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Soft Bornological Group Acts on Soft Bornological Set

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Abstract

In this paper, we introduce the notation of the soft bornological group to solve the problem of boundedness for the soft group. We combine soft set theory with bornology space to produce a new structure which is called soft bornological group. So that both the product and inverse maps are soft bounded. As well as, we study the actions of the soft bornological group on the soft bornological sets. The aim soft bornological set is to partition into orbital classes by acting soft bornological group on the soft bornological set. In addition, we explain the centralizer, normalizer, and stabilizer in details. The main important results are to prove that the product of soft bornological groups is soft bornological group and the action for different elements are the same actions.

Keywords: Soft set, Bounded map, Soft bornological group, Soft bornological set, Soft bounded set.

زمرة برنلوجية لينة تعمل على مجموعة برنلوجية لينة

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الخلاصة

في هذا البحث، نقدم مفهوم الزمرة البرنولوجية اللينة لحل مشكلة التقيد بالنسبة للزمراللينة. حيث قمنا بدمج الجبر مع الفضاء البرنلوجي لإنتاج هيكل جديد يسمى الزمرة البرنلوجية اللينة التي تساعدنا في اجراء دراسات جديدة في المستقبل. أيضًا ، ندرس إجراءات الزمرة البرنلوجية اللينة على مجموعة برنلوجية لينة ، الهدف من هذه الورقة هو أن المجموعة البرنلوجية اللينة مقسمة إلى فئات مدارية من خلال تمثيل زمرة برنلوجية لينة على مجموعة برنلوجية لينة كما وضحنا centralizer,normalizer, stabilizer بالتفصيل. والنتائج الرئيسية المهمة، حيث أثبتنا أن ضرب الزمر البرنلوجية اللينة هو زمرة برنلوجية لينة ايضا والعمل لعناصر مختلفة هو نفس الإجراء.

1.Introduction

In 1977 H. Hogbe-Nlend [1] introduced the concept of bornology. It is a type of space in functional analysis and is extremely helpful in resolving the boundedness problem for a set of elements and functions. Pombo and Bambozzi studied the fundamental construction and category theory for bornological groups, respectively. Anwar N. Imran also started to solve

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many existent problems in algebraic bornology [2] [3] [4]. In 1999 Molodtsov [5] pioneered the notion of soft sets as a new mathematical tool for dealing with engineering uncertainty modelling problems in physics, economics, computer science, social sciences, and medical sciences. Many researchers have contributed to developing soft set theory's algebraic structure to the best. For more details see [6] [7] [8] [9] [10]. To tackle the problems of boundedness for a soft set, one can merge the soft set with bornology to develop a new structure called the soft bornological set [11]. On a soft set $\mathbb{X}$, the soft bornology $\tilde{\beta}$ is a set of soft subsets of $\mathbb{X}$ that covers $\mathbb{X}$ as well as is stable under hereditary and finite soft union see [9]. In this study, we combine the soft set theory with bornology to construct a new structure that is called a soft bornological group, denoted by $(\mathbb{G}, \tilde{\beta})$ to solve the problems of boundedness for the soft groups in such a way that the product map and the inverse map are soft bounded maps. We also go over the definition, examples, and some of the results in detail. In addition, we study the actions of soft bornological groups on soft bornological sets. Furthermore, the letters $\mathbb{G}$ and $\mathbb{X}$ stand for the soft bornological group $(\mathbb{G}, \tilde{\beta})$ and soft bornological set $(\mathbb{X}, \tilde{\beta})$, respectively. The main goal of this research is to look at the actions of soft bornological groups. The process of the soft bornological group action on a soft bornological set is known as soft bornological group action (SBGA) or soft (bounded) action, and the result is that the soft bornological set is partitioned into orbital classes. We prove that the product of soft bornological groups is a soft bornological group. We also prove that the centralizer, normalizer, and stabilizer are soft bornological subgroups of a soft bornological group.

2. Preliminaries:

In this section, we give some definitions and results that are related to the soft set theory and will be used throughout the study. For more details, we suggest the reader to [5] [6] [7] [8] [9] [10].

