية لشبه التجزئة **(11, 7, 5)/ (1, 1, 1)** باستخدام مصفوفة التجزئة جاكوبي ترودي .ايضا تمت دراسة تحلل المميز الصفري لمقاس وايل باستخدام تطبيق كون دون الاعتماد على تحلل المميز الحر لمقاس وايل لشبه التجزئة ذاتها.

1. Introduction:

Let R be a commutative ring with 1 and $\mathcal{F}$ be a free R-module and $\mathcal{D}_i\mathcal{F}$ be the divided power algebra of degree i.

Authors in [1-3] discussed the complex of characteristic zero for the partitions (4,4,4),(8,7,3) and skew-partition (8,6,3)/(u,1) when u=1,2 , respectively. Shaymaa N.A., Haytham R.H. and Nubras in [4,5] exhibited the terms and the exactness of the Weyl resolution in the case of skew- partition (8, 6)/(2.0) , (8,6)/(2.1) and (7,7), (7,7)/(1,0), respectively. As well Artale [6] discussed the terms for the three-rowed skew-partition and almost skew-shape in Lascoux. In this paper, we find the terms and Lascoux resolution for the skew-partition (11, 7, 5) / (1, 1, 1) by using the homological diagrams. We also prove

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the sequence of Lascoux is exact which does not include the characteristic-free resolution of the Weyl module for the same skew-shape by using a mapping Cone.

Authors in [7] defined the Capelli identities as follows:

Let $i, j, k, \ell \in \mathcal{P}^{+}$, so we have the following:

(1) If $k \neq j$, then

$$\partial_{ij}^{(r)}\partial_{jk}^{(s)} = \sum_{\alpha\geq 0}\partial_{jk}^{(s-\alpha)}\partial_{ij}^{(r-\alpha)}\partial_{ik}^{(\alpha)}$$

$$\partial_{jk}^{(s)}\partial_{ij}^{(r)} = \sum_{\alpha\geq 0}(-1)^{\alpha}\partial_{ij}^{(r-\alpha)}\partial_{jk}^{(s-\alpha)}\partial_{ik}^{(\alpha)}$$

(2) If $i \neq k$ and $j \neq \ell$ then $\partial_{ik}^{(s)}\partial_{i\ell}^{(r)} = \partial_{i\ell}^{(r)}\partial_{ik}^{(s)}$

The author in [8] defined the concept of mapping Cone as follows:

The commute diagram

$$\begin{array}{llllllll}
C_0: & C_{n-1} & \xrightarrow{d_{n-1}} & C_n & \xrightarrow{d_n} & C_{n+1} & \xrightarrow{d_{n+1}} & C_{n+2} \ \dots \\
 & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} & & \downarrow f_{n+2} \\
D_0: & D_{n-1} & \xrightarrow{d'_{n-1}} & D_n & \xrightarrow{d'_n} & D_{n+1} & \xrightarrow{d'_{n+1}} & D_{n+2} \ \dots
\end{array}$$

If the sequence of the rows is exact and $\lambda_{n-1}: C_n \otimes D_{n-1} \longrightarrow C_{n+1} \otimes D_n$ definite by $(\alpha, b) \longmapsto (-d_n(\alpha), d'_{n-1}(b)\ t + f_n(\alpha))$ such hat $\lambda_{n-1} \circ \lambda_n = 0; \forall\ n \in \mathbb{Z}^{+}$.

Then the sequence

$C_{n-1} \xrightarrow{\lambda_{n-1}} C_n \otimes D_{n-1} \xrightarrow{\lambda_n} C_{n+1} \otimes D_n \xrightarrow{\lambda_{n+1}} C_{n+2} \otimes D_{n+1} \xrightarrow{\lambda_{n+2}} \dots$ Is exact,

2. The terms of the sequence of Lascous in the skew-shape (11, 7, 5)/ (1, 1, 1)

The positions of the terms of the Lascous are determined by the length of the permutation to which they correspond in [9] and [10].

