Characteristic Zero Resolution (Lascoux Resolution) of Weyl Module in the Case of the Skew-Partition \((11, 7, 5)/ (1, 1, 1)\)

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Abstract:
In this paper, the terms of Lascoux and boundary maps for the skew-partition \((11,7,5)/ (1,1,1)\) are found by using the Jacobi-Trudi matrix of partition. Further, Lascoux resolution is studied by using a mapping Cone without depending on the characteristic-free resolution of the Weyl module for the same skew-partition.

Keywords: Resolution, Lascoux resolution, Weyl module, Characteristic free, mapping Cone, skew-shape.

1. Introduction:
Let \(R\) be a commutative ring with 1 and \(F\) be a free \(R\)-module and \(\mathcal{D}_i F\) be the divided power algebra of degree \(i\).

Authors in [1-3] discussed the complex of characteristic zero for the partitions \((4,4,4),(8,7,3)\) and skew-partition \((8,6,3)/(u,1)\) when \(u=1,2\), respectively. Shaymaa N.A., Haytham R.H. and Nubras in [4,5] exhibited the terms and the exactness of the Weyl resolution in the case of skew-partition \((8,6)/(2,0)\), \((8,6)/(2,1)\) and \((7,7),(7,7)/(1,0)\), respectively. As well Artale [6] discussed the terms for the three-rowed skew-partition and almost skew-shape in Lascoux. In this paper, we find the terms and Lascoux resolution for the skew-partition \((11, 7, 5)/ (1, 1, 1)\) by using the homological diagrams. We also prove

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Consider the homological diagram of $\mathcal{C}$ for the skew-shape by using a mapping Cone.

Authors in [7] defined the Capelli identities as follows:

Let $i, j, k, \ell \in \mathcal{P}^+$, so we have the following:

1. If $k \neq j$, then
   
   \[
   \partial_{ij}^{(r)} = \sum_{\alpha \geq 0} \partial_{ij}^{(s)} = \sum_{\alpha \geq 0} \partial_{jl}^{(s)}
   \]

   \[
   \partial_{ij}^{(s)} = \sum_{\alpha \geq 0} (-1)^{\alpha} \partial_{ij}^{(r)} = \sum_{\alpha \geq 0} (-1)^{\alpha} \partial_{jl}^{(r)}
   \]

2. If $i \neq k$ and $j \neq \ell$, then $\partial_{ik}^{(s)} \partial_{jl}^{(r)} = \partial_{il}^{(r)} \partial_{kj}^{(s)}$

   The author in [8] defined the concept of mapping Cone as follows:

   The commute diagram

   \[
   \begin{array}{c}
   C_0: \quad C_{n-1} & \xrightarrow{d_{n-1}} & C_n & \xrightarrow{d_n} & C_{n+1} & \xrightarrow{d_{n+1}} & C_{n+2} & \cdots \\
   \downarrow & & \downarrow f_n & & \downarrow f_{n+1} & & \downarrow f_{n+2} & \\
   D_0: \quad D_{n-1} & \xrightarrow{d_{n-1}} & D_n & \xrightarrow{d_n} & D_{n+1} & \xrightarrow{d_{n+1}} & D_{n+2} & \cdots
   \end{array}
   \]

   If the sequence of the rows is exact and $\lambda_{n-1}: C_0 \rightarrow C_{n+1}$ definite by $(\alpha, b) \mapsto (-d_0(\alpha), d_{n-1}(b) + f_n(\alpha))$ such that $\lambda_{n-1} \circ \lambda_n = 0$, then $\lambda_n \circ \lambda_{n+1} \circ \lambda_{n+2} \cdots$ is exact.

2. The terms of the sequence of Lascoux in the skew-shape $(11, 7, 5)/ (1, 1, 1)$

   The positions of the terms of the Lascoux are determined by the length of the permutation to which they correspond in [9] and [10].

   In the case of the skew-partition $(11, 7, 5)/ (1, 1, 1)$, we have the pursue matrix:

   \[
   \begin{bmatrix}
   D_{10}F & D_5F & D_2F \\
   D_{11}F & D_6F & D_3F \\
   D_{12}F & D_7F & D_4F
   \end{bmatrix}
   \]

   Then the characteristic zero complexes have the correspondence between their terms as purses:

   \[
   \begin{align*}
   &D_{10}F \otimes D_6F \otimes D_4F \leftrightarrow \text{identity} \\
   &D_{11}F \otimes D_6F \otimes D_4F \leftrightarrow (12) \\
   &D_{10}F \otimes D_7F \otimes D_3F \leftrightarrow (23) \\
   &D_{11}F \otimes D_7F \otimes D_2F \leftrightarrow (132) \\
   &D_{12}F \otimes D_7F \otimes D_3F \leftrightarrow (123) \\
   &D_{12}F \otimes D_6F \otimes D_2F \leftrightarrow (13)
   \end{align*}
   \]

