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Pointwise Estimates for Finding the Error of Best Approximation by Spline, Positive Algebraic Polynomials and Copositive

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Abstract:
The first step in this research is to find some of the necessary estimations in approximation by using certain algebraic polynomials, as well as we use certain specific points in approximation. There are many estimations that help to find the best approximation using algebraic polynomials and geometric polynomials. Throughout this research, we deal with some of these estimations to estimate the best approximation error using algebraic polynomials where the basic estimations in approximation are discussed and proven using algebraic polynomials that are discussed and proven using algebraic polynomials that are specified by the following pointsX_j , $X_j^0$and X_j^1 if $j < \frac{n}{2}$ as well as if $j \geq \frac{n}{2}$.

For the second step of the work, the estimations in the first step are used to find and estimate the error for the best approximation of the weighted function $F \in L_{\psi,p}(I)$. This is done through the use of an algebraic polynomial $P_n \in \prod_n$, whose degree at most is $0 \leq n \leq k-1$ where the sign of the algebraic polynomial is positive.

Further, the error is also found and estimated for the best approximation of the restricted function $F \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_s)$ using the restricted algebraic polynomial$P_n \in \prod_n \cap \Delta^0(\mathbb{J}_s)$, which is copositive with the function $F \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_s)$ in the quasi weighted normed space.

In addition, we deal with the created estimations to estimate the error of the best approximation of the function $F \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_s)$ by using pieces of algebraic polynomials that are of the highest degree$0 \leq n \leq k-1$ .These pieces of algebraic polynomials are connected to each other, so they have formed a spline of the highest degree $0 \leq n \leq k-1$ whose knots are considered the contact areas of the algebraic polynomials.

Keywords: Modulus of Smoothness, Spline, Approximation, Algebraic polynomial, quasi normed space.

تقديرات نقطية لإيجاد خطأ أفضل تقريب بواسطة الشرائح ،ومتعددات الحدود الجبرية الموجبة والمحافظة على الاشارة

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الخلاصة

تناول العمل الأول في هذا البحث الى كيفية إيجاد بعض منالتخمينات الضرورية في نظرية التقريب, وذلك

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باستخدام متعددات الحدود الجبرية المذكورة في المقدمة وباستخدام بعض من النقاط المحددة. كما هو معروف في نظرية التقريب ، هنالك العديد من التخمينات التي تساعد في إيجادالخطأ لأفضل تقدير تقريبي باستخدام متعددات الحدود الجبرية ومتعددات الحدود الهندسية ، لذلك تعاملنا في هذا البحث مع بعض من هذه التخميناتلحاجتنا اليها لتقدير أفضل خطأ تقريبي باستخدام متعددات الحدود الجبرية ، حيث تمت مناقشة التخمينات الأساسية في التقريب وإثباتها باستخدام متعددات الحدود الجبرية من خلال تحديد ما يلي من النقاط X_j^0 , X_jوالنقطة X_j^1 اذا كانت$j < \frac{n}{2}$ وإذا كانت $j \geq \frac{n}{2}$.

المرحلة الثانية من العمل في هذا البحث ، تطرقنا فيه إلى كيفية استخدام هذه التخمينات وإضافتها لإيجاد الخطأ لأفضل تقدير تقريبي للدالة الترجيحية الموزونة$F \in L_{\psi,p}(I)$,حيث تم ذلك من خلال استخدام متعددات الحدود الجبرية$P_n \in \prod_n$ ، والتي تكون درجتها على الأكثر هي $0 \leq n \leq k-1$حيث تكون متعددات الحدود الجبرية موجبة.

أيضًا ، تم ايجاد الخطأ لأفضل تقدير تقريبيللدالة المقيدة $F \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_s)$ ، باستخدام متعددات الحدود الجبرية المقيدة $P_n \in \prod_n \cap \Delta^0(\mathbb{J}_s)$,

التي درجتها على الاكثر هي $0 \leq n \leq k-1$ والتي تكون محافظة على الاشارة مع الدالة المقيدة $F \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_s)$ في الفضاء شبه المعياري الموزون.

كما تطرقنا في هذا البحث الى استخدام وتوظيف التخمينات التي اوجدناها لإيجاد الخطأ لأفضل تقدير تقريبي للدالة المقيدة $F \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_s)$وذلك باستخدام قطع من متعددات الحدود الجبرية المقيدة $P_n \in \prod_n \cap \Delta^0(\mathbb{J}_s)$ التي درجتها على الاكثر $0 \leq n \leq k-1$ والتي تكون متصلة مع بعضها البعض لذلك كونت شريحة درجتها على الاكثر $0 \leq n \leq k-1$ والتي اعتبرت عقدها هي مناطق اتصال متعددات الحدود الجبرية .

1. Introduction and Main Results

Recall that the weighted quasi normed linear space$L_{\psi,p}(I)$ such that $0 < p < 1$ is the set of all functionsF on an interval $I \subset R$, $I = [-b, b]$ that is defined as follows [1]:

$$L_{\psi,p}(I) = \left\{F|F: I \subset R \to R: \left(\int_{-b}^{b} \left|\frac{F(x)}{\psi(x)}\right|^p dx\right)^{\frac{1}{p}} < \infty , 0 < p < 1\right\},$$

and

$$\|F\|_{L_{\psi,p}(I)} = \left(\int_I \left|\frac{F(x)}{\psi(x)}\right|^p dx\right)^{\frac{1}{p}} , x \in [-b, b] .$$

Also, the kth symmetric difference is given by

$$\Delta_h^k(F, x, I)_\psi = \Delta_h^k(F, x)_\psi = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \frac{F(x - \frac{kh}{2} + ih)}{\psi(x + \frac{kh}{2})} , & x \pm \frac{kh}{2} \in \mathbb{J} \\ 0 , & o.w \end{cases}$$

Which is non-negative for a given $k \in N$ for all $x \in I$ .Let Δ^k be the set of functions F for which the kth difference is non-negative on the interval $I = [-b, b]$. Let $\prod_n$, $0 \leq n \leq k-1$ be the set of algebraic polynomials of degree that is not exceeding n. The polynomials and splines that are used in this work are different in form and according to the degree of what we want to achieve in the proof.

The error positive approximation by algebraic polynomial [2] is defined by

$$\mathbb{E}_n^{0)}(F)_{\psi,p} = \inf\|F - P_n\|, P_n \in \prod_n.$$

The Ditzian-Totik modulus of smoothness of $F \in L_{\psi,p}(I)$, $0 < p < 1$ which is defined as follows:

$$\mathcal{W}_k^{\varphi}(F,n^{-1})_{\psi,p} = \sup\left\|\Delta_{h\varphi(.)}^{k}(F,.)\right\|_{L_{\psi,p}(I)}, 0 < h \le \delta, I = [-b,b]$$

where φ be a function of $x \in I$, [2]. We show that the error positive approximation by algebraic polynomial $P_n \in \prod_n, 0 \le n \le k-1$ when $F \in L_{\psi,p}(I)$.

Theorem 1.1. If $F \in L_{\psi,p}(I), 0 < p < 1, k > 1, \mathfrak{m} \ge 2$ and $F(x) \ge 0, x \in I$, then

$$\mathbb{E}_n^{0)}(F)_{\psi,p} \le c_1 \mathfrak{m}^{-C_2} \mathcal{W}_k^{\varphi}(F,n^{-1})_{\psi,p} \quad, 0 \le n \le k-1,$$

where $c_1 = c(p,k,\mathfrak{m},b)$ and $c_2 = k - 2\mathfrak{m} - 1 + \frac{1}{p} \ge 0$.

In the next theorem, we show that if $F \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_s)$, change sign in$(-b,b)$, then we get the error of best copositive approximation by algebraic polynomial $P_n \in \prod_n \cap \Delta^0(\mathbb{J}_s)$, and spline $\mathcal{S}_n \in \mathbb{S}_n, 0 \le n \le k-1$.

Theorem 1.2 If $F \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_s), 0 < p < 1, k > 1$, where $\mathbb{J}_s = \{j_1, \dots, j_s\}$, then

$$\mathbb{E}_n^{0)}(F,\mathbb{J}_s)_{\psi,p} \le c\, \mathcal{W}_k^{\varphi}(F,n^{-1})_{\psi,p} \quad, 0 \le n \le k-1.$$

Where $c = A(p,k,b,\mathfrak{m})$ and $A = \left(\frac{|x-j_r|}{1+|x-j_r|}\right)^{\mathfrak{m}} c(p,k), \mathfrak{m} \ge 2, r = 0, \dots s$.

2. Definitions and Notations

In this paper, we use the following notations [1] , [3]

$\Delta_n = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$, $x \in I$ and Chebyshev notation

$X_j = a\cos\frac{j\pi}{n}, 0 \le j \le n$ and let $X_j^1 = a\cos(\frac{j\pi}{n} - \frac{\pi}{2n}), 1 \le j \le n$

$X_j^0 = a\cos(\frac{j\pi}{n} - \frac{\pi}{4n})$ when $j < \frac{n}{2}$, we also use $X_j^0 = a\cos(\frac{j\pi}{n} - \frac{3\pi}{4n})$ when $j \ge \frac{n}{2}$

$I_j = [X_{j+1}, X_j]$ and $h_j = |I_j| = X_j - X_{j+1}$, $0 \le j \le n$.

Let $t_j(x) = (X - X_j^0)^{-2}\cos^2 2n \arccos x + (X - X_j^1)^{-2}\sin^2 2n \arccos x$ be the algebraic polynomial of degree which is not exceeding $4n-2$ [4], [5].

Let $F \in L_{\psi,p}(I)$ such that F changes its sign infinitely many times say $s \ge 1$ in I at the points $j_r \in \mathbb{J}_s, 0 \le s+1$, and we will write $F \in \Delta^0(\mathbb{J}_s)$.In this paper, we approximate F by algebraic polynomial $P_n \in \Pi_n, 0 \le n \le K_1$, in one time and in a second time by a spline on the knots $\{X_j\}_{j=0}^{n}$.

$\mathbb{J}_s = \{j_1, \dots, j_s \mid -b = j_0 < j_1 < \dots < j_s < j_{s+1} = b\}$ for $\mathbb{J}_s$we set

$\sum_{(X,\mathbb{J}_s)} = \sum_{r=1}^{s}(X - j_r)$ and $\sum_{(X,\mathbb{J}_s)}^{(t)} = \sum_{r=1}^{s}(X - j_r)^t$ we denote $\sum_{(X,\mathbb{J}_s)} = \sum_{(X)}$.

