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More on Result Involution Graphs

Ali Abd Aubad[1]*, Haval M. Mohammed Salih[2]
[1]Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq
[2]Department of Mathematics, Faculty of Science, Soran University, Erbil, Iraq

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Abstract

The result involution graph of a finite group G, denoted by Γ_G^{RI} is an undirected simple graph whose vertex set is the whole group G and two distinct vertices are adjacent if their product is an involution element. In this paper, result involution graphs for all Mathieu groups and connectivity in the graph are studied. The diameter, radius and girth of this graph are also studied. Furthermore, several other graph properties are obtained.

Keywords: Mathieu Group, Result Involution Graph, Connectedness, Girth.

المزيد حول بيانات ناتج الالتفاف

علي عبد عبيد[1]*, هفال محمود محمد صالح[2]
[1] قسم الرياضيات, كلية العلوم, جامعة بغداد, بغداد, العراق
[2] قسم الرياضيات, كلية العلوم, جامعة سوران, اربيل, العراق

الخلاصة

بيان ناتج الالتفاف لزمرة منتهية G, يرمز له Γ_G^{RI} , وهو بيان بسيط غير موجه مجموعة رؤوسه تمثل عناصر الزمرة G وان رأسين مختلفين متجاورين اذا كان حاصل ضربهما عنصر الالتفاف. في هذا البحث, بيان ناتج الالتفاف لجميع زمر ماثيو مع الاتصال تم دراستها. القطر, نصف القطر, الخصر لهذا البيان تم دراسته ايضاً. بالإضافة الى ذلك, العديد من خصائص البيان تم الحصول عليها.

1. Introduction

Many studies have been done in order to show that the connection between graph theory and group theory. For example, if G is a finite group. Then the commuting involution graph is introduced on G, whose vertex set a subset of involution elements in G and two vertices are adjacent if they commute see [1, 2]. Furthermore, in [3, 4, 5] the most popular interest work in this context can be seen. Moreover, involution graphs, which adjacency of two vertices is defined by their product is of order 3, are describe in full details in [6]. In the recent research, Tolue [7] introduced a twin non-commuting graph of a group by partitioning the vertex set of non-commuting graph with special properties. Recently, with the aid of the computer algebra system GAP and YAGs

*Email: ali.abd@sc.uobaghdad.edu.iq

package [8], the result involution graph of all symmetric groups, alternating groups, dihedral groups and quatrain groups were classified in [9]. Let G be a finite group act on a finite set X. The orbit of $x \in X$ is the set $O(x) = \{g * x: g \in G\}$. In this paper, we consider the conjugation action on X, so the orbit can be written as $x^G = \{x^g: g \in G\}$. In particular if the G acts on itself by conjugation, then it is called the conjugacy class of x. The conjugacy classes partition G into some equivalence classes. That is, every element in a group lies exactly in one conjugacy class. According to Atlas notation which used in [2], we are going label the conjugacy class of x_i by the symbol d_iA where d_i is the order of x_i of type A and so on. For a graph Γ by $V(\Gamma)$ and $E(\Gamma)$, we denote the set of all vertices and edges of Γ respectively.

A graph is connected if there is a path connecting any two distinct vertices. The distance between two distinct vertices u and v is the length of the shortest path connecting u and v and denoted by $d(u,v)$ (if such a path does not exist, define $(u,v) = \infty$). The diameter of a graph Γ, is defined by

$$\sup\{d(u,v) \mid u,v \in V(\Gamma)\}$$

The girth of a graph Γ, is the length of the shortest cycle in Γ. A graph with no cycles has infinite girth. The minimum among all the maximum distances between a vertex to all other vertices is considered as the radius of the graph Γ. A graph Γ is complete if each pairof vertices is joined by an edge. We use K_n to denote the complete graph with *n* vertices. A vertex v of a connected graph Γ is called a cut vertex of Γ, if $\Gamma \backslash v$ (delete v from Γ) produce a disconnected graph. The maximum clique size of Γ, denoted by $\omega(\Gamma)$, is the largest integer k for which Γ contains a clique (complete subgraph) of size k. In this paper we go further and extend these results to some finite groups. In particular we focus on the result involution graphs for Mathieu groups.

This paper is organized as follows. In Section 2, we provide some notation and basic facts for result involution graphs. In Section 3, we prove some results concerning for some finite groups which consists of all Mathieu groups. Several graph properties are given.