Definition 2.1 [5] [6] [9]**:** Let U be a universal set and E be the set of parameters. A *soft set* over U is a pair (ξ, A) consisting of a subset A of E and a mapping $\xi: A \longrightarrow P(U)$.
That means$\xi = \{(e, \xi(e)), e \in A, \xi(e) \in P(U)\}$. Note that $S(U)$ will refer to the set of all soft sets under$U$.

Propositions 2.2 [5] [6] [9]**:**
1. If $A = \emptyset$, then $\xi(e) = \emptyset$, this symbol $(\emptyset, \emptyset)$ is referred to an empty soft set, denoted by $\tilde{\emptyset}$.
2. If $A = E$, then $\xi(e) = U$, this symbol (U, E) is referred to a universal soft set, denoted by $\tilde{U}$.
3. If $L \subseteq A$, and $\xi(e) \subseteq \xi'(e)$, this symbol $(\xi, L) \tilde{\subseteq} (\xi', A)$ is referred to a soft subset of a soft set.
4. If $L = A$, $\xi(e) \subseteq \xi'(e)$ and $\xi'(e) \subseteq \xi(e)$, this symbol $(\xi, L) = (\xi', A)$ is referred to soft set equal to another soft set.
5. The soft union and the soft intersection of ξ and $\xi'$are defined by the soft sets, respectively.
$\xi(e) \cup \xi'(e), e \in L \cup A$, we write$(\xi, L) \tilde{\cup} (\xi', A)$.
$\xi(e) \cap \xi'(e), e \in L \cap A$, we write$(\xi, L) \tilde{\cap} (\xi', A)$.
6. The soft complement of (ξ, A) is defined by $(\xi, A)^c = \{\xi(e)^c : e \in E\}$, where $\xi(e)^c = U \backslash \xi(e)$.

Example 2.3 [5] [6] [9]**:** Assume that there are five cars in $U = \{c_1, c_2, c_3, c_4, c_5\}$ under consideration, and let E be set of parameters$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.
The e_i for $1 \leq i \leq 8$ stand for the parameters such that the e_1 is expensive, $e_2$is beautiful,

e_3 is a sport, e_4 is cheap, e_5 is luxury, e_6 is modern, e_7 is in good repair, and e_8 is in bad repair.
Consider the function ξ which is given by ''car (.)''; (.) is to be the filled in by one of the parameters$e_i \in E$.

$$L = \{e_2, e_3, e_4, e_5, e_7\}$$

$\xi(e_2) = \{c_2, c_3, c_5\}, \xi(e_3) = \{c_2, c_4\}, \xi(e_4) = \{c_1\}, \xi(e_5) = U, \xi(e_7) = \{c_3, c_5\}$.

$$A = \{e_1, e_2, \dots, e_7\}$$

$\xi'(e_1) = \{c_3, c_5\}, \xi'(e_2) = \{c_4\}, \xi'(e_3) = \{c_2, c_4\}, \xi'(e_4) = \{c_1\}, \xi'(e_5) = \{c_2, c_3, c_4, c_5\}$,
$\xi'(e_6) = \xi'(e_7) = \{c_3\}$.
$(\xi, L) \tilde{\cap} (\xi', A) = (\xi \cap \xi', \{e_3, e_4, e_5, e_7\})$, where $(\xi \cap \xi')(e_3) = \{c_2, c_4\}$,
$(\xi \cap \xi')(e_4) = \{c_1\}, (\xi \cap \xi')(e_5) = \{c_2, c_3, c_4, c_5\}, (\xi \cap \xi')(e_7) = \{c_3\}$.
$(\xi, L) \tilde{\cup} (\xi', A) = (\xi \cup \xi', \{e_1, e_2, \dots, e_7\})$, where $(\xi \cup \xi')(e_1) = \{c_3, c_5\}$,
$(\xi \cup \xi')(e_2) = \{c_2, c_3, c_4, c_5\}, (\xi \cup \xi')(e_3) = \{c_2, c_4\}, (\xi \cup \xi')(e_4) = \{c_1\}$,
$(\xi \cup \xi')(e_5) = U, (\xi \cup \xi')(e_6) = \{c_3\}, (\xi \cup \xi')(e_7) = \{c_3, c_5\}$.
$(\xi, L)^c = (\xi^c, \{e_1, e_2, e_3, e_4, e_6, e_7, e_8\})$, where $\xi^c(e_1) = U, \xi^c(e_2) = \{c_1, c_4\}$,
$\xi^c(e_3) = \{c_1, c_3, c_5\}, \xi^c(e_4) = \{c_2, c_3, c_4, c_5\}, \xi^c(e_6) = U, \xi^c(e_7) = \{c_1, c_2, c_4\}, \xi^c(e_8) = U$.