Let $\prod_j(a,d,\mathfrak{m}) = \int_{-b}^{b}(y - X_{j+1})^a (X_j - y)^d t_j^{\mathfrak{m}}(y)\,dy$

Finally we put T_j, $Q_j$and $\overline{Q}_j$ are algebraic polynomials which define by

$T_j(x) = \frac{\int_{-b}^{x} t_j^{\mathfrak{m}}(y)\Sigma_{(y)}dy}{\int_{-b}^{b} t_j^{\mathfrak{m}}(y)\Sigma_{(y)}dy}$, note that $T_j(x) = \begin{cases} 0 & ; x = -b \\ 1 & ; x = b \end{cases}$ see that

$$\acute{T}_j(x)\textstyle\sum_{(X)}\sum_{(X_{j+1})} \ge 0, \qquad x \in I$$

$Q_j(x) = \frac{\int_{-b}^{x}(y - X_{j+1}) t_j^{\mathfrak{m}}(y)\Sigma_{(y)}dy}{\int_{-b}^{b}(y - X_{j+1}) t_j^{\mathfrak{m}}(y)\Sigma_{(y)}dy}$ note that $Q_j(x) = \begin{cases} 0 & ; x = -b \\ 1 & ; x = b \end{cases}$

And

$\overline{Q}_j(x) = \frac{\int_{-b}^{x}(X_j - y) t_j^{\mathfrak{m}}(y)\Sigma_{(y)}dy}{\int_{-b}^{b}(X_j - y) t_j^{\mathfrak{m}}(y)\Sigma_{(y)}dy}$ note that $\overline{Q}_j(x) = \begin{cases} 0 & ; x = -b \\ 1 & ; x = b \end{cases}$

The denominators of their polynomials are never zeroes and

$$\Psi_j = \frac{h_j}{|X - X_j| + h_j}, \qquad \text{and } \chi_j(x) = \begin{cases} 0 & ; x < X_j \\ 1 & ; x \ge X_j \end{cases}$$

In fact for all $r = 1, \ldots, \mathfrak{s}$ and $x \in I$.In this work, we take $X_j < X_{j+1}$, hence $X_{j+1} < j_r$. Since $x \in I$,therefore the inequality $|x - X_{j+1}| \le |x - j_r|$ holds.

3. Estimates by algebraic polynomials

There are many estimations that we can list in the research. However, we will limit ourselves to some conjectures that will be used to prove some of the facts that we discussed in the introduction. Therefore, in this section, we introduce the following inequalities which satisfy:

a) $\prod_j(a, d, m) \approx h_j^{2a+2d-2m+1}$,

b) $|\acute{T}_j(x)| \le c\Psi_j^{2m}$,

c) $|\acute{Q}_j(x)| \le c\Psi_j^{2m-1}h_j^{2m}$,

d) $|\acute{\bar{Q}}_j(x)| \le c\Psi_j^{2m-1}h_j^{2m}$,

e) $|\chi_j(x) - T_j(x)| \le \frac{c}{2m-1}\Psi_j^{2m-1}(x)h_j$ if $\chi_j(x) = 0$ and $|\chi_j(x) - T_j(x)| \le 1 + \frac{c}{2m-1}h_j\Psi_j^{2m-1}(x)$ if $\chi_j(x) = 1$.

f) $|\chi_j(x) - Q_j(x)| \le \frac{c}{2m-2}\Psi_j^{2m-2}(x)h_j^{2m+1}$ if $\chi_j(x) = 0$ and $|\chi_j(x) - Q_j(x)| \le 1 + \frac{c}{2m-2}h_j^{2m+1}\Psi_j^{2m-2}(x)$ if $\chi_j(x) = 1$.

g) $|\bar{Q}_j(x) - \chi_j(x)| \le \frac{c}{2m-2}\Psi_j^{2m-2}h_j^{2m+2}$ if $\chi_j(x) = 0$ and $|\bar{Q}_j(x) - \chi_j(x)| \le 1 + \frac{c}{2m-2}h_j^{2m+2}\Psi_j^{2m-2}(x)$ if $\chi_j(x) = 1$.

4. Proof the Estimates for Algebraic Polynomial

a) To proof the estimation $\prod_j(a, d, m) \approx h_j^{2a+2d-2m+1}$.

Write $\prod_j(a, d, m)$ as follows

$$\prod_j(a, d, m) = \left(\int_{-b}^{X_{j+1}} + \int_{X_{j+1}}^{X_j} + \int_{X_j}^{b}\right)(y - X_{j+1})^a (X_j - y)^d t_j^m(y) \sum_{(y)}^{(a+d)} dy$$

$$= I_1 + I_2 + I_3$$

$$I_1 = \int_{-b}^{X_{j+1}} (y - X_{j+1})^a t_j^m(y) \sum_{(y)}^{(a+d)} (X_j - y)^d dy$$

$$|I_1| \le \int_{-b}^{X_{j+1}} (X_{j+1} - y)^a t_j^m(y) \sum_{r=1}^{\mathfrak{s}} (y - y_r)^{(a+d)} (X_j - y)^d dy$$

$$\le \sum_{r=1}^{\mathfrak{s}} \int_{-b}^{X_{j+1}} (X_{j+1} - y)^a t_j^m(y)\, (j_r - y)^{(a+d)} (X_j - y)^d dy$$

$$\le \sum_{r=1}^{\mathfrak{s}} \int_{-b}^{X_{j+1}} (X_{j+1} - y)^a\, (j_r - y)^{(a+d)} (X_j - y)^d \mathrm{Max}\left\{(y - X_j^0)^{-2m}, (y - \bar{X}_j)^{-2m}\right\} dy$$

Since $X_j^0 > \bar{X}_j$, then

$$|I_1| \le \int_{-\infty}^{X_{j+1}} (X_j^0 - y)^a (X_j^0 - y)^{(a+d)} (X_j - y)^d dy$$

By using the inequalities in [6]

$h_j < 2(X_j - X_j^1) < 4(X_j^0 - X_j)$, for $h_j = X_j - X_{j+1}$, $h_j = X_j^0 - X_{j+1}$, hence

$$|I_1| \leq \int_{-\infty}^{X_{j+1}} \left(X_j^0 - y\right)^a \left(X_j^0 - y\right)^{(a+d)} \left(X_j^0 - y\right)^d \left(X_j^0 - y\right)^{-2m} dy$$

$$= \int_{-\infty}^{X_{j+1}} 4^d \left(X_j^0 - y\right)^{2a+2d-2m} dy$$

$$= \frac{4^d}{2m - 2a - 2d - 1} \left(X_j^0 - X_{j+1}\right)^{2a+2d-2m+1}$$

Since $\left(X_j^0 - X_{j+1}\right)^{-1} < 4h_j^{-1}$, we get

$$I_1 \leq \frac{4^{2m-2a-d-1}}{2m - 2a - 2d - 1} h_j^{\,2a+2d-2m+1}$$

$$I_2 = \int_{X_{j+1}}^{X_j} \left(y - X_{j+1}\right)^a \left(X_j - y\right)^d \sum\nolimits_{(y)}^{(a+d)} t_j^m(y) dy$$

By using the relationship $h_j(x) < 10^3 h_j^{-2}$, we get

$$\leq \sum_{r=1}^{s} \int_{X_{j+1}}^{X_j} 10^{3m} h_j^{-2m} \left(X_{j+1} - y\right)^d \left(y - X_{j+1}\right)^a (y - j_r)^{a+d} dy$$

$$|I_2| \leq \int_{X_{j+1}}^{X_j} 10^{3m} h_j^{-2m} \left(X_j - y\right)^{2d+2a} dy$$

$= \frac{10^{3m} h_j^{-2m}}{2a+2d+1} (X_j - X_{j+1})^{2a+2d+1}$, $h_j = X_j - X_{j+1}$. Hence,

$$I_2 \leq \frac{10^{3m}}{2a + 2d + 1} h_j^{\,2a+2d-2m+1}$$

$$I_3 = \int_{X_j}^{b} \left(y - X_{j+1}\right)^a \left(X_j - y\right)^d \sum\nolimits_{(y)}^{(a+d)} t_j^m(y) dy$$

And by the same method in I_1 we found I_3 with only difference that instead of $\left(y - X_j^0\right)$ by $\left(y - X_j^1\right)$

$$I_3 \leq \sum_{r=1}^{s} \int_{X_j}^{b} \left(y - X_{j+1}\right)^a \left(y - X_j\right)^d (y - j_r)^{a+d} \operatorname{Max}\left\{\left(y - X_j^0\right)^{-2m}, \left(y - X_j^1\right)^{-2m}\right\} dy$$

$$|I_3| \leq \sum_{r=1}^{s} \int_{X_j}^{b} \left(y - X_{j+1}\right)^a \left(y - X_j^1\right)^d \left(y - X_j^1\right)^{a+d} \left(y - X_j^1\right)^{-2m} dy$$

$$\leq \int_{X_j}^{\infty} 4^a \left(y - X_j^1\right)^{2a+2d-2m} dy$$

Since $\left(y - X_j^1\right)^{-1} \leq 4h_j^{-1}$, then

$$I_3 \leq \frac{4^a}{2m - 2a - 2d - 1} \left(X_j - X_j^1\right)^{2a+2d-2m+1}$$

$I_3 \leq \frac{4^{2m-a-2d-1}}{2m-2a-2d-1} h_j^{\,2a+2d-2m+1}$. Therefore,

$\prod_j(a, d, m) \leq \left(\frac{4^{2m-2a-d-1}}{2m-2a-2d-1} + \frac{10^{3m}}{2a+2d+1} + \frac{4^{2m-a-2d-1}}{2m-2a-2d-1}\right) h_j^{\,2a+2d-2m+1}$. Hence,

i) $\prod_j(a, d, m) \leq c_1(a, d, m) h_j^{\,2a+2d-2m+1}$,

where

$$c_1(a, d, m) = \frac{4^{2m-2a-d-1}}{2m - 2a - 2d - 1} + \frac{10^{3m}}{2a + 2d + 1} + \frac{4^{2m-a-2d-1}}{2m - 2a - 2d - 1}$$

Now by finding the estimates on the other side.

$$I_1 \geq \sum_{r=1}^{\mathfrak{s}} \int_{-b}^{X_{j+1}} \left(X_{j+1} - y\right)^a (j_r - y)\left(X_j - y\right)^d \min\left\{\left(y - X_j^0\right)^{-2m}, \left(y - X_j^1\right)^{-2m}\right\} dy$$

$$I_1 \geq \sum_{r=1}^{\mathfrak{s}} \int_{-\infty}^{X_{j+1}} \left(X_{j+1} - y\right)^a \left(X_j^1 - y\right)^d \left(X_j - y\right)^{a+d} \left(X_j^1 - y\right)^{-2m} dy$$

$$\geq \int_{-\infty}^{X_{j+1}} 4^a \left(X_j^1 - y\right)^{2a+2d-2m} dy$$

With $(X_j^1 - X_{j+1}) > \frac{1}{4} h_j$

$$I_1 \geq \frac{4^{2m-2a-d-1}}{2m - 2a - 2d - 1} h_j^{2a+2d-2m+1}$$

$$I_2 \geq \sum_{r=1}^{\mathfrak{s}} \int_{X_{j+1}}^{X_j} \left(y - X_{j+1}\right)^a \left(X_j - y\right)^d (y - j_r)^{a+d} \min\left\{\left(y - X_j^0\right)^{-2m}, \left(y - X_j^1\right)^{-2m}\right\} dy$$

$X_j - y > j_r - y$

$$I_2 \geq \sum_{r=1}^{\mathfrak{s}} \int_{X_{j+1}}^{X_j} \left(y - X_{j+1}\right)^a (j_r - y)^d (y - j_r)^{a+d} \left(y - X_j^1\right)^{-2m} dy$$

And $j_r - y \geq X_{j+1} - y$

$$I_2 \geq \sum_{r=1}^{\mathfrak{s}} \int_{X_{j+1}}^{X_j} \left(y - X_{j+1}\right)^a \left(X_{j+1} - y\right)^d (y - j_r)^{a+d} \left(y - X_j^1\right)^{-2m} dy$$

$$I_2 \geq \sum_{r=1}^{\mathfrak{s}} \int_{X_{j+1}}^{X_j} \left(y - X_{j+1}\right)^{a+d} (y - j_r)^{a+d} \left(y - X_j^1\right)^{-2m} dy$$