2.Preliminary

We begin by setting up some definition and stating a few results which will be needed later in the paper. A non-trivial element x in G is said to be an involution element if $x^2 = e$. The set of allinvolution elements in G is denoted by $I(G)$. It is clear that the size of $I(G)$ is the sum of thesize of involution conjugacy classes in G and denoted by t. Recall that the number of vertices of $\Gamma_G^{RI}$is t and the number of edges of Γ_G^{RI} can be obtained in the following result.

Proposition 2.1. [9]Let G be a finite group with t involution elements. Then the number of edges in Γ_G^{RI} is ½ $(t|G| - F)$ where F is the number of elements of order 4.

A resize graph is a graph with vertex set the set of all conjugacy class of the group G and two vertices are adjacent if their conjugacy class representatives are connected in Γ_G^{RI}.

In **Example 2.3**, it is clear that the resize graph of G is $4K_2$, this means that it has 8 vertices (8 conjugacy classes) and 4 edges because the class $1A$ is only adjacent to $2A$ (7 edges) and it represent by one edge and so on.

Proposition 2.2. [9] Let G be a finite group. Then the resize graph of Γ_G^{RI} is connected

if and only if the result involution graph Γ_G^{RI} is connected.
To illustrate these results, this can be seen in the following example.

Example 2.3. The affine general linear group $G = AGL(1,8)$ has eight different conjugacy classes which can be explain in **Table 1**.

Table 1: Vertex set and edge set of $\Gamma_{AGL(1,8)}^{RI}$.

Conjugacy classes	1A	2A	7A	7E	7B	7C	7D	7F
Size of conjugacy class	1	7	8	8	8	8	8	8
Number of Edges in different classes	7		56		56		56	
Number of edges in the same class			21					

From **Table 1**, we deduce that the result involution graph for G is the three copy of 7-regular graph over 16 vertices union 7-regular graph over 8 vertices. It gives 4 components. A straightforward calculation shows that it is 7-regular graph with 56 vertices and 196 edges, with girth 3. However, the resize graph of it is $4K_2$.

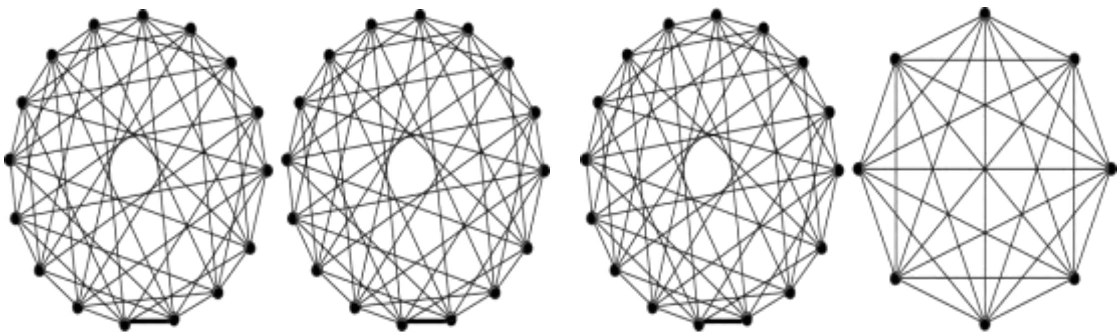

Figure 1: The result involution graph of $AGL(1,8)$.

Definition 2.4. [10] Let a *and* b *be* positive integers, $1 \leq a < b$. A graph Γ is said to be (a, b) −biregular if its vertices have degree either a or b, and if it possesses vertices of degree a and b.

Theorem 2.5. [11] [Kuratowski's Theorem] A graph is non-planar if and only if it contains asubgraph homeomorphic to $K_{3,3}$ or K_5.

3.Main Results

In this section, we present some properties of the result involution graphs which can be seen in the following:

Proposition 3.1. For all $v \in V(\Gamma_G^{RI})$,

$$\deg(v) = \begin{cases} t & otherwise \\ t-1 & if\ |v| = 4 \end{cases}$$

Proof. Since $E(\Gamma_G^{RI})$ is non-empty, then we can take $v \in V(\Gamma_G^{RI})$. Then there is an element in G which is adjacent with v that is $gv = x$ for some involution element x. From this wehave $g = xv^{-1}$. If $|v| \neq 4$. Then it has degree t. If $|v| = 4$, then the degree of it is $t-1$ because v^2 is an involution element by taking $x = v^2$. This force $g = v$ which is impossible because g and v are distinct vertices.