In the case of the skew-partition (11, 7, 5)/ (1, 1, 1)**,** we have the pursue matrix:

$$\begin{bmatrix} \mathcal{D}_{10}\mathcal{F} & \mathcal{D}_5\mathcal{F} & \mathcal{D}_2\mathcal{F} \\ \mathcal{D}_{11}\mathcal{F} & \mathcal{D}_6\mathcal{F} & \mathcal{D}_3\mathcal{F} \\ \mathcal{D}_{12}\mathcal{F} & \mathcal{D}_7\mathcal{F} & \mathcal{D}_4\mathcal{F} \end{bmatrix}$$

Then the characteristic zero complexes have the correspondence between their terms as pursues:

$\mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \leftrightarrow identity$

$\mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \leftrightarrow (12)$

$\mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \leftrightarrow (23)$

$\mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \leftrightarrow (132)$

$\mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \leftrightarrow (123)$

$\mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \leftrightarrow (13)$

So the sequence of Lascoux is

$$0 \to D_{12}F \otimes D_6F \otimes D_2F \to \begin{array}{c} D_{12}F \otimes D_5F \otimes D_3F \\ \oplus \\ D_{11}F \otimes D_7F \otimes D_2F \end{array} \to \begin{array}{c} D_{10}F \otimes D_7F \otimes D_3F \\ \oplus \\ D_{11}F \otimes D_5F \otimes D_4F \end{array}$$
$$\to D_{10}F \otimes D_6F \otimes D_4\boldsymbol{F} \qquad \dots \textbf{(1)}$$

2.1 The homological diagram of the Lascoux sequence.

Consider the following diagram

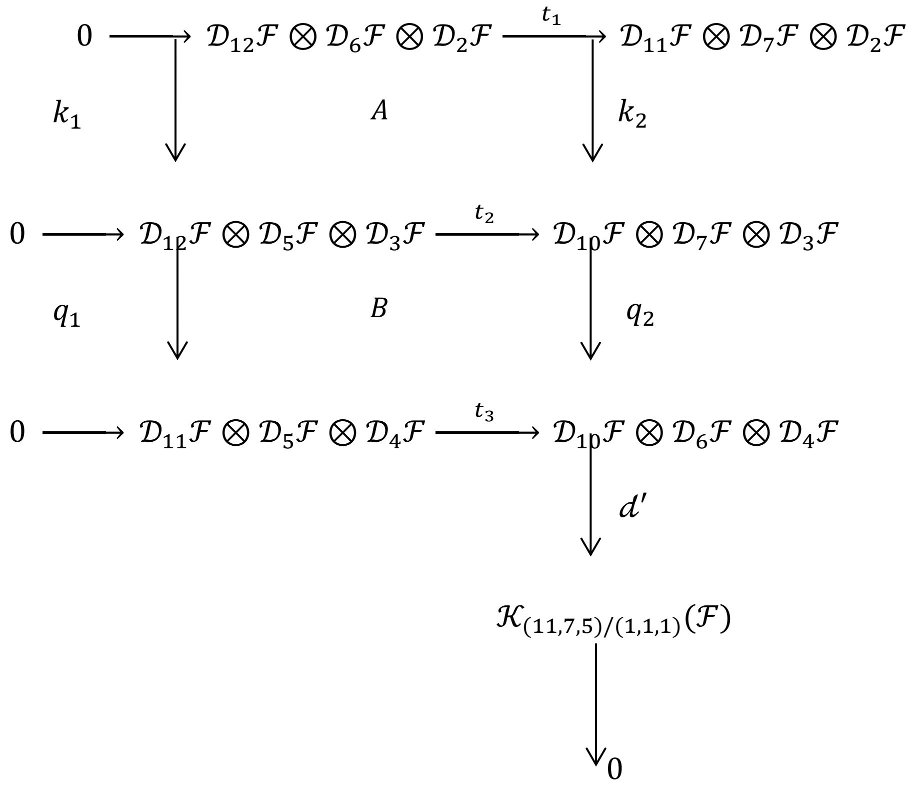

Diagram (2.1)

Define the maps as follows:

$t_1(v): \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \to \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$, by $t_1(v) = \partial_{21}(v)$;
$v \in \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$

$t_2(v): \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \to \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$, by
$t_2(v) = \partial_{21}^{(2)}(v)$; $v \in \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$

$q_1(v): \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \to \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_4\mathcal{F}$, by
$q_1(v) = \frac{1}{2}\partial_{21}\partial_{32} + \partial_{31}(v); v \in \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$;

$q_2(v): \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \to \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_4\mathcal{F}$, by
$q_2(v) = \partial_{21}(v); v \in \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$
$k_1(v): \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \to \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$, by
$k_1(v) = \partial_{32}(v); v \in \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$

$k_2(v): \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \to \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$, by
$k_2(v) = \frac{1}{2}\partial_{32}\partial_{21} - \partial_{31}$; $v \in \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$, and
$t_3(v): \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \to \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_4\mathcal{F}$, by
$t_3(v) = \partial_{21}(v)$; $v \in \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_4\mathcal{F}$

2.2 The commutative of the diagrams

Proposition (2.2.1):

The diagram A in (2.1) is commutative.