We have $y - X_{j+1} > y - j_r$

$$I_2 \geq \sum_{r=1}^{\mathfrak{s}} \int_{X_{j+1}}^{X_j} (y - j_r)^{2a+2d} \left(y - X_j^1\right)^{-2m} dy$$

$y - j_r > y - X_j^1$

$I_2 \geq \sum_{r=1}^{\mathfrak{s}} \left(y - X_j^1\right)^{2a+2d-2m} dy$ .Hence,

$$I_2 \geq \frac{2}{(2a + 2d - 2m + 1)4^{2a+2d-2m+1}} h_j^{2a+2d-2m+1}$$

$$I_3 \geq \sum_{r=1}^{\mathfrak{s}} \int_{X_j}^{b} \left(y - X_{j+1}\right)^a \left(y - X_j\right)^d (y - j_r)^{a+d} \min\left\{\left(y - X_j^0\right)^{-2m}, \left(y - X_j^1\right)^{-2m}\right\} dy$$

$$I_3 \geq \int_{X_j}^{b} \left(y - X_j^1\right)^a \left(y - X_j\right)^d \left(y - X_j^1\right)^{a+d} \left(y - X_j^1\right)^{-2m} dy$$

$$\geq \int_{X_j}^{b} 4^d \left(y - X_j^1\right)^{2a+2d-2m} dy$$

Since $X_j - X_j^1 > h_j$, then

$$I_3 \geq \frac{4^{2m-2a-d-1}}{2m - 2a - 2d - 1} h_j^{2a+2d-2m+1}$$

So we get

ii) $\prod_j(a, d, m) \geq c_2(a, d, m) h_j^{2a+2d-2m+1}$,

where

$$c_2(a,d,m) = \frac{4^{2m-a-2d-1}+4^{2m-2a-d-1}}{2m-2a-2d-1} + \frac{2}{(2a+2d-2m+1)4^{2a+2d-2m+1}}$$

From (i) and (ii) we get
$c_2(a,d,m)h_j^{2a+2d-2m+1} \le \prod_j(a,d,m) \le c_1(a,d,m)h_j^{2a+2d-2m+1}$. Hence, we get the first estimation

$$\prod_j(a,d,m) \approx h_j^{2a+2d-2m+1}$$

b) To proof the estimation $\left|\acute{T}_j(x)\right| \le c\Psi_j^{2m}$.

We have $\acute{T}_j(x) = \frac{\frac{d}{dx}\int_{-b}^{x} t_j^{m}(y)\Sigma_{(y)}dy}{\int_{-b}^{b} t_j^{m}(y)\Sigma_{(y)}dy} = \frac{\frac{d}{dx}\int_{-b}^{x} t_j^{m}(y)\sum_{r=1}^{s}(y-j_r)dy}{\int_{-b}^{b} t_j^{m}(y)\sum_{r=1}^{s}(y-j_r)\,dy}$.

Take the numerator

$$\int_{-b}^{x} t_j^{m}(y)\sum_{r=1}^{s}(y-j_r)\,dy = \int_{-b}^{x} t_j^{m}(y)[(y-j_1)+\cdots+(y-j_s)]\,dy$$

So that we get

$\int_{-b}^{x} t_j^{m}(y)\sum_{r=1}^{s}(y-j_r)\,dy = \sum_{r=1}^{s}(x-j_r)t_j^{m}(x)$, since $c_0h_j^{1-2b} < \frac{d_j}{\Pi(X_j)}$.

Hence, the numerator is

$$\int_{-b}^{x} t_j^{m}(y)\sum_{r=1}^{s}(y-j_r)\,dy = \sum_{(y)} t_j^{m}(x)$$

Take the place

$$\int_{-b}^{b} t_j^{m}(y)\sum_{r=1}^{s}(y-j_r)\,dy \le 10^{3m}h_j^{-2m}\int_{-b}^{b}\sum_{r=1}^{s}(y-j_r)\,dy$$

$$= \sum_{r=1}^{s}\int_{-b}^{b}(y-j_r)dy$$

$= -b\sum_{r=1}^{s} j_r$ hence $\acute{T}_j(x) \le \frac{\Sigma_{(y)} t_j^{m}(x)}{-b\sum_{r=1}^{s} j_r}$

$$\acute{T}_j(x) \le \frac{\Sigma_{(x)}}{\Sigma_{(0)}} t_j^{m}(x)$$

$\left|\acute{T}_j(x)\right| \le c\left|\frac{\Sigma_{(x)}}{\Sigma_{(0)}}\right| t_j^{m}(x)$, for all $r = 1,\ldots,s$

$$\left|\frac{x-j_r}{0-j_r}\right| \le \frac{|x-j_r|}{\left|X_j-X_{j+1}\right|} \le 4\left(\frac{\left|x-X_{j+1}\right|+h_j}{h_j}\right)$$

$$\left|\acute{T}_j(x)\right| \le 4cs\left(\frac{\left|x-X_{j+1}\right|+h_j}{h_j}\right)t_j^{m}(x)$$

By using the inequality in [3]
$t_j(x) = \text{Max}\left\{\left(x-X_j^0\right)^{-2m},\left(x-X_j^1\right)^{-2m}\right\}$, for all $x \in I$. Hence,

$$\left|\acute{T}_j(x)\right| \le c(b,m,s)\left(\frac{\left|x-X_{j+1}\right|+h_j}{h_j}\right)\text{Max}\left\{\left(x-X_j^0\right)^{-2m},\left(x-X_j^1\right)^{-2m}\right\}$$

$\le c\left(\frac{|x-X_{j+1}|+h_j}{h_j}\right)^{1-2m} = c\Psi_j^{2m-1}$. Hence, we get the second estimation

$$\left|\acute{T}_j(x)\right| \le c\Psi_j^{2m-1} \;, m > 0\;, c = c(b,m,s)\;.$$

C) To proof the estimate $\left|\acute{Q}_j(x)\right| \le c\Psi_j^{2m-1}h_j^{2m}$

We have $\acute{Q}_j(x) = \frac{\frac{d}{dx}\int_{-b}^{x}(y-X_{j+1})t_j^{m}(y)\Sigma_{(y)}dy}{\int_{-b}^{b}(y-X_{j+1})t_j^{m}(y)\Sigma_{(y)}dy}$, by take the numerator

$$\frac{d}{dx}\int_{-b}^{x}(y-X_{j+1})t_j^{m}(y)\Sigma_{(y)}dy=\frac{d}{dx}\int_{-b}^{x}\sum_{r=1}^{\mathfrak{s}}(y-X_{j+1})t_j^{m}(y)(y-j_r)dy$$

Perforating integration and abbreviations, we get

$$\frac{d}{dx}\int_{-b}^{x}(y-X_{j+1})t_j^{m}(y)\Sigma_{(y)}dy = \Sigma_{(x)}(x-X_{j+1})t_j^{m}(x)$$

Take the place

$\int_{-b}^{b}(y-X_{j+1})t_j^{m}(y)\Sigma_{(y)}dy$, by the inequality

$$t_j(x)\le 10^3 h_j^{-2}, x\in I$$

$$\sum_{r=1}^{\mathfrak{s}}\int_{-b}^{b}(y-X_{j+1})(y-j_r)t_j^{m}(y)\,dy\le\sum_{r=1}^{\mathfrak{s}}\int_{-b}^{b}(y-X_{j+1})(y-j_r)\,10^{3m}h_j^{-2m}dy$$

$$=\sum_{r=1}^{\mathfrak{s}}10^{3m}h_j^{-2m}\left(\frac{2}{3}b^3\right)$$

$$=c(b,m)h_j^{-2m}$$

$$\acute{Q}_j(x)\le\frac{(x-X_{j+1})t_j^{m}(x)\Sigma_{(x)}}{c(b,m)h_j^{-2m}}$$

$$\left|\acute{Q}_j(x)\right|\le c\left|(x-X_{j+1})\right||(x-j_r)|h_j^{2m}\,t_j^{m}(x)$$

Since $\left|x-X_{j+1}\right|\le|x-j_r|$, $r=1,\dots,\mathfrak{s}$. Hence,

$$\left|(x-X_{j+1})(x-j_r)\right|\le|x-j_r|$$

$$\le 4\left(\frac{\left|x-X_{j+1}\right|+h_j}{h_j}\right)$$

$$\left|\acute{Q}_j(x)\right|\le 4c\mathfrak{s}h_j^{2m}\,t_j^{m}(x)\left(\frac{\left|x-X_{j+1}\right|+h_j}{h_j}\right)$$

$$\le ch_j^{2m}\left(\frac{|x-X_{j+1}|+h_j}{h_j}\right)\mathrm{Max}\left\{(x-X_j^0)^{-2m},(x-X_j^1)^{-2m}\right\}$$

$$\le ch_j^{2m}\left(\frac{|x-X_{j+1}|+h_j}{h_j}\right)^{1-2m}$$. Hence, we get the third estimation

$$\left|\acute{Q}_j(x)\right|\le ch_j^{2m}\Psi_j^{2m-1}\ ,m>0$$

d) To proof the estimate $\left|\acute{\bar{Q}}_j(x)\right|\le c\Psi_j^{2m-1}h_j^{2m}$

We have $\acute{\bar{Q}}_j(x)=\frac{\int_{-b}^{x}(X_j-y)t_j^{m}(y)\Sigma_{(y)}dy}{\int_{-b}^{b}(X_j-y)t_j^{m}(y)\Sigma_{(y)}dy}$, by taking the numerator

$$\frac{d}{dx}\int_{-b}^{x}\sum_{r=1}^{\mathfrak{s}}(X_j-y)\,(y-j_r)t_j^{m}(y)dy=\sum_{r=1}^{\mathfrak{s}}\int_{-b}^{x}(X_j-y)(y-j_r)t_j^{m}(y)dy$$

And perforating integration and abbreviations, we get

$$\frac{d}{dx}\int_{-b}^{x}(X_j-y)t_j^{m}(y)\Sigma_{(y)}dy = \sum_{r=1}^{\mathfrak{s}}(X_j-x)(x-j_r)t_j^{m}(x)$$

$$=(X_j-x)t_j^{m}(x)\Sigma_{(x)}$$

Take the place

$\int_{-b}^{b}(X_j-y)t_j^{m}(y)(y-j_r)dy$, by the inequality $t_j(x)\le 10^3h_j^{-2}, x\in I$

$$\int_{-b}^{b}(X_j - y)t_j^{m}(y)(y - j_r)dy \le \int_{-b}^{b}(X_j - y)(y - j_r)10^{3m}h_j^{-2m}\,dy$$
$$= -(2)10^{3m}h_j^{-2m}\left(bj_rX_j + \tfrac{1}{3}b^3\right) \quad ,$$

hence

$$\acute{\bar{Q}}_j(x) \le \frac{(X_j - x)t_j^{m}(x)\sum_{(x)}}{(2)10^{3m}h_j^{-2m}\left(-bj_rX_j - \frac{1}{3}b^3\right)}$$
$$\left|\acute{\bar{Q}}_j(x)\right| \le \frac{|X_j - x||x - j_r|}{\left|bj_rX_j + \frac{1}{3}b^2\right|}t_j^{m}(x)h_j^{2m}$$
$$\le c|x - j_r|h_j^{2m}\mathrm{Max}\left\{\left(x - X_j^0\right)^{-2m}, \left(x - X_j^1\right)^{-2m}\right\}$$
$$\le 4c\mathfrak{s}\left(\frac{|x - X_{j+1}| + h_j}{h_j}\right)^{1-2m} h_j^{2m}$$