   So the sequence of Lascoux is

   \[
   0 \rightarrow D_{12}F \otimes D_6F \otimes D_2F \rightarrow \oplus \rightarrow D_{10}F \otimes D_7F \otimes D_3F \\
   \rightarrow D_{11}F \otimes D_7F \otimes D_2F \rightarrow D_{11}F \otimes D_5F \otimes D_4F \rightarrow D_{10}F \otimes D_6F \otimes D_4F \cdots \quad (1)
   \]

2.1 The homological diagram of the Lascoux sequence.

   Consider the following diagram
Define the maps as follows:

\[ t_1(v) : \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_2 \rightarrow \mathcal{D}_{11} \otimes \mathcal{D}_7 \otimes \mathcal{D}_2, \quad t_1(v) = \partial_{21}(v); \]
\[ t_2(v) : \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3 \rightarrow \mathcal{D}_{10} \otimes \mathcal{D}_7 \otimes \mathcal{D}_3, \quad t_2(v) = \partial_{21}^{(2)}(v); \]
\[ q_1(v) : \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_3 \rightarrow \mathcal{D}_{11} \otimes \mathcal{D}_5 \otimes \mathcal{D}_4, \quad q_1(v) = \frac{1}{2} \partial_{21} \partial_{32} + \partial_{31}(v); \]
\[ q_2(v) : \mathcal{D}_{10} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3 \rightarrow \mathcal{D}_{10} \otimes \mathcal{D}_6 \otimes \mathcal{D}_4, \quad q_2(v) = \partial_{21}(v); \]
\[ k_1(v) : \mathcal{D}_{12} \otimes \mathcal{D}_6 \otimes \mathcal{D}_2 \rightarrow \mathcal{D}_{12} \otimes \mathcal{D}_5 \otimes \mathcal{D}_3, \quad k_1(v) = \partial_{32}(v); \]
\[ k_2(v) : \mathcal{D}_{11} \otimes \mathcal{D}_5 \otimes \mathcal{D}_2 \rightarrow \mathcal{D}_{10} \otimes \mathcal{D}_7 \otimes \mathcal{D}_3, \quad k_2(v) = \frac{1}{2} \partial_{32} \partial_{21} - \partial_{31}; \]
\[ t_3(v) : \mathcal{D}_{11} \otimes \mathcal{D}_5 \otimes \mathcal{D}_4 \rightarrow \mathcal{D}_{10} \otimes \mathcal{D}_6 \otimes \mathcal{D}_4, \quad t_3(v) = \partial_{21}(v); \]

2.2 The commutative of the diagrams

**Proposition (2.2.1):**

The diagram A in (2.1) is commutative.
Proof:
We have to prove that \((t_2 \circ k_1)(v) = (k_1 \circ t_1)(v)\) so
\[
(t_2 \circ k_1)(v) = \partial_{21}^{(2)} \partial_{32}(v) = \partial_{32} \partial_{21}^{(2)} - \partial_{21}^{(1)} \partial_{31},
\]

\[
= \left(\frac{1}{2} \partial_{32} \partial_{21} - \partial_{21} \partial_{31}\right)(v)
\]

But \((\partial_{21} \partial_{31})(x) = (\partial_{31} \partial_{21})(x)\), then
\[
(t_2 \circ k_1)(v) = \left(\frac{1}{2} \partial_{32} \partial_{21} - \partial_{21} \partial_{31}\right)(v)
\]

Proposition (2.2.2):
The diagram B in (2.1) is commutative.
Proof: We have to prove that \((q_2 \circ t_2)(v) = (t_3 \circ q_1)(v)\) so
\[
(q_2 \circ t_2)(v) = \partial_{32} \partial_{21}^{(2)}(v) = \partial_{21}^{(2)} \partial_{32} + \partial_{21}^{(1)} \partial_{31},
\]

\[
= \left(\frac{1}{2} \partial_{21} \partial_{32} + \partial_{21} \partial_{31}\right)(v)
\]

\[
(q_2 \circ t_2)(v) = \partial_{21}^{(2)} \left(\frac{1}{2} \partial_{32} \partial_{21} - \partial_{21} \partial_{31}\right)(v) = (t_3 \circ q_1)(v)
\]