Proof:

We have to prove that $(t_2 \circ k_1)(v) = (\mathrm{k}_1 \circ t_1)(\mathrm{v})$ so

$$(t_2 \circ k_1)(\mathrm{v}) = \partial_{21}^{(2)}\, \partial_{32}(\mathrm{v}) = \partial_{32}\, \partial_{21}^{(2)} - \partial_{21}^{(1)}\, \partial_{31},$$

$$= \left(\tfrac{1}{2}\, \partial_{32}\, \partial_{21}\, \partial_{21} - \partial_{21}\, \partial_{31}\right)(\mathrm{v})$$

But $(\partial_{21}\, \partial_{31})(x) = (\partial_{31}\, \partial_{21})(x)$, then

$$(t_2 \circ k_1)(\mathrm{v}) = \left(\tfrac{1}{2}\, \partial_{32}\, \partial_{21}\, \partial_{21} - \partial_{31}\partial_{21}\right)(\mathrm{v})$$

$$= \left(\tfrac{1}{2}\, \partial_{32}\, \partial_{21} - \partial_{31}\right)\partial_{21}(\mathrm{v}) = (\mathrm{k}_2 \circ t_1)(\mathrm{v})$$ Type equation here.

Proposition (2.2.2):

The diagram B in (2.1) is commutative.

Proof: We have to prove that $(q_2 \circ t_2)(v) = (\mathrm{t}_3 \circ q_1)(\mathrm{v})$ so

$$(q_2 \circ t_2)(v) = \partial_{32}\, \partial_{21}^{(2)}(\mathrm{v}) = \partial_{21}^{(2)}\, \partial_{32} + \partial_{21}^{(1)}\, \partial_{31},$$

$$= \left(\tfrac{1}{2}\, \partial_{21}\, \partial_{21}\, \partial_{32} + \partial_{21}\, \partial_{31}\right)(\mathrm{v})$$

$$(q_2 \circ t_2)(\mathrm{v}) = \partial_{21}\left(\tfrac{1}{2}\, \partial_{32}\, \partial_{21} - \partial_{31}\right)(\mathrm{v}) = (\mathrm{t}_3 \circ q_1)(\mathrm{v})$$

By apply the mapping Cone to the following diagram

$$\begin{array}{ccccc}
0 \longrightarrow & \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} & \xrightarrow{k_1} & \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\
k_1 & \downarrow & A & \downarrow & k_2 \\
0 \longrightarrow & \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} & \xrightarrow{t_2} & \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}
\end{array}$$

Diagram (2.2)

We obtain the subsequence

$$0 \to \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \xrightarrow{\propto_3} \begin{array}{c} \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{array} \xrightarrow{\eta_1} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F}$$

… (2)

Where $\alpha_3(x) = \left(-\partial_{32}\,(x), \partial_{21}\,(x)\right)$ and

$$\eta_1(x_1, x_2) = \partial_{21}^{(2)}(x_1) + \left(\tfrac{1}{2}\, \partial_{32}\, \partial_{21} - \partial_{31}\right)(x_2)$$

Proposition (2.2.3):

$$(\eta_1 \circ \alpha_3)(v) = 0$$

Proof:

$$(\eta_1 \circ \alpha_3)(v) = \eta_1(-\partial_{32}\,(v), \partial_{21}(v))$$

$$= -\partial_{21}^{(2)}(\partial_{32}(v)) + \left(\tfrac{1}{2}\, \partial_{32}\, \partial_{21} - \partial_{31}\right)(\partial_{21}\,(v))$$

$$= -\partial_{21}^{(2)}\partial_{32}(v) + \tfrac{1}{2}\, \partial_{32}\, \partial_{21}\partial_{21}\,(v) - \partial_{21}\,\partial_{31}\,(v)$$

$$= -\partial_{21}^{(2)}\partial_{32}(v) + \partial_{32}\, \partial_{21}^{(2)}(v) - \partial_{21}\,\partial_{31}\,(v)$$

$= -\partial_{21}^{(2)}\partial_{32}(v) + \partial_{21}^{(2)}\partial_{32}(v) + \partial_{21}\partial_{31}(v) - \partial_{21}\partial_{31}(v) = 0$

From above, we obtain the subsequence (2) is complex.