$= ch_j^{2m}\Psi_j^{2m-1}$. Hence, we get the third estimates

$$\left|\acute{\bar{Q}}_j(x)\right| \le c(m, b, \mathfrak{s})h_j^{2m}\Psi_j^{2m-1} \quad , m > 0$$

e) To proof the estimate $\left|\chi_j(x) - T_j(x)\right| \le \frac{c}{2m-1}\Psi_j^{2m-1}h_j$

We have $\left|\chi_j(x) - T_j(x)\right| = \left|T_j(x)\right|$, if $\chi_j(x) = 0$

$$\left|\chi_j(x) - T_j(x)\right| = \left|\int_{-b}^{x}\acute{T}_j(y)dy\right|$$
$$\le \int_{-b}^{x}\left|\acute{T}_j(y)\right|dy$$, from estimate (b), we have

$$\left|\acute{T}_j(x)\right| = \frac{c\,h_j^{\,2m}}{\left(|x - X_{j+1}| + h_j\right)^{2m}}$$

$\le c\int_{-b}^{x}\left|\frac{c\,h_j^{\,2m}}{\left(|y - X_{j+1}| + h_j\right)^{2m}}\right|dy$ if $y \ge X_j$ then

$$\left|\chi_j(x) - T_j(x)\right| \le c\int_{x}^{\infty}\left|\frac{c\,h_j^{\,2m}}{\left(y - X_{j+1} + h_j\right)^{2m}}\right|dy$$
$$= c\int_{x}^{\infty}h_j^{\,2m}\left(y - X_{j+1} + h_j\right)^{-2m}dy$$

Performing integration and abbreviations, we get

$\left|x - X_j\right| + h_j = h_j\Psi_j^{-1}(x)$. So we get

$$\left|\chi_j(x) - T_j(x)\right| \le \frac{c}{2m - 1}h_j\Psi_j^{2m-1}(x)$$

If $y < X_j$ then by the same way in the above, we get the same result.

Now, if $\chi_j(x) = 1$ and $y \ge X_j$, then

$$\left|\chi_j(x) - T_j(x)\right| = \left|1 - \int_{x}^{b}\acute{T}_j(x)dy\right|$$
$$\le 1 + \left|\int_{x}^{b}\acute{T}_j(y)\right|dy$$
$$\le 1 + c\int_{x}^{\infty}\frac{c\,h_j^{\,2m}}{\left(|y - X_{j+1}| + h_j\right)^{2m}}dy$$

$$\leq 1 + c{h_j}^{2m} \int_x^{\infty} \left(y - X_{j+1} + h_j\right)^{-2m} dy$$

Performing integration and abbreviations, we get

$$\left|\chi_j(x) - T_j(x)\right| \leq 1 + \frac{c}{2m-1}\left(\frac{h_j}{\left|x - X_j\right| + h_j}\right)^{2m} h_j \Psi_j^{-1}(x)$$

$= 1 + \frac{c}{2m-1} h_j \Psi_j^{2m-1}(x)$. If $y < X_j$, then by the same way in the above, we get the same result.

f)) In order to prove this estimation, we use the same method as for the proof of estimate (e) , in the case of $\chi_j(x) = 0$, we take $y \geq X_j$ and $y < X_j$ then we get the result $\frac{c}{2m-2} h_j^{2m+1} \Psi_j^{2m-2}(x)$ by using estimate (c), and in the case $\chi_j(x) = 1$, we take $y \geq X_j$ and $y < X_j$ then we get the result $1 + \frac{c}{2m-2} h_j^{2m+1} \Psi_j^{2m-2}(x)$.

g)) In order to prove this estimation, we use the same method as for the prove of estimate (f) by taking estimation (d), we get the result we take $y \geq X_j$ and $y < X_j$ then we get the result $\frac{c}{2m-2} h_j^{2m+2} \Psi_j^{2m-2}(x)$, in case $\chi_j(x) = 0$ .In case $\chi_j(x) = 1$ we get $1 + \frac{c}{2m-2} h_j^{2m+2} \Psi_j^{2m-2}(x)$.

5. Auxiliary Results

To prove the main results in this paper, we need the following lemmas.

Lemma 5.1. For every $n \in N$, $j = 0, \dots, n$ and $m \geq 2$, there exists a polynomial $T_j(x)$ of degree $\leq n$, $0 < n \leq k - 1$, such that for $x \in I = [-b, b]$, the following inequalities satisfy:

1) $\left|\chi_j(x) - T_j(x)\right| \leq \frac{c}{2m-1} h_j \Psi_j^{2m-1}(x)$, for $x \geq X_j$ and $x < X_j$

2) $\left|\chi_j(x) - T_j(x)\right| \leq 1 + \frac{c}{2m-1} h_j \Psi_j^{2m-1}(x)$, for $x \geq X_j$ and $x < X_j$

Where

$$\chi_j(x) = \begin{cases} 0 & ; x < X_j \\ 1 & ; x \geq X_j \end{cases} , \qquad x \in I , j = 0, \dots, n$$

By the section (3), we get the above result.

Lemma 5.2. If $F \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_s)$, $0 < p < 1$ and $k \in N$, then we have

$$\left(\sum_{j=0}^{n} \mathcal{W}_k^{\varphi}(F, h_j, I_j^*)_{\psi,p}^{p}\right)^{\frac{1}{p}} \leq c(p,k) \mathcal{W}_k^{\varphi}(F, n^{-1})_{\psi,p}$$

Where $I_j \subset I_j^* = \left[X_{j-1}, X_{j+1}\right]$.

Proof: If $r \geq 0$ and $X_{j-1} + r + 2k\delta \leq X_{j+1}$ for $1 \leq j \leq n$.

Let $0 < \delta \leq \frac{X_{j+1} - X_{j-1}}{k}$, $k \geq 1$ then

$$\sup_{0<h\leq\delta} \int_{X_{j-1}}^{X_{j-1}+r} \left|\Delta_{\varphi(x)h}^{k} \frac{f(x)}{\psi(x)}\right|^{p} dx \leq \frac{c}{\delta} \int_0^{\delta} \int_{X_{j-1}}^{X_{j+1}} \left|\Delta_{\varphi(x)h}^{k} \frac{f(x)}{\psi(x)}\right|^{p} dxdt \qquad \dots (1$$

Where c= c(k, p) , by using the identity

$\Delta_{\varphi(x)h}^{k} \frac{f(x)}{\psi(x)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} \left\{\Delta_{\frac{i(\varphi t - \varphi h)}{k}}^{k} \frac{F(x+ih\varphi)}{\psi(x+ih\varphi)} - \Delta_{\varphi h + \frac{i(\varphi t - \varphi h)}{k}}^{k} \frac{F(x)}{\psi(x)}\right\}$,

we have

$$\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\Delta^{k}_{\varphi h+\frac{r(\varphi t-\varphi h)}{k}}\frac{F(x)}{\psi(x)}$$

$$=\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\frac{F\left(x+i\left(\frac{kh\varphi+r(\varphi t-\varphi h)}{k}\right)\right)}{\psi\left(x+i\left(\frac{kh\varphi+r(\varphi t-\varphi h)}{k}\right)\right)}$$

$$=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\frac{F\left(x+ih\varphi+\frac{ri(\varphi t-\varphi h)}{k}\right)}{\psi\left(x+i\,h\varphi+\frac{ri(\varphi t-\varphi h)}{k}\right)}$$

$$\left|\Delta^{k}_{\varphi(x)h}\frac{f(x)}{\psi(x)}\right|^{p}\le c(p,k)\sum_{i=1}^{k}\left(\left|\Delta^{k}_{\frac{i(\varphi t-\varphi h)}{k}}\frac{F(x+ih\varphi)}{\psi(x+ih\varphi)}\right|^{p}+\left|\Delta^{k}_{\varphi h+\frac{i(\varphi t-\varphi h)}{k}}\frac{F(x)}{\psi(x)}\right|^{p}\right)$$

By integrating to $t\in[0,\delta]$ and dividing by δ, we get

$$\left|\Delta^{k}_{\varphi(x)h}\frac{f(x)}{\psi(x)}\right|^{p}\le\frac{c(p,k)}{\delta}\sum_{i=1}^{k}\left\{\int_{t=0}^{\delta}\left|\Delta^{k}_{\frac{i(\varphi t-\varphi h)}{k}}\frac{F(x+ih\varphi)}{\psi(x+ih\varphi)}\right|^{p}dt+\int_{0}^{\delta}\left|\Delta^{k}_{\varphi h+\frac{i(\varphi t-\varphi h)}{k}}\frac{F(x)}{\psi(x)}\right|^{p}dt\right\}$$

$$x\in[X_{j-1},X_{j-1}+r],\ h\in[0,\delta]$$

Now, by integrating on $x\in[X_{j-1},X_{j-1}+r]$, and for $h\in[0,\delta]$, we get

$\int_{X_{j-1}}^{X_{j-1}+r}\left|\Delta^{k}_{\varphi(x)h}\frac{f(x)}{\psi(x)}\right|^{p}dx\le$

$\frac{c(p,k)}{\delta}\sum_{i=1}^{k}\left\{\int_{t=0}^{\delta}\int_{X_{j-1}}^{X_{j-1}+r}\left|\Delta^{k}_{\frac{i(\varphi t-\varphi h)}{k}}\frac{F(x+ih\varphi)}{\psi(x+ih\varphi)}\right|^{p}dxdt+\int_{0}^{\delta}\int_{X_{j-1}}^{X_{j-1}+r}\left|\Delta^{k}_{\varphi h+\frac{i(\varphi t-\varphi h)}{k}}\frac{F(x)}{\psi(x)}\right|^{p}dxdt\right\}$

… (2

We shall estimate each term in the sum on the right side of (2).Substituting $x+ih\varphi=u_1$ and $\frac{i(\varphi t-\varphi h)}{k}=u_2$, we get

$$I_1=\int_{t=0}^{\delta}\int_{X_{j-1}}^{X_{j-1}+r}\left|\Delta^{k}_{\frac{i(\varphi t-\varphi h)}{k}}\frac{F(x+ih\varphi)}{\psi(x+ih\varphi)}\right|^{p}dxdt$$

$$=\int_{\frac{-i\varphi h}{k}}^{\delta-\frac{i\varphi h}{k}}\int_{X_{j-1}+ih\varphi}^{X_{j-1}+r+ih\varphi}\left|\Delta^{k}_{u_2}\frac{F(u_1)}{\psi(u_1)}\right|^{p}du_1du_2$$

$$\le\int_{\frac{-i\varphi h}{k}}^{\frac{i\varphi(\delta-h)}{k}}\int_{X_{j-1}+ih\varphi}^{X_{j-1}+r+ih\varphi}\left|\Delta^{k}_{u_2}\frac{F(u_1)}{\psi(u_1)}\right|^{p}du_1du_2$$