By apply the mapping Cone to the following diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & D_{12}F \otimes D_6F \otimes D_2F & \rightarrow & D_{12}F \otimes D_5F \otimes D_3F & \rightarrow & 0 \\
\downarrow{k_1} & & \downarrow{A} & & \downarrow{k_2} & \\
0 & \rightarrow & D_{11}F \otimes D_7F \otimes D_2F & \rightarrow & D_{10}F \otimes D_7F \otimes D_3F & \rightarrow & 0
\end{array}
\]

Diagram (2.2)

We obtain the subsequence

\[
0 \rightarrow D_{12}F \otimes D_6F \otimes D_2F \xrightarrow{\alpha_3} D_{12}F \otimes D_5F \otimes D_3F \oplus D_{11}F \otimes D_7F \otimes D_2F \xrightarrow{\eta_1} D_{10}F \otimes D_7F \otimes D_3F \rightarrow \ldots
\]

(2)

Where \(\alpha_3(x) = \left(-\partial_{32}(x), \partial_{21}(x)\right)\) and
\[
\eta_1(x_1, x_2) = \partial_{21}^{(2)}(x_1) + \left(\frac{1}{2} \partial_{32} \partial_{21} - \partial_{31}\right)(x_2)
\]

Proposition (2.2.3):

\((\eta_1 \circ \alpha_3)(v) = 0\)

Proof:
\[
(\eta_1 \circ \alpha_3)(v) = \eta_1(-\partial_{32}(v), \partial_{21}(v))
\]

\[
= -\partial_{21}^{(2)}(\partial_{32}(v)) + \left(\frac{1}{2} \partial_{32} \partial_{21} - \partial_{31}\right)(\partial_{21}(v))
\]

\[
= -\partial_{21}^{(2)}(\partial_{32}(v)) + \partial_{21} \partial_{32} \partial_{21}(v) - \partial_{21} \partial_{31}(v)
\]

\[
= -\partial_{21}^{(2)}(\partial_{32}(v)) + \partial_{32} \partial_{21}^{(2)}(v) - \partial_{21} \partial_{31}(v)
\]

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\[= -\partial_{21}^{(2)} \partial_{32}(v) + \partial_{21}^{(2)} \partial_{32}(v) + \partial_{21} \partial_{31}(v) - \partial_{21} \partial_{31}(v) = 0\]

From above, we obtain the subsequence (2) is complex.

Now, consider the following diagram (2.3) so we have

\[
\begin{array}{c}
0 \longrightarrow \mathcal{D}_{12}F \otimes \mathcal{D}_{6}F \otimes \mathcal{D}_{2}F \\
\mathcal{D}_{11}F \otimes \mathcal{D}_{7}F \otimes \mathcal{D}_{2}F \longrightarrow \mathcal{D}_{12}F \otimes \mathcal{D}_{5}F \otimes \mathcal{D}_{3}F \\
\mathcal{D}_{11}F \otimes \mathcal{D}_{5}F \otimes \mathcal{D}_{4}F \longrightarrow \mathcal{D}_{10}F \otimes \mathcal{D}_{7}F \otimes \mathcal{D}_{3}F \\
\mathcal{D}_{11}F \otimes \mathcal{D}_{5}F \otimes \mathcal{D}_{4}F \longrightarrow \mathcal{D}_{10}F \otimes \mathcal{D}_{6}F \otimes \mathcal{D}_{4}F \\
\mathcal{K}_{(11,7,5/(1,1,1)}(\mathcal{F}) \\
0
\end{array}
\]

**Diagram (2.3)**

\[\eta_2: \mathcal{D}_{11}F \otimes \mathcal{D}_{7}F \otimes \mathcal{D}_{2}F \longrightarrow \mathcal{D}_{12}F \otimes \mathcal{D}_{5}F \otimes \mathcal{D}_{4}F \]

By

\[\eta_2(a, b) = \left(\frac{1}{2} \partial_{21} \partial_{32} + \partial_{31}\right)(a) + \partial_{32}^{(2)}(b)\]

**Proposition (2.2.4):**

The diagram W in (2.3) is commutative.