Now, consider the following diagram (2.3) so we have

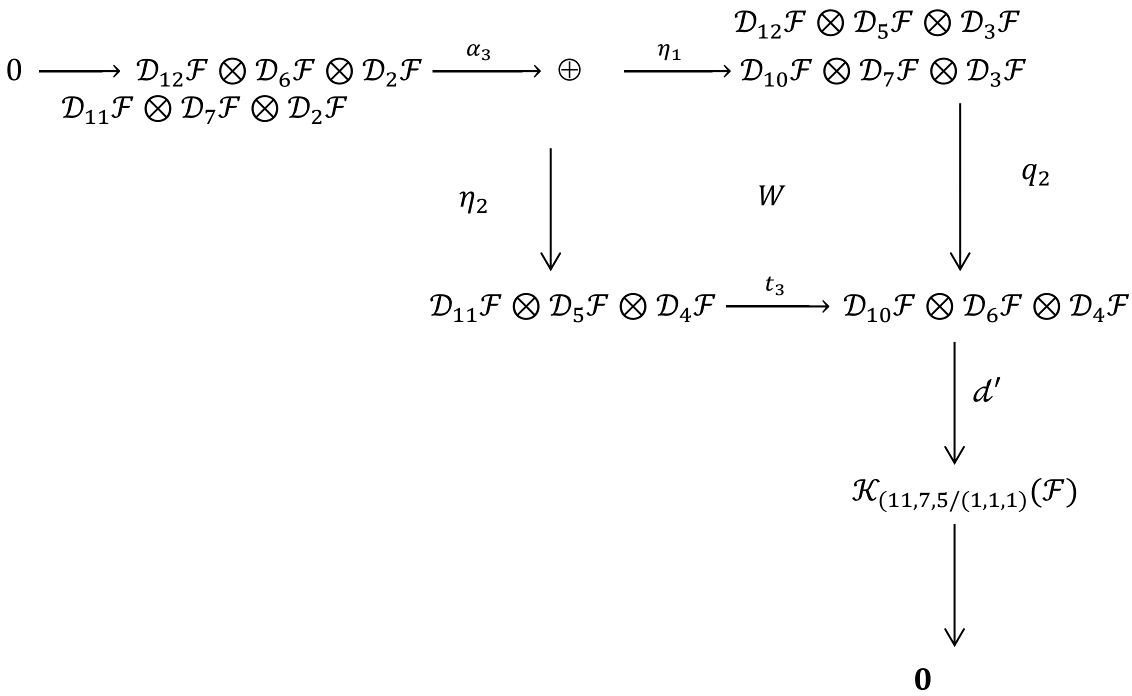

Diagram (2.3)

$$\eta_2: \begin{array}{c} \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{array} \longrightarrow \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \text{ By}$$

$\eta_2(\mathrm{a,b}) = \left(\frac{1}{2}\,\partial_{21}\,\partial_{32} + \partial_{31}\right)(\mathrm{a}) + \partial_{32}^{(2)}(\mathrm{b})$

Proposition (2.2.4):
The diagram W in (2.3) is commutative.

Proof:
We have to prove that $(q_2 \circ \eta_1)(a,b) = (t_3 \circ \eta_2)(a,b)$ so
$(q_2 \circ \eta_1)(a,b) = q_2(\partial_{21}^{(2)}(a) + (\frac{1}{2}\,\partial_{32}\,\partial_{21} - \partial_{31})(b))$
$= (\partial_{32}\partial_{21}^{(2)})(a) + (\partial_{32}^{(2)}\partial_{21} - \partial_{32}\partial_{31})(b)$
$= (\partial_{21}^{(2)}\,\partial_{32} + \partial_{21}\partial_{31})(a) + (\partial_{21}\,\partial_{32}^{(2)} + \partial_{32}\partial_{31} - \partial_{32}\partial_{31})(b)$
$= (\partial_{21}^{(2)}\,\partial_{32} + \partial_{21}\partial_{31})(a) + (\partial_{21}\,\partial_{32}^{(2)})(b)$Where
$(t_3 \circ \eta_2)(a,b) = \partial_{21}\left((\frac{1}{2}\partial_{21}\,\partial_{32} + \partial_{31})(a) + (\partial_{31}^{(2)})(b)\right)$
$= \left(\partial_{21}^{(2)}\,\partial_{32} + \partial_{21}\partial_{31}\right)(a) + \left(\partial_{21}\,\partial_{32}^{(2)}\right)(b),$
This implies that$(q_2 \circ \eta_1)(a,b) = (t_3 \circ \eta_2)(a,b)$, then the diagram w is commute