Since $\Delta^{k}_{u_2}\frac{F(u_1)}{\psi(u_1)}=(-1)^{k}\Delta^{k}_{-u_2}\frac{F(u_1+ku_2)}{\psi(u_1+ku_2)}$ and $0<\frac{i\varphi}{k}h\le\frac{i\varphi}{k}\delta$ then by substitution $u_1+ku_2=x$, we get

$$\int_{\frac{-i\varphi h}{k}}^{0}\int_{X_{j-1}+ih\varphi}^{X_{j-1}+r+ih\varphi}\left|\Delta^{k}_{u_2}\frac{F(u_1)}{\psi(u_1)}\right|^{p}du_1du_2=\int_{0}^{\frac{i\varphi h}{k}}\int_{X_{j-1}+ih\varphi}^{X_{j-1}+r+ih\varphi}\left|\Delta^{k}_{u_2}\frac{F(x-ku_2)}{\psi(x-ku_2)}\right|^{p}dxdu_2$$

$$=\int_{0}^{\frac{i\varphi h}{k}}\int_{X_{j-1}+ih\varphi-ku_2}^{X_{j-1}+r+ih\varphi-ku_2}\left|\Delta^{k}_{u_2}\frac{F(x)}{\psi(x)}\right|^{p}dxdu_2$$

For $ku_2=i\varphi t-i\varphi h$, $i\varphi h\le k\delta$ and $X_{j-1}+r+2k\delta\le X_{j+1}$ we get

$$I_1\le\int_{0}^{\delta}\int_{X_{j-1}+k\delta-ku_2}^{X_{j-1}-it\varphi}\left|\Delta^{k}_{\varphi t}\frac{F(x)}{\psi(x)}\right|^{p}dxdt$$

$$\leq \int_0^{\delta}\int_{X_{j-1}}^{X_{j-1}-it\varphi}\left|\Delta_{\varphi t}^{k}\frac{F(x)}{\psi(x)}\right|^{p} dxdt$$

Now, let

$$I_2 = \int_0^{\delta}\int_{X_{j-1}}^{X_{j-1}+r}\left|\Delta^{k}_{\varphi h+\frac{i(\varphi t-\varphi h)}{k}}\frac{F(x)}{\psi(x)}\right|^{p} dxdt$$

The above integrals are estimate by the same way .So it can be estimate (1) by

$$\sup_{0<h\leq\delta}\int_{X_{j-1}}^{X_{j-1}+r}\left|\Delta_{\varphi(x)h}^{k}\frac{f(x)}{\psi(x)}\right|^{p} dx \leq \frac{c}{\delta}\int_0^{\delta}\int_{X_{j-1}}^{X_{j+1}-kt\varphi}\left|\Delta_{\varphi(x)t}^{k}\frac{f(x)}{\psi(x)}\right|^{p} dxdt$$

By using integral moduli of smoothness for with $F \in L_{\psi,p}(I)\cap\Delta^{0}(\mathbb{J}_{s}), 0 < p < 1$ we obtain

$$\mathcal{W}_k^{\varphi}(F,h_j,I_j^*)_{\psi,p}^{p} \leq c^p n^p \int_0^{n^{-1}}\int_{X_{j-1}}^{X_{j+1}}\left|\Delta_{\varphi(x)t}^{k}\frac{f(x)}{\psi(x)}\right|^{p} dxdt$$

For $j = 0, \dots, n$, we have $I_j^* = [X_{j-1}, X_{j+1}] \approx [X_{j-1}, X_{j+1} - kt\varphi]$

As $k > 1$ and $\delta \to 0$, we have regular integral, hence

$$\mathcal{W}_k^{\varphi}(F,h_j,I_j^*)_{\psi,p}^{p} \leq c^p n^p \int_0^{n^{-1}}\int_{X_{j-1}}^{X_{j+1}-kt\varphi}\left|\Delta_{\varphi(x)t}^{k}\frac{f(x)}{\psi(x)}\right|^{p} dxdt$$

Finally , for $x \in I_1^* \cup I_2^* \cup \dots \cup I_n^*$ we get

$$\sum_{j=0}^{n}\mathcal{W}_k^{\varphi}(F,h_j,I_j^*)_{\psi,p}^{p} \leq c^p n^p \int_0^{n^{-1}}\int_{-b}^{b}\left|\Delta_{\varphi(x)t}^{k}\frac{f(x)}{\psi(x)}\right|^{p} dxdt$$

$$\leq c^p n^p \int_0^{n^{-1}}\left\|\Delta_{\varphi t}^{k}(f,.)\right\|_{L_{\psi,p}(I)}^{p} dt$$

$$\leq c^p t^p n^p \mathcal{W}_k^{\varphi}(F,n^{-1})_{\psi,p}^{p}\Big|_0^{n^{-1}}$$

$$= c^p \mathcal{W}_k^{\varphi}(F,n^{-1})_{\psi,p}^{p} \text{ , hence}$$

$$\left(\sum_{j=0}^{n}\mathcal{W}_k^{\varphi}(F,h_j,I_j^*)_{\psi,p}^{p}\right)^{\frac{1}{p}} \leq c\mathcal{W}_k^{\varphi}(F,n^{-1})_{\psi,p}.$$

Where c depends on p and k .

Lemma 5.3. Let $r \in N$ and S_r be a piecewise spline of the degree r , $0 < r \leq k-1$ on knots sequence $\{X_j\}_{j=0}^{n}$, then for every interval $I_j^* \subset [-b, b]$ we have the inequality

$$\left|\frac{S_r(X_{j+1})}{\psi(X_{j+1})} - \frac{S_r(X_{j-1})}{\psi(X_{j-1})}\right| \leq c(p,k)\left(\left|I_j^*\right|^{-(k-1)-\frac{1}{p}} + \left|I_j^*\right|^{-2k+3-\frac{2}{p}}\right)\mathcal{W}_k^{\varphi}(S_r,\left|I_j^*\right|,I_j^*)_{\psi,p}$$

$$\leq c(p,k)\left|I_j^*\right|^{-(k-1)-\frac{1}{p}}\mathcal{W}_k^{\varphi}(S_r,\left|I_j^*\right|,I_j^*)_{\psi,p}$$

Proof: By using [1], there is a polynomial $P_n \in \prod_n \cap \Delta^0(\mathbb{J}_s)$, $0 \leq n \leq k-1$ such that for a spline S_r of an order $< k$ implies

$$\|S_r - P_n\|_{L_{\psi,p}(I_j^*)} \leq c(p,k)\mathcal{W}_k^{\varphi}(S_r,\left|I_j^*\right|,I_j^*)_{\psi,p} , 0 < p < 1 \quad \dots (3$$

By using [7] and by the following inequality for $r = 0$ and an interval I_j^*

$\|P_n\|_{L_{\psi,p}(I_j^*)} \leq c(p,k)\left|I_j^*\right|^{-(k-1)-\frac{1}{p}}\|P_n\|_{L_{\psi,p}(I_j^*)}$. Then, we get

$$\left|\frac{P_r(X_j)}{\psi(X_j)}\right| \le c\left|I_j^*\right|^{-(k-1)-\frac{1}{p}}\|P_n\|_{L_{\psi,p}(I_j^*)}$$

Let $X_j \in I_j^*$, since for every $0 \le j \le n$, the inequality $\left|I_j^*\right|^{(k-1)+\frac{1}{p}} \le \left|I_j\right| = h_j$ holds, the interval I_j^* does not contain more than

$1 + \frac{|X_{j+1}-X_{j-1}|}{\left|I_j^*\right|^{(k-1)+\frac{1}{p}}}$, intervals I_j , then we have

$$\left|\frac{S_r(X_j^+)}{\psi(X_j^+)} - \frac{P_n(X_j)}{\psi(X_j)}\right| \le c\left|I_j^*\right|^{-(k-1)-\frac{1}{p}}\|S_r - P_n\|_{L_{\psi,p}(I_j^*)}$$

$X_j \in (-b, b)$ we also have

$\left|\frac{S_r(X_j^-)}{\psi(X_j^-)} - \frac{P_n(X_j)}{\psi(X_j)}\right| \le c\left|I_j^*\right|^{-(k-1)-\frac{1}{p}}\|S_r - P_n\|_{L_{\psi,p}(I_j^*)}$. Hence,

$$\left|\frac{S_r(X_j^+)}{\psi(X_j^+)} - \frac{S_r(X_j^-)}{\psi(X_j^-)}\right| \le \left|\frac{S_r(X_j^+)}{\psi(X_j^+)} - \frac{P_n(X_j)}{\psi(X_j)}\right| + \left|\frac{S_r(X_j^-)}{\psi(X_j^-)} - \frac{P_n(X_j)}{\psi(X_j)}\right|$$
$$\le c\left|I_j^*\right|^{-(k-1)-\frac{1}{p}}\|S_r - P_n\|_{L_{\psi,p}(I_j^*)}$$

By inequality (3), we get

$\left|\frac{S_r(X_j^+)}{\psi(X_j^+)} - \frac{S_r(X_j^-)}{\psi(X_j^-)}\right| \le c\left|I_j^*\right|^{-(k-1)-\frac{1}{p}}\mathcal{W}_k^{\varphi}(S_r, \left|I_j^*\right|, I_j^*)_{\psi,p}$. Therefore,

$$\left|\frac{S_r(X_{j+1})}{\psi(X_{j+1})} - \frac{S_r(X_{j-1})}{\psi(X_{j+1})}\right| \le \sum_{j=0}^{n}\left|\frac{S_r(X_j^+)}{\psi(X_j^+)} - \frac{S_r(X_j^-)}{\psi(X_j^-)}\right|$$
$$\le c(p,k)\Big(1 + \frac{|X_{j+1}-X_{j-1}|}{\left|I_j^*\right|^{(k-1)+\frac{1}{p}}}\Big)\left|I_j^*\right|^{-(k-1)-\frac{1}{p}}\mathcal{W}_k^{\varphi}(S_r, \left|I_j^*\right|, I_j^*)_{\psi,p}$$
$$\le c(p,k)\left|I_j^*\right|^{-(k-1)-\frac{1}{p}}\mathcal{W}_k^{\varphi}(S_r, \left|I_j^*\right|, I_j^*)_{\psi,p}$$

Lemma 5.4 For every $j_r \in \mathbb{J}_s = \{j_1, \dots, j_s\}$ and $m \ge 2$, there exists a polynomial $P_n(j_r, x) \in \prod_n \cap \Delta^0(\mathbb{J}_s)$, such that $0 \le n \le k-1$, copositive with $sgn(x - j_r)$ in $I = [-b, b]$, such that

$$|sgn(x - j_r) - P_n(j_r, x)| \le c\left(\frac{I_j^*}{|j_r - x| + I_j^*}\right)^m \quad \dots (4$$

Proof: The inequality (4) is proven in [3] for $m = 2$ with $\Delta_n(y_k)$ instead of $\delta_n(y_k, x)$, we have the inequality (4) , $\Delta_n(j_r) = \Delta_n(y_k), 0 \le n \le k-1$, and $\delta_n(y_k, x) = I_j^*$,also in [8] is proved for $m \ge 2$ and the proof is similar for $m = 2$. The following inequalities hold for all $x, y \in I$, [9], [10] , hence in $I = [-b, b]$ and $0 \le j \le n$, the following holds.

$$(\Delta_n(y_k))^2 < 4\Delta_n(x)\big(|x - y| + \Delta_n(x)\big)$$
$$\frac{1}{2}\big(|x - y| + \Delta_n(x)\big) < |x - y| + \Delta_n(y) < 2\big(|x - y| + \Delta_n(x)\big)$$

For any $x, y \in I$, we have

$$\frac{\Delta_n(j_r)}{|j_r - x| + \Delta_n(j_r)} \le \frac{2\sqrt{\Delta_n(x)\big(|j_r - x| + \Delta_n(x)\big)}}{\frac{1}{2}\big(|j_r - x| + \Delta_n(x)\big)}$$
$$= 4\sqrt{\frac{\Delta_n(x)}{|j_r - x| + \Delta_n(x)}}$$

For $P_n(j_r, x)$, we have

$$|sgn(x-\mathrm{j}_r)-P_n(\mathrm{j}_r,x)|\le c\left(\frac{\Delta_n(\mathrm{j}_r)}{|\mathrm{j}_r-x|+\Delta_n(\mathrm{j}_r)}\right)^{2\mathfrak{m}}$$

Hence

$$|sgn(x-\mathrm{j}_r)-P_n(\mathrm{j}_r,x)|\le c\left(\frac{I_\mathrm{j}^*}{|\mathrm{j}_r-x|+I_\mathrm{j}^*}\right)^{\mathfrak{m}}$$

$\mathfrak{m}\ge 2\,, r=0,\dots,n$.