**Proof:**

We have to prove that \((q_2 \circ \eta_1)(a, b) = (t_3 \circ \eta_2)(a, b)\) so

\[(q_2 \circ \eta_1)(a, b) = q_2(\partial_{21}^{(2)}(a) + \left(\frac{1}{2} \partial_{32} \partial_{21} - \partial_{31}\right)(b))\]

\[= (\partial_{32} \partial_{21}^{(2)})(a) + \left(\partial_{32}^{(2)} \partial_{21} - \partial_{32} \partial_{31}\right)(b)\]

\[= (\partial_{21}^{(2)} \partial_{32} + \partial_{21} \partial_{31})(a) + \left(\partial_{21} \partial_{32}^{(2)} + \partial_{32} \partial_{31} - \partial_{32} \partial_{31}\right)(b)\]

\[= (\partial_{21}^{(2)} \partial_{32} + \partial_{21} \partial_{31})(a) + \left(\partial_{21} \partial_{32}^{(2)}\right)(b)\]

Where

\[(t_3 \circ \eta_2)(a, b) = \partial_{21} \left(\frac{1}{2} \partial_{21} \partial_{32} + \partial_{31}\right)(a) + \left(\partial_{31}^{(2)}\right)(b)\]

\[= (\partial_{21}^{(2)} \partial_{32} + \partial_{21} \partial_{31})(a) + \left(\partial_{21} \partial_{32}^{(2)}\right)(b),\]

This implies that \((q_2 \circ \eta_1)(a, b) = (t_3 \circ \eta_2)(a, b)\), then the diagram w is commute.
In this part, we obtain the sequence is a complex

\[ 0 \to \mathcal{D}_{12} \mathcal{F} \otimes \mathcal{D}_6 \mathcal{F} \otimes \mathcal{D}_2 \mathcal{F} \xrightarrow{\alpha_2} \mathcal{D}_{12} \mathcal{F} \otimes \mathcal{D}_5 \mathcal{F} \otimes \mathcal{D}_3 \mathcal{F} \oplus \mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_7 \mathcal{F} \otimes \mathcal{D}_3 \mathcal{F} \xrightarrow{\alpha_1} \mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_6 \mathcal{F} \otimes \mathcal{D}_4 \mathcal{F}, \]

Now, define the map \( \alpha_2(a, b) = (-\eta_1(a, b), \eta_2(a, b)) \)

\[ = (-\partial^{(2)}_{21}(a) - \left( \frac{1}{2} \partial_{32} \partial_{21} - \partial_{31} \right)(b), \left( \frac{1}{2} \partial_{21} \circ \partial_{32} + \partial_{31} \right)(a) + \partial^{(2)}_{32}(b)) \]

And the map \( \alpha_1: \mathcal{D}_{11} \mathcal{F} \otimes \mathcal{D}_7 \mathcal{F} \otimes \mathcal{D}_2 \mathcal{F} \to \mathcal{D}_{10} \mathcal{F} \otimes \mathcal{D}_6 \mathcal{F} \otimes \mathcal{D}_4 \mathcal{F} \)

\( \alpha_1(a, b) = \partial_{32}(a) + \partial_{21}(b) \)

**Proposition (2.2.5):**

\((\alpha_2 \circ \alpha_3)(a) = 0\)

**Proof:**

\((\alpha_2 \circ \alpha_3)(a) = \alpha_2(-\partial_{32}(a), \partial_{21}(a)), \text{ where } a \in \mathcal{D}_{12} \mathcal{F} \otimes \mathcal{D}_6 \mathcal{F} \otimes \mathcal{D}_2 \mathcal{F} \)

\[ = (-\partial^{(2)}_{21}(-\partial_{32})(a) - \left( \frac{1}{2} \partial_{32} \partial_{21} - \partial_{31} \right) \partial_{21}(a), \left( \frac{1}{2} \partial_{21} \circ \partial_{32} + \partial_{31} \right) (-\partial_{32})(a) + \partial^{(2)}_{32} \partial_{21}(a)) \]

By using Capelli identities, we have

\[
\begin{align*}
\partial^{(2)}_{32} \partial_{21} & = \partial^{(2)}_{21} \partial_{32} + \partial_{21} \partial_{31} \partial_{32} = \partial_{21} \partial_{31} \\
\partial^{(2)}_{32} \partial_{21} & = \partial_{21} \partial^{(2)}_{32} + \partial_{21} \partial_{32} \\
\partial^{(2)}_{32} \partial_{21} & = \partial^{(2)}_{21} \partial_{32} + \partial_{21} \partial_{31} \\
= & \left( \partial^{(2)}_{21} \partial_{32} - \partial^{(2)}_{32} \partial_{21} \right)(a) + (\partial_{21} \partial_{31})(a), \left( \partial^{(2)}_{21} \partial_{32} \right)(a) - (\partial_{31} \partial_{32})(a) - \left( \partial^{(2)}_{21} \partial_{32} \right)(a) + (\partial_{31} \partial_{32})(a) \\
= & (0, 0)
\end{align*}
\]