Definition 2.4 [10]: Let G be a group and A be a nonempty subset of parameters$E$. For the soft set (ξ, A) over G, (ξ, A) is said to be a soft group over G if and only if $\xi(e) \leq G$, for all $e \in A$.

Example 2.5 [8]**:** Assume that $G = A = S_3 = \{e, (12), (13), (23), (123), (132)\}$ and $\xi(e) = \{e\}, \xi(12) = \{e, (12)\}, \xi(13) = \{e, (13)\}, \xi(23) = \{e, (23)\}$,
$\xi(123) = \xi(132) = \{e, (123), (132)\}$. Then (ξ, A) is a soft group over$G$, since $\xi(e)$ is a subgroup of G for all $e \in A$.

Remarks: Let G be a group and (ξ, A) be a soft group over$G$. For each $e \in A$

i. The (ξ, A) is referred to be the identity soft group over G if $\xi(e) = \{\varepsilon\}$, where ε is the identity element of G.
ii. The (ξ, A) is referred to be an absolute soft group over G if $\xi(e) = G$, i.e. $A = E$.
iii. We denote the soft group by $\mathbb{G} = (\xi, A)$.

Definition 2.6 [8]: Let $\mathbb{G} = (\xi, A)$ be a soft group over G and $\mathbb{H} = (\xi', B)$ be a soft subgroup of a soft group $\mathbb{G} = (\xi, A)$. If $\xi'(e)$ is a normal subgroup of $\xi(e)$ for all $e \in B$, then $\mathbb{H}$ is said to be a normal soft subgroup of $\mathbb{G}$ which is written as $\mathbb{H} \lhd \mathbb{G}$,

Definition 2.7 [11]**:** Let $\mathbb{X}$ be a soft set. A bornology on $\mathbb{X}$ is a family $\tilde{\beta} \sqsubseteq P(\mathbb{X})$, if the following conditions hold:

i. $\tilde{\beta}$ covers $\mathbb{X}$, i.e. $\mathbb{X} = \tilde{\bigcup}_{\tilde{B} \in \tilde{\beta}} \tilde{B}$, where $\tilde{B}$ is a soft bounded set;
ii. $\tilde{\beta}$ is stable under hereditary. If $\tilde{B} \in \tilde{\beta}$, $\exists\ \tilde{A} \tilde{\sqsubseteq} \tilde{B}$, then $\tilde{A} \in \tilde{\beta}$;
iii. $\tilde{\beta}$ is stable under finite soft union.

Then $\tilde{\beta}$ is called soft bornology on $\mathbb{X}$ and the pair $(\mathbb{X}, \tilde{\beta})$ is a *soft bornological set*.
The elements of soft bornology are called soft bounded sets. We denote a soft bornological set by $(\mathbb{X}, \tilde{\beta})$.
Example 2.8 [11]**:** Suppose that $\mathbb{X}$ is an infinite soft set and let $\tilde{\beta}$ be a family of all subsets of $\mathbb{X}$ which has an infinite complement, such that $\tilde{\beta} = \{\tilde{B} \tilde{\sqsubseteq} \mathbb{X}: \tilde{B}^c\ is\ infinite\}$.

Then $\tilde{\beta}$ is a soft bornology on $\mathbb{X}$. To prove that $\tilde{\beta}$ is a soft bornology on $\mathbb{X}$. We must satisfy three conditions:

i. $\forall\, x \in \mathbb{X}, \{x\}$ is a finite soft set belonging to $\tilde{\beta}$. Also, then $\{x\}^c$ is infinite soft set. So, $\tilde{\beta}$ covers $\mathbb{X}$;

ii. Let $\tilde{B} \in \tilde{\beta}, \tilde{A} \tilde{\subseteq} \tilde{B}$ then $\tilde{A}$ is a finite soft set (every subset of finite soft set is finite soft set), $\tilde{A}^c$ is infinite soft set. So, $\tilde{A} \in \tilde{\beta}$.(Hereditary property);

iii. Le t $\tilde{B}_1, \tilde{B}_2, \dots, \tilde{B}_n \in \tilde{\beta}$, then $\tilde{B}_1^c, \tilde{B}_2^c, \dots, \tilde{B}_n^c$ is infinite soft sets. To prove that

$\tilde{\bigcup}_{1 \le i \le n} \tilde{B}_i$ hat means, we have to show that $(\tilde{\bigcup}_{1 \le i \le n} \tilde{B}_i$ is an infinite soft set.