In this part, we obtain the sequence is a complex

$$0 \to \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \xrightarrow{\alpha_3} \begin{matrix} \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{matrix} \xrightarrow{\alpha_2} \begin{matrix} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \end{matrix}$$
$$\xrightarrow{\alpha_1} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_4\,\mathcal{F},$$

Now, define the map $\alpha_2(a,b)$: $\begin{matrix} \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_2\mathcal{F} \end{matrix} \to \begin{matrix} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \end{matrix}$

$\alpha_2(a,b) = \left(-\eta_1(a,b), \eta_2(a,b)\right)$
$=(-\partial_{21}^{(2)}(a) - \left(\frac{1}{2}\,\partial_{32}\,\partial_{21} - \partial_{31}\right)(b), (\frac{1}{2}\,\partial_{21} \circ \partial_{32} + \partial_{31})(a) + \partial_{32}^{(2)}(b))$

And the map α_1: $\begin{matrix} \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_7\mathcal{F} \otimes \mathcal{D}_3\mathcal{F} \\ \oplus \\ \mathcal{D}_{11}\mathcal{F} \otimes \mathcal{D}_5\mathcal{F} \otimes \mathcal{D}_4\mathcal{F} \end{matrix} \to \mathcal{D}_{10}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_4\,\mathcal{F}$ by
$\alpha_1(a,b) = \partial_{32}(a) + \partial_{21}(b)$

Proposition (2.2.5):
$(\alpha_2 \circ \alpha_3)(a) = 0$

Proof:
$(\alpha_2 \circ \alpha_3)(a) = \alpha_2(-\partial_{32}\,(a), \partial_{21}(a))$, where a $\in \mathcal{D}_{12}\mathcal{F} \otimes \mathcal{D}_6\mathcal{F} \otimes \mathcal{D}_2\mathcal{F}$
$=(-\partial_{21}^{(2)}(-\partial_{32})(a) - \left(\frac{1}{2}\,\partial_{32}\partial_{21} - \partial_{31}\right)\partial_{21}\,(a), \left(\frac{1}{2}\,\partial_{21}\partial_{32} + \partial_{31}\,)(-\partial_{32}\right)(a) +$
$\partial_{32}^{(2)}\partial_{21}(a)$

By using Capelli identities, we have
$\partial_{32}\,\partial_{21}^{(2)} = \partial_{21}^{(2)}\,\partial_{32} + \partial_{21}\,\partial_{31}\,, \partial_{31}\,\partial_{21} = \partial_{21}\,\partial_{31}$
$\partial_{32}^{(2)}\partial_{21} = \partial_{21}\,\partial_{32}^{(2)} + \partial_{32}\partial_{31}$
$=\left(\partial_{21}^{(2)}\partial_{32} - \partial_{32}\,\partial_{21}^{(2)}\right)(a)+(\partial_{21}\,\partial_{31})(a), \left(\partial_{21}\,\partial_{32}^{(2)}\right)(a) - (\partial_{31}\partial_{32})(a) -$
$\left(\partial_{21}\,\partial_{32}^{(2)}\right)(a) + (\partial_{31}\partial_{32})(a)$, which implies that

$=\left(\partial_{21}^{(2)}\partial_{32} - \partial_{21}^{(2)}\,\partial_{32} - \partial_{21}\,\partial_{31}\right)(a)+(\partial_{21}\,\partial_{31})(a), \left(\partial_{21}\,\partial_{32}^{(2)}\right)(a) - (\partial_{31}\partial_{32})(a) -$
$\left(\partial_{21}\,\partial_{32}^{(2)}\right)(a) + (\partial_{31}\partial_{32})(a)$