Corollary 3.2. The result involution graph for a finite group G is either t-regular or $(t, t-1)$-biregular.

Proof. The proof follows from **Proposition 3.1**.

Example 3.3. If k is a positive integer and $G = D_{2n}$, then

$$\Gamma_G^{RI} = \begin{cases} t - regular & if\ n \neq 4k \\ (t, t-1) - biregular & if\ n = 4k. \end{cases}$$

Proposition 3.4. The result involution graph of G is K_{2^l} for some l if and only if $G \cong \underbrace{C_2 \times C_2 \times \ldots \times C_2}_{l-times}$, where C_2 is a cyclic group of order 2.

Proof.
Let Γ_G^{RI} be the result involution graph of G where $G \cong \underbrace{C_2 \times C_2 \times \ldots \times C_2}_{l-times}$. It is clear that $|V(\Gamma_G^{RI})| = 2^l$. Let $a, b \in G$. Then $a = (a_1, \ldots, a_l)$ and $b = (b_1, \ldots, b_r)$. So $ab = (a_1b_1, \ldots, a_lb_l)$. We need to show that $(ab)^2 = e$. Now have the following $(ab)(ab) = (a_1b_1a_1b_1, \ldots, a_lb_la_lb_l) = (a_1^2b_1^2, \ldots, a_l^2b_l^2) = (e, \ldots, e) = e$.
Thus a and b are adjacent in Γ_G^{RI}, but they are arbitrary elements in G. Therefore, Γ_G^{RI} is complete graph of order 2^l.
Conversely, let $a, b \in G$ and $\Gamma_G^{RI} = K_{2^l}$. This implies that $(ab)^2 = e$. We obtain G is an abelian group of order 2^l. Thus G is an elementary abelian 2-group of order 2^l. Hence $G \cong \underbrace{C_2 \times C_2 \times \ldots \times C_2}_{l-times}$.

Lemma 3.5. The resize graph of the result involution graph Γ_G^{RI} is a subgraph of Γ_G^{RI}.
Proof. The proof is clear.

In **Table 2**, the affine special linear group ASL (3, 2) has 11 conjugacy classes. So, we have 91 involution elements which is equal to $|2A| + |2B| + |2C|$ and 60942 edges as we see in **Proposition 2.1** . It is hard to draw its graph. Therefore, we draw the resize graph of it just by taking one representative from each pair of conjugacy classes and joining them together according **Table 2,** if it exists.

Table 2: The edges number of $\Gamma_{ASL(3,2)}^{RI}$.

G-Class	The edges numbers
1A	1A(0),2C(7),2A(42),2B(42),
2A	2A(189),2B(256),4C(336),4A(1008),4B(336),3B(672),3A(672)
2B	2B(189),4C(336),4A(336),4B(1008),3B(672),3A(672)
2C	2C(21),2A(126),2B(126),4C(336)
4A	4A(504),4B(672),3B(2688),3A(2688),7A(2688),7B(2688)
4C	4C(588),4A(1344),4B(1344),3B(1344),3A(1344)

4B	4B(504),3B(2688),3A(2688),7A(2688),7B(2688)
3A	3A(2352),7A(2688),7B(2688)
3B	3B(2352),3A(2240),7A(2688),7B(2688)
7A	7A(2688),7B(1344)
7B	7B(2688)

In the above table and the tables follows in this paper we use the following notations. The class of the table start with G-class nX, and the next class have $mY(r)$ for a G-class mY means there are r-edges between the classes nX and nY. For Example, there are 126-edges between the classes 2C and 2B as we see in the five class of **Table 2**.

Also, we describe in the next figure the result involution graph of $ASL(3,2)$.

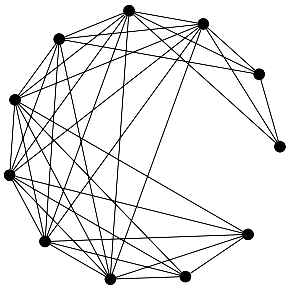

Figure 2: The resize graph of result involution graph of $ASL(3,2)$.