Lemma 5.5. If $\mathrm{F}\in L_{\psi,p}(I)\cap\Delta^0(\mathbb{J}_\mathfrak{s}), 0<p<1$, then there is copositive a piecewise spline S_{k-1} with F on the knots sequence $\{t_r\}_{r=1}^{n}$ that satisfies:
$\|f-S_{k-1}\|_{L_{\psi,p}(I)}\le c(p,k)\,\mathcal{W}_k^{\varphi}(\mathrm{F},n^{-1})_{\psi,p}$, if $t_r=X_{n-r}\,, 0\le r\le n$.

Proof: We will call the interval $I_r=[t_{r+1},t_{r-1}]$ is defiled if $t_r<\mathrm{j}_i\le t_{r+1}$ for $\mathrm{j}_i\in\mathbb{J}_\mathfrak{s}\,, 1\le i\le\mathfrak{s}$.
There is exactly one j_i in each of the defiled intervals I_i^* , $1\le i\le\mathfrak{s}$, and there is at least one non-defiled interval I_r between I_i^* and I_{i+1}^* ,$0\le i\le\mathfrak{s}$.
Note that, F does not change sign between $I_i^*$and I_{i+1}^* .Define a linear operator [3] with a weight $\psi(x)$ by

$$\mathbb{L}\mathrm{F}=\sum_{i=1}^{\mathfrak{s}}\mathrm{C}_i\mathbb{N}_i\quad,\quad \mathrm{C}_i=\mathrm{C}_i(\mathrm{F})=\mathbb{d}_i^{-1}\int_{I_r}\frac{\mathrm{F}(x)}{\psi(x)}dx$$

Where $\mathbb{d}_i$ is an absolute constant and $\mathbb{N}_i(x)=\mathbb{N}_i(x,t_r,t_{r+1})$ is the B-spline. Consider that $\mathbb{L}$ preserves linearity, hence by [1] , we have

$$|\mathrm{C}_i|\le\mathbb{d}_i^{-\frac{1}{p}}\|\mathrm{F}\|_{L_{\psi,p}(I_r)}$$

We define

$$S_{k-1}=\begin{cases}0 & [t_{i-1}^*\,,t_{i+1}^*)\\ \mathbb{L}\mathrm{F}\quad, & [t_{r-1}^*\,,t_{r+1}^*)=\mathfrak{T}_r^*\ ,r\ne i\ for\ 1\le i\le\mathfrak{s}\end{cases}$$

Let ℓ_{k-1} be copositive with F in an interval I_r and the best approximation on I_r , $0\le r\le n-1$, by [1] we have

$$\|\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(I_r)}\le c(p,k)\,\mathcal{W}_k^{\varphi}(\mathrm{F},|I_r|,I_r)_{\psi,p}\qquad\dots(5$$

Now

$$\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I_i^*)}=\|\mathrm{F}-\mathbb{L}\mathrm{F}\,\|_{L_{\psi,p}(I_i^*)}\quad\le c\|\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(I_i^*)}+c\|\mathbb{L}\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(I_i^*)}$$
$$=c\|\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(I_i^*)}+c\|\mathbb{L}(\mathrm{F}-\ell_{k-1})\ \|_{L_{\psi,p}(I_i^*)}\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I_i^*)}\le c\|\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(I_i^*)}$$
$$\le c(p,k)\|\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(\mathfrak{T}_r^*)}$$

Where $\mathfrak{T}_r^*=[t_{r-1}^*\,,t_{r+1}^*)$ and $I_r\subset\mathfrak{T}_r^*$, by inequality (5), we get

$$\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I_i^*)}\le c(p,k)\mathcal{W}_k^{\varphi}(\mathrm{F},|\mathfrak{T}_r^*|,\mathfrak{T}_r^*)_{\psi,p}\qquad\dots(6$$

$$\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I_r)}\le c\|\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(I_r)}+c\|\ell_{k-1}-S_{k-1}\|_{L_{\psi,p}(I_r)}$$

Where ℓ_{k-1} is the best $L_{\psi,p}$ −approximation of F on I_i^* , hence S_{k-1} is the best approximation of ℓ_{k-1} on I_i^* for $I_i^*\subset I_r$, hence

$$\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I_r)}\le c\|\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(I_i^*)}+c\|\ell_{k-1}-\mathrm{S}_{k-1}\|_{L_{\psi,p}(I_i^*)}$$
$$\le c\|\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(I_i^*)}+c\|\mathrm{F}-\ell_{k-1}\|_{L_{\psi,p}(I_i^*)}+c\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I_i^*)}$$

By inequality (5) and (6), we get

$$\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I_r)}\le c(p,k)\mathcal{W}_k^{\varphi}(\mathrm{F},|I_r|,I_r)_{\psi,p}+c(p,k)\mathcal{W}_k^{\varphi}(\mathrm{F},|\mathfrak{T}_r^*|,\mathfrak{T}_r^*)_{\psi,p}$$

For $\mathfrak{T}_r^*\subset\mathfrak{T}_r$, we get

$$\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I_r)} \le c(p,k)\mathcal{W}_k^{\varphi}(\mathrm{F},|\mathfrak{T}_r|,\mathfrak{T}_r)_{\psi,p} \qquad \dots (7$$

That is mean S_{k-1} copositive with F in $\mathfrak{T}_r$.
There are at least two X_{j} in each $(\mathrm{j}_i,\mathrm{j}_{i+1})$, for all i , by the inequality (7) and lemma (4.2), we get

$$\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I)} = \sum_{r=0}^{n-1}\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I_r)}$$

$\le c\sum_{r=0}^{n-1}\mathcal{W}_k^{\varphi}(\mathrm{F},|\mathfrak{T}_r|,\mathfrak{T}_r)_{\psi,p}$. Hence, we get

$$\|\mathrm{F}-S_{k-1}\|_{L_{\psi,p}(I)} \le \mathcal{W}_k^{\varphi}(\mathrm{F},n^{-1})_{\psi,p}$$

Where c depends on p and k .

6. Positive Approximation by Algebraic Polynomial

Proof of Theorem 1.1.

We shall prove Theorem (1.1) by a piecewise function P_1 it follows from [9] that $P_1$is bounded by F .Let

$$P_1(\mathrm{F}\,,x) = P_n(-b) + \sum_{\mathrm{j}=0}^{n-1}\left(P_{\mathrm{j}+1}-P_{\mathrm{j}+2}\right)\chi_{\mathrm{j}}(x)$$

Where P_n is constant function and $P_{\mathrm{j}+1}$ is the best $L_{\psi,p}$ −approximation of F on I_{j} , $\mathrm{j} = 0,\dots,n-1$. Therefore, it is clear that $\chi_{\mathrm{j}}(x) = 0$, when $I_{\mathrm{j}} = \left[X_{\mathrm{j}+1},\mathrm{X}_{\mathrm{j}}\right], \mathrm{j} = 0,\dots,n$ that means $P_1(\mathrm{F}\,,x) = P_{\mathrm{j}+1}(-b)$ on I_{j} .
Since
$P_{\mathrm{j}+1}(-b)\mathrm{F}(x) \ge 0$ then $P_1(\mathrm{F}\,,x)\mathrm{F}(x) \ge 0$ and since $\mathrm{F}(x) \ge 0$, hence $P_1(\mathrm{F}\,,x) \ge 0$, $x \in I$ .By [11], we have

$$\left\|\mathrm{F}-P_{\mathrm{j}+1}\right\|_{L_{\psi,p}(I_{\mathrm{j}})} \le c(p)\mathcal{W}_k(\mathrm{F},\left|I_{\mathrm{j}}\right|,I_{\mathrm{j}})_p$$

For $L_p \subset L_{\psi,p}$, $0 < p < 1$, hence
$\left\|\mathrm{F}-P_{\mathrm{j}+1}\right\|_{L_{\psi,p}(I_{\mathrm{j}})} \le c(p)\mathcal{W}_k^{\varphi}(\mathrm{F},\left|I_{\mathrm{j}}\right|,I_{\mathrm{j}})_{\psi,p}$, $\mathrm{j} = 0,\dots,n-1$.
By Lemma (4.2), we have

$$\|\mathrm{F}-P_1(\mathrm{F})\|_{L_{\psi,p}(I_{\mathrm{j}})}^p = \sum_{\mathrm{j}=0}^{n-1}\int_{I_{\mathrm{j}}}\left|\frac{\mathrm{F}(x)-P_{\mathrm{j}+1}(x)}{\psi(x)}\right|^p dx$$

$$\le c^p\sum_{\mathrm{j}=0}^{n-1}\mathcal{W}_k^{\varphi}(\mathrm{F},\left|I_{\mathrm{j}}\right|,I_{\mathrm{j}})_{\psi,p}^p$$

$$\le c^p\mathcal{W}_k^{\varphi}(\mathrm{F},n^{-1})_{\psi,p}^p$$

$$\|\mathrm{F}-P_1(\mathrm{F})\|_{L_{\psi,p}(I)} \le c(p,k)\mathcal{W}_k^{\varphi}(\mathrm{F},n^{-1})_{\psi,p} \qquad \dots (8$$

Now, we define

$$P_2(\mathrm{F}\,,x) = P_n(-b) + \sum_{\mathrm{j}=0}^{n-1}\mathbb{B}_{\mathrm{j}}\frac{sgn\left(\mathbb{B}_{\mathrm{j}}\right)+1}{2}T_{\mathrm{j}}(x)$$

Such that $\mathbb{B}_{\mathrm{j}} = P_{\mathrm{j}+1}-P_{\mathrm{j}+2}$, and $T_{\mathrm{j}}(x) = \frac{\int_{-b}^{x} t_{\mathrm{j}}^{\mathrm{m}}(y)\Sigma_{(y)}dy}{\int_{-b}^{b} t_{\mathrm{j}}^{\mathrm{m}}(y)\Sigma_{(y)}dy}$

The polynomial $P_2(\mathrm{F}\,,x) \ge 0$, since $P_2(\mathrm{F}\,,x) \ge P_1(\mathrm{F}\,,x) \ge 0$ $\qquad \dots (9$
From the inequality (8) and (9), we have

$$\|\mathrm{F}-P_2(\mathrm{F})\|_{L_{\psi,p}(I)} \le c(p,k)\mathcal{W}_k^{\varphi}(\mathrm{F},n^{-1})_{\psi,p}$$