**Proposition (2.2.6):**

\((\alpha_1 \circ \alpha_2)(a, b) = 0\)

**Proof:**

\[(\alpha_1 \circ \alpha_2)(a, b) = \alpha_1(-\partial^{(2)}_{21}(a) - \left( \frac{1}{2} \partial_{32} \partial_{21} - \partial_{31} \right)(b), \left( \frac{1}{2} \partial_{21} \circ \partial_{32} + \partial_{31} \right)(a) + \partial^{(2)}_{32}(b))\)
By using Capelli identities
\[ (-\partial_{21}^{(2)}\partial_{32})(a) - (\partial_{21}\partial_{31})(a) - (\partial_{21}\partial_{32}^{(2)})(b) - (\partial_{32}\partial_{31})(b) + (\partial_{21}\partial_{31})(b) + (\partial_{21}\partial_{32}^{(2)})(a) = 0 \]

**Theorem (2.2.7):**

The complex
\[ 0 \rightarrow \mathcal{D}_{12}F \otimes \mathcal{D}_6F \otimes \mathcal{D}_2F \xrightarrow{\alpha_3} \mathcal{D}_{11}F \otimes \mathcal{D}_5F \otimes \mathcal{D}_4F \]
\[ \mathcal{D}_{12}F \otimes \mathcal{D}_5F \otimes \mathcal{D}_3F \xrightarrow{\alpha_2} \mathcal{D}_{10}F \otimes \mathcal{D}_7F \otimes \mathcal{D}_3F \]
\[ \mathcal{D}_{10}F \otimes \mathcal{D}_6F \otimes \mathcal{D}_4F \]

is exact.

**Proof:** The diagrams, A and B in (2.1) are commutes and the maps.
\[ t_1(v): \mathcal{D}_{12}F \otimes \mathcal{D}_6F \otimes \mathcal{D}_2F \rightarrow \mathcal{D}_{11}F \otimes \mathcal{D}_7F \otimes \mathcal{D}_2F \]
\[ t_2(v): \mathcal{D}_{12}F \otimes \mathcal{D}_5F \otimes \mathcal{D}_3F \rightarrow \mathcal{D}_{10}F \otimes \mathcal{D}_7F \otimes \mathcal{D}_3F \]

are injective [7], and from proposition (2.2.3) \( \eta_1 \circ \alpha_3(v) = 0 \), then by using the mapping Cone we get the complex

\[ 0 \rightarrow \mathcal{D}_{12}F \otimes \mathcal{D}_6F \otimes \mathcal{D}_2F \xrightarrow{\alpha_3} \mathcal{D}_{12}F \otimes \mathcal{D}_5F \otimes \mathcal{D}_3F \]
\[ \mathcal{D}_{11}F \otimes \mathcal{D}_7F \otimes \mathcal{D}_2F \]
\[ \mathcal{D}_{10}F \otimes \mathcal{D}_7F \otimes \mathcal{D}_3F \]
\[ \mathcal{D}_{10}F \otimes \mathcal{D}_6F \otimes \mathcal{D}_4F \]

exact. Diagram W in the diagram (2.3) is commute and
\[ t_3(v): \mathcal{D}_{11}F \otimes \mathcal{D}_5F \otimes \mathcal{D}_4F \rightarrow \mathcal{D}_{10}F \otimes \mathcal{D}_6F \otimes \mathcal{D}_4F \]
\[ t_3(v) = \partial_{21}(v) ; v \in \mathcal{D}_{11}F \otimes \mathcal{D}_5F \otimes \mathcal{D}_4F \]

Is injective [7] we have the diagram (2.3) commute with the exact rows. Although \( (\alpha_2 \circ \alpha_3)(a) = 0 \) and \( (\alpha_1 \circ \alpha_2)(a, b) = 0 \), again by the conditions of mapping Cone, we get the complex

\[ 0 \rightarrow \mathcal{D}_{12}F \otimes \mathcal{D}_6F \otimes \mathcal{D}_2F \xrightarrow{\alpha_3} \mathcal{D}_{11}F \otimes \mathcal{D}_7F \otimes \mathcal{D}_2F \]
\[ \mathcal{D}_{12}F \otimes \mathcal{D}_5F \otimes \mathcal{D}_3F \]
\[ \mathcal{D}_{10}F \otimes \mathcal{D}_7F \otimes \mathcal{D}_3F \]
\[ \mathcal{D}_{10}F \otimes \mathcal{D}_6F \otimes \mathcal{D}_4F \]

is exact.

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