Lemma 3.6. The result involution graph of $ASL(3,2)$ is connected graph with girth 3, diameter 3 and radius 2.

proof. From **Table 2**, there is a path between any pair of vertices, so the result involution graph is connected. It is clear that it has three classes of involution and there is an edge between any two of them. So, it has girth three. The trivial element is only adjacent to the involution elements. In **Figure 2**, the greatest distance between any pair of vertices in the resize graph of Γ_G^{RI}, is either 2 or 3. Hence it has diameter 3 and radius 2. From **Lemma 3.5** and the result follows.

Proposition 3.7. If $V_{8n} =< a, b \mid a^{2n} = b^2 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b >$, then the result involution graph is a $(t, t-1)$-biregular connected graph of size $n(4t-1)$ with girth 3, radius 3 and diameter 3, if $n > 1$ is an odd positive integer.

Proof. We are going to prove that it has size $n(4t - 1)$. Now there are $2n$ vertices of degree $t - 1$ and $6n$ vertices of degree t. From the hand shake lemma, we have $2|E(\Gamma_G^{RI})|=2n(t - 1) + 6n(t)$. So $|E(\Gamma_G^{RI})| = n(4t - 1)$. The proof of remaining is similar to **Proposition 3.6**.

Proposition 3.8. If G is a finite group and it has a unique conjugacy class of involution say $2A$. Then $2A$ is a cut vertex in the resize graph of Γ_G^{RI}.

Proof. Since G contains only one involution class $2A$ and it is only adjacent to the trivial class, then the number of components increase in the resize result involution graph if we remove the vertex $2A$. Therefore, $2A$ is a cut vertex.

For the rest, we try to find the associated result involution graphs for Mathieu groups and then several graph properties of them are given. In the following figure provides the resize graphs of $\Gamma_{M_{11}}^{RI}$.

Figure 3: The resize graphs of $\Gamma_{M_{11}}^{RI}$.

Also we provide information about the vertices and edges set for the $\Gamma_{M_{11}}^{RI}$

Table 3: The number of vertices and edges of $\Gamma_{M_{11}}^{RI}$.

G-Class	The edges numbers
1A	2A(165)
2A	2A(990),3A(5280),4A(3960),5A(7920),6A(7920)
3A	3A(3960),4A(7920),5A(23760),6A(3960),8A(3960),8B(3960),11A(7920),11B(7920)
4A	4A(19800),5A(23760),6A(15840),8A(27720),8B(27720),11A(7920),11B(7920)
5A	5A(35640),6A(39600),8A(23760),8B(23760),11A(23760),11B(23760)
6A	6A(15840),8A(27720),8B(27720),11A(31680),11B(31680)
8A	8A(20295),8B(7920),11A(15840),11B(15840)
8B	8B(20295),11A(15840),11B(15840)
11A	11A(3960),11B(7920)
11B	11B(3960)

From above table and figure we have the following result:

Proposition 3.9. If $G = M_{11}$, then Γ_G^{RI} is a connected graph with diameter 3, radius 2 and girth 3.

Proof. According to **Table 3**, there is a path between any pair of vertices. So Γ_G^{RI} is a connected graph. Also note that the classes $2A, 3A$ and $4A$ produce the cycle C_3. So, it has girth 3. Furthermore, the identity element only adjacent to the involution elements. In **Figure 3,** the greatest distance between any pair of vertices in the resize graphof Γ_G^{RI} is either 2 or 3. Hence it has diameter 3 and radius 2. From **Lemma 3.5** and the result follows.

The next figure information about the resize graph for $\Gamma_{M_{12}}^{RI}$.

Figure 4: The resize graphs of $\Gamma^{RI}_{M_{12}}$.