By using estimate (e) when $\chi_{\mathrm{j}}(x)=0$
$\left|\chi_{\mathrm{j}}(x)-T_{\mathrm{j}}(x)\right| \leq \frac{c}{2\mathrm{m}-1} h_{\mathrm{j}} \Psi_{\mathrm{j}}^{2m-1}(x)$ when $X<X_{\mathrm{j}}$, $\mathrm{j}=0,\ldots,n-1$ then
$\|P_2-P_1\|_{L_{\psi,p}(I)}=\left\|\sum_{\mathrm{j}=0}^{n-1} \mathbb{B}_{\mathrm{j}} \frac{sgn(\mathbb{B}_{\mathrm{j}})+1}{2} T_{\mathrm{j}}-\sum_{\mathrm{j}=0}^{n-1} \mathbb{B}_{\mathrm{j}} \chi_{\mathrm{j}}\right\|_{L_{\psi,p}(I)}$,
for $X<X_{\mathrm{j}}$, we get
$\|\mathrm{P}_2-P_1\|_{L_{\psi,p}(I)}=\left\|\sum_{\mathrm{j}=0}^{n-1} \mathbb{B}_{\mathrm{j}} T_{\mathrm{j}}\right\|_{L_{\psi,p}(I)}$,

$$\leq c\left\|\sum_{\mathrm{j}=0}^{n-1} \mathbb{B}_{\mathrm{j}} T_{\mathrm{j}}\right\|_{L_{\psi,p}(I)}$$

$$=c\left(\int_{-b}^{b}\left|\sum_{\mathrm{j}=0}^{n-1} \frac{\mathbb{B}_{\mathrm{j}}(x)\, T_{\mathrm{j}}(x)}{\psi(x)}\right|^{p} dx\right)^{\frac{1}{p}}$$

$$\|P_2-P_1\|_{L_{\psi,p}(I)}^{p} \leq c^{p} \int_{-b}^{b}\left|\sum_{\mathrm{j}=0}^{n-1} \frac{\mathbb{B}_{\mathrm{j}}(x)\, T_{\mathrm{j}}(x)}{\psi(x)}\right|^{p} dx$$

$$\leq c^{p} \int_{-b}^{b} \sum_{\mathrm{j}=0}^{n-1}\left|\frac{\mathbb{B}_{\mathrm{j}}(x)}{\psi(x)}\right|^{p} \frac{c^{p}}{(2\mathrm{m}-1)^{p}} \Psi_{\mathrm{j}}^{(2\mathrm{m}-1)p} h_{\mathrm{j}}^{p} dx$$

By using the inequality [7], [12], [11] , and since $\Psi_{\mathrm{j}}=\frac{h_{\mathrm{j}}}{|X-X_{\mathrm{j}}|+h_{\mathrm{j}}}$ for $X<X_{\mathrm{j}}$, $\mathrm{j}=0,\ldots,n-1$, hence

$$\|P_2-P_1\|_{L_{\psi,p}(I)}^{p} \leq \frac{c^{2p}}{(2\mathrm{m}-1)^{p}} \sum_{\mathrm{j}=0}^{n-1} h_{\mathrm{j}}^{-kp+2p-1}\left\|\mathbb{B}_{\mathrm{j}}\right\|_{L_{\psi,p}(I_{\mathrm{j}})}^{p} \int_{-b}^{b} \Psi_{\mathrm{j}}^{(2\mathrm{m}-1)p} dx$$

$$=\frac{c^{2p}}{(2\mathrm{m}-1)^{p}} \sum_{\mathrm{j}=0}^{n-1} h_{\mathrm{j}}^{-kp+2p-1}\left\|\mathbb{B}_{\mathrm{j}}\right\|_{L_{\psi,p}(I_{\mathrm{j}})}^{p} \int_{-b}^{b} \frac{h_{\mathrm{j}}^{(2\mathrm{m}-1)p}}{\left(\left|X-X_{\mathrm{j}}\right|+h_{\mathrm{j}}\right)^{(2\mathrm{m}-1)p}} dx$$

$$=\frac{c^{2p}}{(2\mathrm{m}-1)^{p}((2m-1)p-1)} \sum_{\mathrm{j}=0}^{n-1} h_{\mathrm{j}}^{\left(-\left(k-2m-1+\frac{1}{p}\right)\right)p} \mathcal{F}_{\mathrm{j}}^{p}(b,\mathrm{m})\left\|\mathbb{B}_{\mathrm{j}}\right\|_{L_{\psi,p}(I_{\mathrm{j}})}^{p}$$

Such that

$$\mathcal{F}_{\mathrm{j}}^{p}(b,\mathrm{m})=\left[\left(X_{\mathrm{j}}-b+h_{\mathrm{j}}\right)^{-(2\mathrm{m}-1)p+1}-\left(X_{\mathrm{j}}+b+h_{\mathrm{j}}\right)^{-(2\mathrm{m}-1)p+1}\right]$$

We note that the union of an intervals $I_{\mathrm{j}} \cup I_{\mathrm{j}-1}$ gives $\left[X_{\mathrm{j}-1}, X_{\mathrm{j}+1}\right]$ at $\mathrm{j}=0,\ldots,n-1$, which is not true because it is outside the an interval $I=[-b,b]$ .Therefore, we will take the union of an interval $I_{\mathrm{j}-1} \cup I_{\mathrm{j}-2}$, it gives $[X_n, X_{n-2}]$ (here I_{-1}, I_{-2} , the empty set) , and let $\mathrm{t}=Max_{0 \leq \mathrm{j} \leq n}\left\{\mathrm{j}: h_{\mathrm{j}}<t \text{ , } as \ \mathrm{t} \rightarrow \infty\right\}$. Hence, we get

$$\begin{aligned}\|P_2-P_1\|_{L_{\psi,p}(I)}^{p} & \\ & \leq A^{p}(p,\mathrm{m}) \sum_{\mathrm{j}=0}^{n} \mathcal{F}_{\mathrm{j}}^{p}(b,\mathrm{m}) \mathrm{t}^{\left(-\left(k-2m-1+\frac{1}{p}\right)\right)p}\left(\left\|\mathrm{F}-P_{\mathrm{j}+1}\right\|_{L_{\psi,p}(I_{\mathrm{j}-1} \cup I_{\mathrm{j}-2})}^{p}\right. \\ & \left.+\left\|\mathrm{F}-P_{\mathrm{j}+2}\right\|_{L_{\psi,p}(I_{\mathrm{j}-1} \cup I_{\mathrm{j}-2})}^{p}\right)\end{aligned}$$

Where $A^{p}(p,\mathrm{m})=\frac{c^{2p}}{(2\mathrm{m}-1)^{p}((2m-1)p-1)}$

$$\|P_2 - P_1\|^p_{L_{\psi,p}(I)} \le A^p(p,\mathfrak{m}) \sum_{\mathrm{j}=0}^{n} \mathcal{F}^p_{\mathrm{j}}(b,\mathfrak{m}) \mathrm{t}^{\left(-\left(k-2m-1+\frac{1}{p}\right)\right)p} \| \mathrm{F} - P_{\mathrm{j}+1}\|^p_{L_{\psi,p}(I_{\mathrm{j}-1} \cup I_{\mathrm{j}-2})}$$

By using the inequality (5) and Lemma (5.2), we get

$$\|P_2 - P_1\|^p_{L_{\psi,p}(I)} \le A^p(p,\mathfrak{m}) \mathcal{F}^p_n(b,\mathfrak{m}) \mathrm{t}^{\left(-\left(k-2m-1+\frac{1}{p}\right)\right)p} \mathcal{W}^{\varphi}_k(\mathrm{F}, n^{-1})^p_{\psi,p}$$

$$\|P_2 - P_1\|_{L_{\psi,p}(I)} \le c(p,k,b,\mathfrak{m}) \mathrm{t}^{\left(-\left(k-2m-1+\frac{1}{p}\right)\right)p} \mathcal{W}^{\varphi}_k(\mathrm{F}, n^{-1})_{\psi,p}$$

$$c(p,k,b,\mathfrak{m}) = A(p,k,\mathfrak{m}) \mathcal{F}_n(b,\mathfrak{m})$$

We have $k + \frac{1}{p} \ge 2\mathfrak{m} - 1$, when $k > 1$, $0 < p < 1$, and by lemma(4.4) , we take $m \ge 2$.

The error of the best positive approximation to $\mathrm{F} \in L_{\psi,p}(I)$, [13] , [14], by algebraic polynomial $P_n \in \prod_n$, $0 \le n \le k-1$ is given by

$$\mathbb{E}^{(0)}_n(\mathrm{F})_{\psi,p} \le \|P_2 - P_1\|_{L_{\psi,p}(I)}$$
$$\le c_1 \mathrm{t}^{-c_2} \mathcal{W}^{\varphi}_k(\mathrm{F}, n^{-1})_{\psi,p}.$$

Where $c_1 = c(p,k,b,\mathfrak{m})$ and $c_2 = k - 2m - 1 + \frac{1}{p} \ge 0$.

7. Copositive Approximation by Algebraic Polynomials and Spline

Proof of Theorem (1.2)

Let the result of theorem (1.2) be valid for every with $\mathrm{F} \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_{\mathfrak{s}-1})$, then there exists a polynomial $q_n(\mathrm{F}) \in \prod_n \cap \Delta^0(\mathbb{J}_{\mathfrak{s}-1}), 0 \le n \le k-1$, we interpolate F inside $[-b, b]$.By using lemma (5.5), there exists a piecewise spline $S \in \Delta^0(\mathbb{J}_{\mathfrak{s}})$, with knots $\{X_{\mathrm{j}}\}^n_{\mathrm{j}=0}$ copositive with F in I and satisfies

$$\|\mathrm{F} - S\|_{L_{\psi,p}(I)} \le c(p,k) \mathcal{W}^{\varphi}_k(\mathrm{F}, n^{-1})_{\psi,p} \qquad \ldots (10$$

Let $S(x) = \frac{sgn(x-\mathrm{j}_r)+1}{2} S_1(x) + \frac{1-sgn(x-\mathrm{j}_r)}{2} S_2(x)$. Then, $S_1, S_2 \in L_{\psi,p}(I) \cap \Delta^0(\mathbb{J}_{\mathfrak{s}-1})$ and it satisfies

$$\|S_1 - q_n(\mathrm{F})\|_{L_{\psi,p}(I)} \le c(p,k) \mathcal{W}^{\varphi}_k(S_1, n^{-1})_{\psi,p} \qquad \ldots (11$$

And

$$\|S_2 - q_n(\mathrm{F})\|_{L_{\psi,p}(I)} \le c(p,k) \mathcal{W}^{\varphi}_k(S_2, n^{-1})_{\psi,p} \qquad \ldots (12$$

Let $P_n(\mathrm{F}) \in \prod_n \cap \Delta^0(\mathbb{J}_{\mathfrak{s}}), 0 \le n \le k-1$, which is defined by the following form

$$P_n(x) = q_n(x,\mathrm{F}) \frac{sgn(x-\mathrm{j}_r)+1}{2} T_n(\mathrm{j}_r, x) + q_n(x,\mathrm{F}) \frac{1-sgn(x-\mathrm{j}_r)}{2} T_n(\mathrm{j}_r, x)$$

$0 \le n \le k-1$, $x \in I$. Where $T_n(\mathrm{j}_r, x)$ is the algebraic polynomial in $\prod_n \cap \Delta^0(\mathbb{J}_{\mathfrak{s}})$, $r = 0, \ldots, \mathfrak{s}$ and copositive with $(x - \mathrm{j}_r)$.Now to find the estimate $\|S - P_n\|_{L_{\psi,p}(I)}$ by using the inequality (11) and (12), we get

$\|S - P_n\|_{L_{\psi,p}(I)} = \left\| \frac{sgn(x-\mathrm{j}_r)+1}{2} S_1(x) + \frac{1-sgn(x-\mathrm{j}_r)}{2} S_2(x) - q_n(x,\mathrm{F}) \frac{sgn(x-\mathrm{j}_r)+1}{2} T_n(\mathrm{j}_r, x) - q_n(x,\mathrm{F}) \frac{1-sgn(x-\mathrm{j}_r)}{2} T_n(\mathrm{j}_r, x) \right\|_{L_{\psi,p}(I)}$, by adding $\pm \frac{sgn(x-\mathrm{j}_r)+1}{2} q_n(x,\mathrm{F})$ and $\pm \frac{1-sgn(x-\mathrm{j}_r)}{2} q_n(x,\mathrm{F})$, then we get

$\|S - P_n\|_{L_{\psi,p}(I)} \le c \left| \frac{sgn(x-\mathrm{j}_r)+1}{2} \right| \|S_1 - q_n\|_{L_{\psi,p}(I)} + c \left| \frac{1-sgn(x-\mathrm{j}_r)}{2} \right| \|S_2 - q_n\|_{L_{\psi,p}(I)} + c|2(1-T)| \|q_n\|_{L_{\psi,p}(I)}$, for $x \ge \mathrm{j}_r$, $r = 0, \ldots, \mathfrak{s}$.