$\tilde{\bigcup}_{1 \le i \le n} \in \tilde{B}\tilde{\beta}$ is an finite soft set (The finite union of the finite soft set is a finite soft set).

Since $(\tilde{\bigcup}_{1 \le i \le n} \tilde{B}_i)^c = \tilde{\bigcap}_{1 \le i \le n} \tilde{B}_i^c$ infinite soft set.

(The finite union of the infinite soft set is infinite soft set).(By De Morgan: The complement of the union of sets is equal to the intersection of their complements).

So, $(\tilde{\bigcup}_{1 \le i \le n} \tilde{B}_i)^c$ infinite soft. Thus, $\tilde{\beta}$ stable under finite soft union, $(\mathbb{X}, \tilde{\beta})$ is soft bornological set.

3.Soft bornological group

To solve the problem of soft groups we construct a new structure that is called a soft bornological group.

Definition 3.1: Let $\mathbb{G}$ be a soft group. We say that $(\mathbb{G}, \tilde{\beta})$ is a ***soft bornological group*** if $\tilde{\beta}$ is a soft bornology on $\mathbb{G}$, and the following conditions hold:

1. The product mapping $(\mathbb{G}, \tilde{\beta}) \times (\mathbb{G}, \tilde{\beta}) \to (\mathbb{G}, \tilde{\beta})$ is soft bounded;
2. The inverse mapping $(\mathbb{G}, \tilde{\beta}) \to (\mathbb{G}, \tilde{\beta})$ is soft bounded.

Example 3.2: Assume that $\mathbb{G}$ is absolute soft group over G, where $G = (Z_3, +_3), E = (Z_3, +_3)$. We can define a soft finite bornology $\tilde{\beta}$ in this group. Which is the collection of all finite soft subsets of $\mathbb{G}$. To prove $\mathbb{G}$ with soft finite bornology $\tilde{\beta}$ is a soft bornological group $(\mathbb{G}, \tilde{\beta})$. We have to prove that the product map and inverse map are soft bounded.

Now, let $\psi: (\mathbb{G}, \tilde{\beta}) \times (\mathbb{G}, \tilde{\beta}) \to (\mathbb{G}, \tilde{\beta})$ and $\tilde{B}_x$, $\tilde{B}_y$ be two soft bounded sets so that we have to prove that $\psi(\tilde{B}_x \times \tilde{B}_y)$ is a soft bounded, where

$\psi(\tilde{B}_x \times \tilde{B}_y) = \{x \tilde{+} y \in \tilde{B}_x \tilde{+} \tilde{B}_y \tilde{\subseteq} \tilde{B}_{x+y} : x \in \tilde{B}_x, y \in \tilde{B}_y\}$.

Since $\mathbb{G}$ is absolute soft group. Thus, the image for every two soft bounded sets $\tilde{B}_x, \tilde{B}_y$ under ψ is soft bounded set;

Let the inverse map $\psi^{-1}: (\mathbb{G}, \tilde{\beta}) \to (\mathbb{G}, \tilde{\beta})$ and $\tilde{B}_x \in \tilde{\beta}, \tilde{B}_x = \{x : x \in \tilde{B}_x\}$. So that $\tilde{B}_{x^{-1}} = \{x^{-1}: x^{-1} \in (\mathbb{G}, \tilde{\beta})\}$. Thus, ψ^{-1} is soft bounded. Therefore, $(\mathbb{G}, \tilde{\beta})$ is a soft bornological group.

Definition 3.3: Let $(\mathbb{G}_1, \tilde{\beta}_1)$ and $(\mathbb{G}_2, \tilde{\beta}_2)$ be two soft bornological groups. A homomorphism of soft bornological groups is a group homomorphism that is also soft bounded.