Also we provide information about the vertices and edges set for the $\Gamma^{RI}_{M_{12}}$

Table 4: The number of vertices and edges of $\Gamma^{RI}_{M_{12}}$

G-Class	The edges numbers
1A	2A(396),2B(495)
2A	2A(6930),2B(5940),3B(23760),4A(23760),4B(23760),5A(47520),6A(71280), 8A(47520),8B(47520),10A(47520)
2B	2B(7425),3A(31680),3B(7920),4A(35640),4B(35640),5A(47520),6A(23760), 6B(95040),8A(47520),8B(47520),10A(47520)
3A	3A(39600),4A(47520),4B(47520),5A(190080),6B(221760),8A(190080), 8B(190080),10A(190080),11A(190080),11B(190080)
3B	3A(55440),4A(47520),4B(47520),5A(237600),6A(245520),6B(443520), 8A(285120),8B(285120),10A(237600),11A(190080),11B(190080)
4A	4A(95040),4B(112860),5A(237600),6A(142560),6B(522720),8A(546480), 8B(118800),10A(237600),11A(190080),11B(190080)
4B	4B(95040),5A(237600),6A(142560),6B(522720),8A(118800),8B(546480), 10A(237600),11A(190080),11B(190080)
5A	5A(689040),6A(712800),6B(1330560),8A(1045440),8B(1045440), 10A(437184),11A(760320),11B(760320)
6A	6A(411840),6B(1330560),8A(855360),8B(855360),10A(712800), 11A(570240),11B(570240)
6B	6B(1116720),8A(1710720),8B(1710720),10A(1330560),11A(1330560), 11B(1330560)
8A	8A(778140),8B(1235520),10A(1045440),11A(950400),11B(950400)
8B	8B(778140),10A(1045440),11A(950400),11B(950400)
10A	10A(689040),11A(760320),11B(760320)
11A	11A(190080),11B(1235520)
11B	11B(190080)

From above table and figure we can conclude the following result:

Proposition 3.10. If $G = M_{12}$, then Γ^{RI}_G is a connected graph with diameter 3, radius 2 and girth 3.

Proof. The proof is similar as **Proposition 3.9**.

In the following the result involution graph resize graph are provided with full information.

Figure 5: The resize graphs of $\Gamma^{RI}_{M_{22}}$.

Also we provide information about the vertices and edges set for the $\Gamma^{RI}_{M_{22}}$

Table 5: The number of vertices and edges of $\Gamma^{RI}_{M_{22}}$.

G-Class	The edges numbers
1A	2A(1155)
2A	2A(28875),3A(221760),4A(166320),4B(221760),5A(443520),6A(221760)
3A	3A(720720),4A(887040),4B(1552320),5A(4435200),6A(369600),7A(1774080), 7B(1774080),8A(887040),11A(443520),11B(443520)
4A	4A(706860),4B(443520),5A(3991680),6A(1330560),7A(1774080),7B(1774080), 8A(2439360),11A(887040),11B(887040)
4B	4B(1580040),5A(7096320),6A(1995840),7A(5322240),7B(5322240),8A(3326400), 11A(1774080),11B(1774080)
5A	5A(13970880),6A(6209280),7A(14192640),7B(14192640),8A(9757440), 11A(7096320),11B(7096320)
6A	6A(2531760),7A(5322240),7B(5322240),8A(7096320),11A(4878720), 11B(4878720)
7A	7A(7096320),7B(9313920),8A(7096320),11A(7096320),11B(7096320)
7B	7B(7096320),8A(7096320),11A(7096320),11B(7096320)
8A	8A(5183640),11A(7983360),11B(7983360)
11A	11A(2661120),11B(3991680)
11B	11B(2661120)

From above table and figure we conclude the following result:

Proposition 3.11. If $G = M_{22}$, then Γ^{RI}_{G} is a connected graph with diameter 3, radius 2 and girth 3.

Proof. The proof is similar as **Proposition 3.9**.

In the next full details on the results involution graph edges for $\Gamma^{RI}_{M_{23}}$ are given as a resize graph for $\Gamma^{RI}_{M_{23}}$.

Figure 6: The resize graphs of $\Gamma^{RI}_{M_{23}}$.

Also we provide information about the vertices and edges set for the $\Gamma^{RI}_{M_{23}}$

Table 6: The number of vertices and edges of $\Gamma_{M_{23}}^{RI}$.