We get

$$\|S - P_n\|_{L_{\psi,p}(I)} \le c\|S_1 - q_n\|_{L_{\psi,p}(I)} + c|2(1-T)| \|q_n\|_{L_{\psi,p}(I)}$$

By using lemma (5.4)for $x \geq \mathrm{j}_r$, we get

$$\|S-P_n\|_{L_{\psi,p}(I)} \leq c\|S_1-q_n\|_{L_{\psi,p}(I)} + c\left(\frac{|x-\mathrm{j}_r|}{1+|x-\mathrm{j}_r|}\right)^{\mathfrak{m}} \|q_n\|_{L_{\psi,p}(I)}$$
$$\leq c\|S_1-q_n\|_{L_{\psi,p}(I)} + \mathbb{A}\left(\|q_n-S\|_{L_{\psi,p}(I)} + \|S\|_{L_{\psi,p}(I)}\right)$$

Where $\mathbb{A} = c\left(\frac{|x-\mathrm{j}_r|}{1+|x-\mathrm{j}_r|}\right)^{\mathfrak{m}}$, $\mathfrak{m} \geq 2$, $x \geq \mathrm{j}_r$, $r = 0, \ldots, \mathfrak{s}$.

$$\|S-P_n\|_{L_{\psi,p}(I)} \leq c\|S_1-q_n\|_{L_{\psi,p}(I)} + \mathbb{A}\left(\|\mathrm{F}-q_n\|_{L_{\psi,p}(I)} + \|\mathrm{F}-S\|_{L_{\psi,p}(I)} + \|S\|_{L_{\psi,p}(I)}\right)$$

By using the inequality (10),(11) and (5) then we get

$$\|S-P_n\|_{L_{\psi,p}(I)} \leq c\mathcal{W}_k^{\varphi}(S_1,n^{-1})_{\psi,p} + \mathbb{A}\mathcal{W}_k^{\varphi}(\mathrm{F},n^{-1})_{\psi,p} + \mathbb{A}\,\|S\|_{L_{\psi,p}(I)}$$

For $x \geq \mathrm{j}_r$ we get from definition of $S(x)$ that $S(x) = S_1(x)$ and $S_1(\mathrm{j}_{\mathfrak{s}}) = 0$, hence $S(\mathrm{j}_{\mathfrak{s}}) = 0$ then

$$\|S\|^p_{L_{\psi,p}(I)} = \int_I \left|\frac{S(x)-S(\mathrm{j}_{\mathfrak{s}})}{\psi(x)}\right|^p dx$$
$$\leq c^p(p,k)\int_{-b}^{b}\left(|x-\mathrm{j}_{\mathfrak{s}}|^{-(k-1)-\frac{1}{p}} + |x-\mathrm{j}_{\mathfrak{s}}|^{-2k+3-\frac{2}{p}}\right)^p \mathcal{W}_k^{\varphi}(S,n^{-1})^p_{\psi,p}$$
$$\leq c^p(p,k)\int_{-b}^{b}\left(|x-\mathrm{j}_{\mathfrak{s}}|^{-(k-1)-\frac{1}{p}}\right)^p \mathcal{W}_k^{\varphi}(S,n^{-1})^p_{\psi,p}$$
$$= c^p(p,k)(2b)^p\mathcal{W}_k^{\varphi}(S,n^{-1})^p_{\psi,p}$$

$= c^p(p,k,b)\mathcal{W}_k^{\varphi}(S,n^{-1})^p_{\psi,p}$. Hence,

$$\|S\|_{L_{\psi,p}(I)} \leq c(p,k,b)\mathcal{W}_k^{\varphi}(S,n^{-1})_{\psi,p} \qquad \ldots (13$$

Then

$$\|S-P_n\|_{L_{\psi,p}(I)} \leq c(p,k)\mathcal{W}_k^{\varphi}(S_1,n^{-1})_{\psi,p} + \mathbb{A}\mathcal{W}_k^{\varphi}(\mathrm{F},n^{-1})_{\psi,p} + c(p,k,b)\mathbb{A}\mathcal{W}_k^{\varphi}(S,n^{-1})_{\psi,p}$$

From lemma (5.5), we have $\mathcal{W}_k^{\varphi}(S,n^{-1})_{\psi,p} \leq \mathcal{W}_k^{\varphi}(\mathrm{F},n^{-1})_{\psi,p}$

$$\|S-\mathrm{P}_n\|_{L_{\psi,p}(I)} \leq c(p,k)\mathcal{W}_k^{\varphi}(S_1,n^{-1})_{\psi,p} + c(p,k,b)\mathbb{A}\mathcal{W}_k^{\varphi}(\mathrm{F},n^{-1})_{\psi,p}$$

For $\mathrm{F} \in L_{\psi,p}(I)\cap\Delta^0(\mathbb{J}_{\mathfrak{s}})$ and $S \in \Delta^0(\mathbb{J}_{\mathfrak{s}})$ which is copositive with F in I.

Now we need to show that $\mathcal{W}_k^{\varphi}(S_1,n^{-1})_{\psi,p} \leq c\mathcal{W}_k^{\varphi}(S,n^{-1})_{\psi,p}$

Since S_1 does not change sign on $[\mathrm{j}_{\mathfrak{s}-1}, b]$ and $S(x) = S_1(x)$, also $|S(x)| = |S_1(x)|$, then

$$\left\|\Delta^k_{\varphi h}(S_1,.,I)\right\|_{L_{\psi,p}(I)} = \left(\int_I \left|\Delta^k_{\varphi h}\frac{(S_1,x)}{(\psi,x)}\right|^p dx\right)^{\frac{1}{p}}$$
$$= \left(\int_{\mathrm{I}} \left|\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\frac{S_1(x-\frac{kh}{2}+ih)}{\psi(x+\frac{kh}{2})}\right|^p dx\right)^{\frac{1}{p}}$$
$$= \left(\int_I \left|\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\right|^p \left|\frac{S_1(x-\frac{kh}{2}+ih)}{\psi(x+\frac{kh}{2})}\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq c(p)\sum_{i=0}^{k}\left|\binom{k}{i}\right|\left(\int_I\left|\frac{S_1(x-\frac{kh}{2}+ih)}{\psi(x+\frac{kh}{2})}\right|^p dx\right)^{\frac{1}{p}}$$

$$= c(p)\|S\|_{L_{\psi,p}(I)}$$

Then

$$sup_{0<h\leq\delta}\left\|\Delta^k_{\varphi h}(S_1,.,I)\right\|_{L_{\psi,p}(I)} \leq sup_{0<h\leq\delta}\left(c(p,k)\|S\|_{L_{\psi,p}(I)}\right)$$

$$\mathcal{W}^{\varphi}_k(S_1,n^{-1})_{\psi,p} \leq c(p,k)\|S\|_{L_{\psi,p}(I)}$$

By using the inequality (13), we get

$$\mathcal{W}^{\varphi}_k(S_1,n^{-1})_{\psi,p} \leq c(p,k,b)\mathcal{W}^{\varphi}_k(S,n^{-1})_{\psi,p}$$

$\leq c(p,k,b)\mathcal{W}^{\varphi}_k(\mathrm{F},n^{-1})_{\psi,p}$. Hence,

$$\|S-P_n\|_{L_{\psi,p}(I)} \leq c(p,k,b)\mathcal{W}^{\varphi}_k(\mathrm{F},n^{-1})_{\psi,p} + \mathcal{B}\mathcal{W}^{\varphi}_k(\mathrm{F},n^{-1})_{\psi,p}$$

Where $\mathcal{B} = c(p,k,b)\mathbb{A}$. Therefore,

$$\|S-P_n\|_{L_{\psi,p}(I)} \leq \mathcal{B}(p,k,b,\mathfrak{m})\mathcal{W}^{\varphi}_k(\mathrm{F},n^{-1})_{\psi,p}$$

The error of the best copositive approximation by algebraic polynomial and spline for $\mathrm{F} \in L_{\psi,p}(I)\cap\Delta^0(\mathbb{J}_{\mathfrak{s}})$ is given by

$\mathbb{E}^{(0)}_{\mathrm{n}}(\mathrm{F},\mathbb{J}_{\mathfrak{s}})_{\psi,\mathrm{p}} \leq \|\mathrm{S}-\mathrm{P_n}\|_{\mathrm{L}_{\psi,\mathrm{p}}(\mathrm{I})}$. Hence,

$\mathbb{E}^{(0)}_{\mathrm{n}}(\mathrm{F},\mathbb{J}_{\mathfrak{s}})_{\psi,\mathrm{p}} \leq \mathcal{B}(\mathrm{p},\mathrm{k},\mathrm{b},\mathfrak{m})\mathcal{W}^{\varphi}_{\mathrm{k}}(\mathrm{F},\mathrm{n}^{-1})_{\psi,\mathrm{p}}$, where $\mathcal{B}$ depends on $\mathrm{p},\mathrm{k},\mathrm{b}$ and $\mathbb{A}$, $\mathrm{k} > 1$, $m \geq 2$, b positive integer.

Conclusions

In this paper, we firstly find some of the necessary estimations in approximation by using certain algebraic polynomials. Certain specific points in approximation are used. algebraic polynomials and geometric polynomials are used in many estimations that help to find the best approximation. We also deal with these estimations to estimate the best approximation error using algebraic polynomials. In addition, the basic estimations in approximation have been discussed and shown. Secondly, the best approximation of the weighted function $\mathrm{F} \in \mathrm{L}_{\psi,\mathrm{p}}(\mathrm{I})$ has been found and estimated. Furthermore, the best approximation of the restricted function $\mathrm{F} \in \mathrm{L}_{\psi,\mathrm{p}}(\mathrm{I}) \cap \Delta^0(\mathbb{J}_{\mathfrak{s}})$ has been found and estimated. In addition, the error of the best approximation of the function $\mathrm{F} \in \mathrm{L}_{\psi,\mathrm{p}}(\mathrm{I}) \cap \Delta^0(\mathbb{J}_{\mathfrak{s}})$ by using pieces of algebraic polynomials that are of the highest degree$0 \leq \mathrm{n} \leq \mathrm{k}-1$ has been estimated by using the created estimations. Finally, some other results and outcomes are given.

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