Remark 3.4: If $(\mathbb{G}, \tilde{\beta})$ is a soft bornological group, then $\tilde{B} \in \tilde{\beta}$ if and only if $\tilde{B}^{-1} \in \tilde{\beta}$.

Theorem 3.5: Let $(\mathbb{G}, \tilde{\beta})$ be a soft bornological group if and only if for each $g_1, g_2 \in \mathbb{G}$ and each soft bounded sets $\tilde{B}_1, \tilde{B}_2$ containing g_1, g_2 respectively, there is a soft bounded set $\tilde{B}$ contains $g_1 * {g_2}^{-1}$ in $\mathbb{G}$ such that $\tilde{B}_1 * \tilde{B}_2{}^{-1} \tilde{\subseteq} \tilde{B}$.

Proof: Since $(\mathbb{G}, \tilde{\beta})$ is a soft bornological group with respect to soft bounded set, that means (g_1, g_2) is to be a point in $\mathbb{G} \times \mathbb{G}$, let $\tilde{B}_1, \tilde{B}_2$ both soft bounded sets containing g_1, g_2, respectively. Then there exist a soft bounded set $\tilde{B}$ containing $\psi(g_1, g_2) = g_1 * {g_2}^{-1}$ in $\mathbb{G}$, such that $\tilde{B}_1 * \tilde{B}_2{}^{-1} \tilde{\subseteq} \tilde{B}$. From Remark 3.4, $\tilde{B}_2{}^{-1}$is a soft bounded set containing ${g_2}^{-1}$, and $\tilde{B}_1 \times \tilde{B}_2$ is soft bounded set containing (g_1, g_2). So, $\psi(\tilde{B}_1 \times \tilde{B}_2) \tilde{\subseteq} \tilde{B}$. This means that the product map is a soft bounded map.

To show the inverse map ϕ is a soft bounded map, let $g \in \mathbb{G}$ and let $\tilde{B}$ be a soft bounded set containing g such that $\phi(g) = g^{-1}$. Thus, by Remark 3.4 , $\tilde{B}^{-1}$ is soft bounded set containing g^{-1} satisfies $\phi(\tilde{B}) = \tilde{B}^{-1}$ which means that ϕ is a soft bounded map at g, soon $\mathbb{G}$ is a soft bornological group. The converse is clear from the concept of the soft bornological group.

Proposition 3.6: If g is an arbitrary element of a soft group $\mathbb{G}$, then

i. $\psi_L(g): \mathbb{G} \to \mathbb{G}$; $x \longmapsto gx$ (respectively $\psi_R(g): \mathbb{G} \to \mathbb{G}; x \longmapsto xg$) is a soft bornological isomorphism.

ii. $\phi(g): \mathbb{G} \to \mathbb{G}, x \longmapsto gxg^{-1}$ is a soft bornological isomorphism.

Proof: It is known that the left translation is bijective so it has an inverse. It is enough to show that the left is a soft bounded set. For any $x, y \in \mathbb{G}$ and a soft bounded sets $\tilde{B}_x$ and $\tilde{B}_y$ containing x, y respectively such that $\tilde{B}_x \cdot \tilde{B}_y \tilde{\subseteq} \tilde{B}$ by definition of soft bornological group, there exist a soft bounded set $\tilde{B}$ contains xy. Hence, we have $\psi_L(x)(\tilde{B}_y) = x \cdot \tilde{B}_y \tilde{\subseteq} \tilde{B}_x \cdot \tilde{B}_y \tilde{\subseteq} \tilde{B}$.

Proposition 3.7: Let $\mathbb{H} \tilde{\subseteq} \mathbb{G}$ be a subgroup. $\tilde{\mathbb{H}}$ is a soft bounded set if and only if there exist $h \in \mathbb{H}$ and a soft bounded $\tilde{B}$ contains h such that $\tilde{\mathbb{H}} \tilde{\subseteq} \tilde{B}$.

Proof: From the hereditary property, the proof can be get.

Definition 3.8: Let $(\mathbb{G}, \tilde{\beta})$ be a soft bornological group, if $\mathbb{H} \tilde{\subseteq} \mathbb{G}$, then $(\mathbb{H}, \tilde{\beta}_{\mathbb{H}})$ is called a ***soft bornological subgroup*** of $\mathbb{G}$ if the following conditions hold:

i. $(\mathbb{H}, *)$ is a soft subgroup of a soft group $(\mathbb{G}, *)$.