= (0, 0)

Proposition (2.2.6):
$(\alpha_1 \circ \alpha_2)(a,b) = 0$

Proof:
$(\alpha_1 \circ \alpha_2)(a,b) = \alpha_1(-\partial_{21}^{(2)}(a) - \left(\frac{1}{2}\,\partial_{32}\,\partial_{21} - \partial_{31}\right)(b), \left(\frac{1}{2}\,\partial_{21}\partial_{32} + \partial_{31}\right)(a) +$
$\partial_{32}^{(2)}(b))$

$=(-\partial_{32}\,\partial_{21}^{(2)}(a)-\partial_{32}^{(2)}\partial_{21}(b))+\partial_{32}\,\partial_{31}(b)+\left(\partial_{21}^{(2)}\partial_{32}\right)(a)+(\,\partial_{21}\partial_{31}\,+\partial_{21}\partial_{31}^{(2)})(a))$

By using Capelli identities

$=\left(-\partial_{21}^{(2)}\partial_{32}\right)(a)-(\partial_{21}\partial_{31})(a)-\left(\partial_{21}\partial_{32}{}^{(2)}\right)(b)-(\partial_{32}\partial_{31})(b)+(\partial_{32}\partial_{31})(b)+\left(\partial_{21}^{(2)}\partial_{32})(a)+(\partial_{21}\partial_{31}\,)(a)+(\partial_{21}\partial_{31}^{(2)}\right)(b)=0$

Theorem (2.2.7):

The complex

$$0\to\mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\xrightarrow{\alpha_3}\begin{matrix}\mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\\ \oplus\\ \mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_5\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}\end{matrix}\xrightarrow{\alpha_2}\begin{matrix}\mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_5\mathcal{F}\otimes\mathcal{D}_4\mathcal{F}\\ \oplus\\ \mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}\end{matrix}$$
$$\xrightarrow{\alpha_1}\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_4\,\mathcal{F}$$

Is exact.

Proof: The diagrams, A and B in (2.1) are commutes and the maps.
$t_1(v)\colon\mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\to\mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}$, such that
$t_1(v)=\partial_{21}(v)\quad;v\in\mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}$, and
$t_2(v)\colon\mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_5\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}\to\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}$, such that $t_2(v)=\partial_{21}^{(2)}(v)$;
$v\in\mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_5\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}$;
are injective [7] , and from proposition(2.2.3) $(\eta_1\circ\alpha_3)(v)=0$, then by using the mapping Cone we get the complex

$$0\to\mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\xrightarrow{\alpha_3}\begin{matrix}\mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_5\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}\\ \oplus\\ \mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}\end{matrix}$$
$$\xrightarrow{\eta_1}\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}$$ is

exact. Diagram W in the diagram (2.3) is commute and
$t_3(v)\colon\mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_5\mathcal{F}\otimes\mathcal{D}_4\mathcal{F}\to\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_4\mathcal{F}$, by
$t_3(v)=\partial_{21}(v)\,;v\in\mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_5\mathcal{F}\otimes\mathcal{D}_4\mathcal{F}$

Is injective [7] we have the diagram (2.3) commute with the exact rows. Although $(\alpha_2\circ\alpha_3)(a)=0$ And $(\alpha_1\circ\alpha_2)(a,b)=0$, again by the conditions of mapping Cone, we get the complex

$$\mathbf{0}\to\boldsymbol{\mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}}\xrightarrow{\alpha_3}\begin{matrix}\boldsymbol{\mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_2\mathcal{F}}\\ \oplus\\ \boldsymbol{\mathcal{D}_{12}\mathcal{F}\otimes\mathcal{D}_5\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}}\end{matrix}\xrightarrow{\alpha_2}\begin{matrix}\boldsymbol{\mathcal{D}_{11}\mathcal{F}\otimes\mathcal{D}_5\mathcal{F}\otimes\mathcal{D}_4\mathcal{F}}\\ \oplus\\ \boldsymbol{\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_7\mathcal{F}\otimes\mathcal{D}_3\mathcal{F}}\end{matrix}$$
$$\xrightarrow{\alpha_1}\boldsymbol{\mathcal{D}_{10}\mathcal{F}\otimes\mathcal{D}_6\mathcal{F}\otimes\mathcal{D}_4\,\mathcal{F}}$$

Is exact.

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