G-Class	The edges numbers
1A	2A(3795)
2A	2A(185955),3A(1700160),4A(3825360),5A(3400320), 6A(5100480)
3A	3A(5100480),4A(30602880),5A(44204160),6A(14451360), 7A(25502400),7B(25502400),8A(25502400),11A(10200960), 11B(10200960),14A(5100480),14B(5100480),15A(3400320), 15B(3400320)
4A	4A(41760180),5A(153014400),6A(132612480),7A(132612480),7B(132612480),8A(173416320),11A(71406720),11B(71406720),14A(51004800),14B(51004800),15A(40803840),15B(40803840),23A(20401920),23B(20401920)
5A	5A(205719360),6A(193818240),7A(265224960),7B(265224960),8A(326430720),11A(234622080),11B(234622080),14A(102009600),14B(102009600),15A(61205760),15B(61205760),23A(61205760),23B(61205760)
6A	6A(174266400),7A(219320640),7B(219320640),8A(382536000),11A(265224960),11B(265224960),14A(239722560),14B(239722560),15A(207419520),15B(207419520),23A(142813440),23B(142813440)
7A	7A(168315840),7B(219320640),8A(295827840),11A(285626880),11B(285626880),14A(91808640),14B(158843520),15A(142813440),15B(142813440),23A(81607680),23B(81607680)
7B	7B(168315840),8A(295827840),11A(285626880),11B(285626880),14A(158843520),14B(91808640),15A(142813440),15B(142813440),23A(81607680),23B(81607680)
8A	8A(313041960),11A(510048000),11B(510048000),14A(377435520),14B(377435520),15A(306028800),15B(306028800),23A(163215360),23B(163215360)
11A	11A(158114880),11B(316229760),14A(204019200),14B(204019200),15A(234622080),15B(234622080),23A(173416320),23B(173416320)
11B	11B(158114880),14A(204019200),14B(204019200),15A(234622080),15B(234622080),23A(173416320),23B(173416320)
14A	14A(168315840),14B(219320640),15A(224421120),15B(224421120),23A(163215360),23B(163215360)
14B	14B(168315840),15A(224421120),15B(224421120),23A(163215360),23B(163215360)
15A	15A(98609280),15B(275425920),23A(142813440),23B(142813440)
15B	15B(98609280),23A(142813440),23B(142813440)
23A	23A(51004800),23B(71406720)
23B	23B(51004800)

From above table and figure we conclude the following result:

Proposition 3.12. If $G = M_{23}$, then Γ_G^{RI} is a connected graph with diameter 3, radius 2 and girth 3.

Proof. The proof is similar as **Proposition 3.9**.
Finally, the resize graph for $\Gamma_{M_{24}}^{RI}$ are provided in the next figure

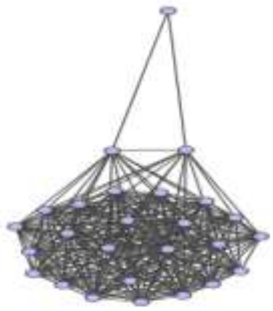

Figure 7: The resize graphs of $\Gamma_{M_{24}}^{RI}$.

Proposition 3.13. If $G = M_{24}$, then Γ_G^{RI} is a connected graph with diameter 3, radius 2 and girth 3.
Proof. The proof is similar as **Proposition 3.9**.

Example 3.14. If G is isomorphic to one of the following groups: M_{11}, M_{22}, M_{23}, then the vertex $2A$ in the resize result involution graph is a cut vertex.

Proposition 3.15. If G is isomorphic to one of the following groups: $\{M_{11}, M_{12}, M_{22}, M_{23}, M_{24}\}$. Then the clique numbers of resize graphs of Γ_G^{RI} are 8,11,10, 14 and 22 respectively.

Proof. According to **Table 3**, the complete subgraphs obtained by removing the classes $1A$ and $2A$. Therefore, the clique number of resize graph of $\Gamma_{M_{11}}^{RI}$ is 8. The proof of the remaining is similar.
Corollary 3.16. If G is isomorphic to one of the following groups $\{M_{11}, M_{12}, M_{22}, M_{23}, M_{24}\}$.Then the graph Γ_G^{RI} is non-planer.

Proof. From **Proposition 3.11**, we obtain the resize graph of Γ_G^{RI} contains a subgraph which is isomorphic to K_5 and it lies in Γ_G^{RI}. So, by **Theorem 2.5** and the result follows.

4. Conclusion

We showed that the result involution graph for a finite group is either regular or bi-regular. In particular, there is a one to one correspondence between elementary abelian 2-groups and complete graphs. Furthermore, the result involution graph for the Mathieu simple groups have been investigated. Many results have obtained. For example, the diameter, girth, radius and clique number for the result involution graph are calculated.

5. References

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