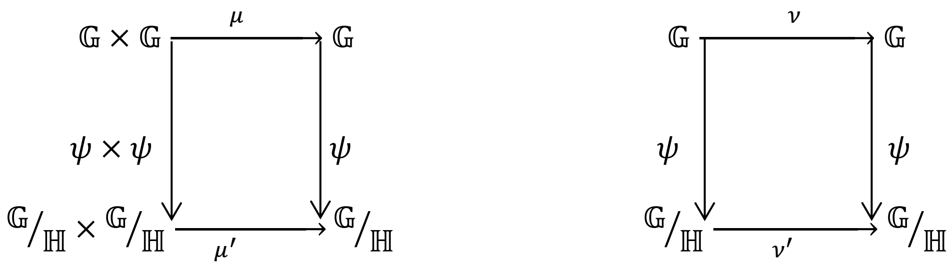
ii. $(\mathbb{H}, \tilde{\beta}_{\mathbb{H}})$ is a soft bornological subspace of $(\mathbb{G}, \tilde{\beta})$.

Theorem 3.9: If $(\mathbb{G}, \tilde{\beta})$ is a soft bornological group, and $\mathbb{H}$ is a normal soft subgroup of $\mathbb{G}$, then ${}^{\mathbb{G}}/_{\mathbb{H}}$ is a soft bornological group.

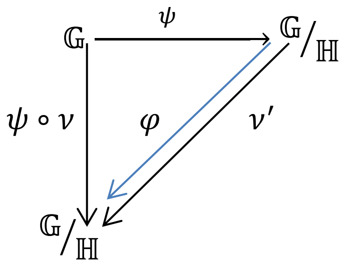
Proof:Let $\mathbb{G}$ be a soft bornological group and $\mathbb{H}$ be a normal soft subgroup of $\mathbb{G}$ then ${}^{\mathbb{G}}/_{\mathbb{H}}$ has a quotient group structure and the function $\psi: \mathbb{G} \to {}^{\mathbb{G}}/_{\mathbb{H}}$, $g \longmapsto g\mathbb{H}$ defines a soft quotient bornology on ${}^{\mathbb{G}}/_{\mathbb{H}}$.

As well as, the quotient function ψ is soft bounded, if $\tilde{A}$ is soft bounded in $\mathbb{G}$, then $\psi(\tilde{A}) = \tilde{A}\mathbb{H}$.Therefore $\psi(\tilde{A})$ is soft bounded in ${}^{\mathbb{G}}/_{\mathbb{H}}$.

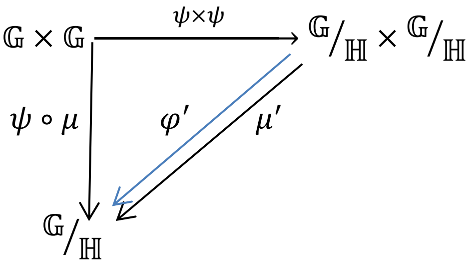
If μ and μ' are the multiplication in $\mathbb{G}$ and $\mathbb{G}/_{\mathbb{H}}$, and $\nu, \nu'$are the inversion in $\mathbb{G}$ and $\mathbb{G}/_{\mathbb{H}}$, respectively. Then, μ' andν' are uniquely defined by the following commutative diagrams:



To demonstrate that μ', ν' are soft bounded function. By using the following diagram:



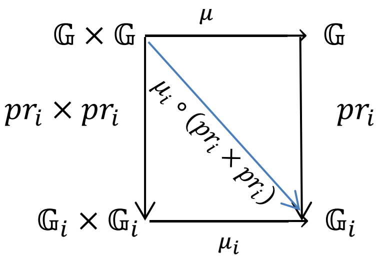
$\psi \circ \nu$ is soft bounded and ψ is soft quotient function, so by the universal property of ψ, there exists a unique soft bounded function φ: $\mathbb{G}/_{\mathbb{H}} \to \mathbb{G}/_{\mathbb{H}}$ making $\varphi \circ \psi = \psi \circ \nu$.
But ν' satisfies this condition $\nu' \circ \psi = \psi \circ \nu$, so $\nu' = \varphi$. So, ν' is soft bounded.
By using the following diagram:



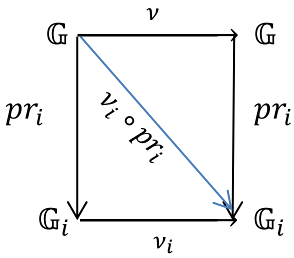
Since $\psi \times \psi$ is soft bounded and onto function, hence $\psi \times \psi$ is soft quotient function since $\psi \circ \mu$ is soft bounded, so by universal property of $\psi \times \psi$ there exists a unique soft bounded function $\varphi' \circ (\psi \times \psi) = \psi \circ \mu$. But μ' satisfies the condition $\mu' \circ (\psi \times \psi) = \psi \circ \mu$. Then, $\mu' = \varphi'$, and so μ' is soft bounded.

Definition 3.10: The soft bornological group $\mathbb{G}/_{\mathbb{H}}$ is called the soft quotient bornological group.

Theorem 3.11: The product of soft bornological groups is a soft bornological group.
Proof: Let $\{\mathbb{G}_i\}_{i\in I}$ be a family of soft bornological groups, their product $\mathbb{G} = \prod_{i\in I} \mathbb{G}_i$ has a naturally occurring group structure (product of groups). And a delicate natural bornology (product of soft bornologies). By utilizing the commutative diagram below:



Then $\mu_i \circ (pr_i \times pr_i)$ is soft bounded. Since μ_i and $pr_i \times pr_i$ are soft bounded. Therefore $pr_i \circ \mu = \mu_i \circ (pr_i \times pr_i)$ is soft bounded.
Now, by using the following commutative diagram:



$\nu_i \circ pr_i$ is soft bounded since ν_i and pr_i are soft bounded. Hence, $pr_i \circ \nu = \nu_i \circ pr_i$ is soft bounded. Therefore ν is soft bounded. So, $\mathbb{G} = \prod_{i \in I} \mathbb{G}_i$ is a soft bornological group.

Definition 3.12: The soft bornological group $\mathbb{G} = \prod_{i \in I} \mathbb{G}_i$ is called the product soft bornological group.

4. Soft bornological group action

In this section, we show that the soft bornological set is partitioned into orbital classes by acting soft bornological group on the soft bornological set, In addition, we explain the centralizer, the normalizer, and the stabilizer in details.

Definition 4.1: A soft bornological group action is a triple $(\mathbb{G}, \mathbb{X}, \theta_e)$ where $(\mathbb{G}, \tilde{\beta})$ a soft bornological group and $(\mathbb{X}, \tilde{\beta}')$ a soft bornological set and $\theta_e: \mathbb{G} \times \mathbb{X} \to \mathbb{X}$ is soft bounded if the following properties are satisfied:

i. $\theta_e(\varepsilon, x) = x$, for all $x \in \mathbb{X}, \varepsilon \in \mathbb{G}$, where ε is the identity soft element of $\mathbb{G}$.
ii. $\theta_e(g, \theta_e(h, x)) = \theta_e(gh, x)$, for all $g, h \in \mathbb{G}, x \in \mathbb{X}$.

We say that the soft bornological group $\mathbb{G}$ acts on a soft bornological set. $\mathbb{X}$ is called $\mathbb{G}$-soft bornological set. Further, the notation $g \cdot x$ (or gx) will be used for $\theta_e(g, x)$, so that (i) and (ii) in Definition 4.1, become $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $\varepsilon \cdot x = x$. If $\tilde{B}$ is soft bounded of soft bornological group $\mathbb{G}$ and $\tilde{A}$ is a soft bounded subset of soft bornological set $\mathbb{X}$, we put

$$\tilde{B} \cdot \tilde{A} = \{g \cdot x : g \in \tilde{B}, x \in \tilde{A}\} \tilde{\subseteq} \mathbb{X}.$$

Definition 4.2: If a soft bornological set $(\mathbb{X}, \tilde{\beta}')$ is a soft bornological set, then $(\mathbb{G}, \tilde{\beta}')$ is said to be a soft bornological transformation group on $(\mathbb{X}, \tilde{\beta}')$.

Example 4.3: If $\mathbb{G}$ is soft bornological group and $\mathbb{X} = \mathbb{G}$, then define the mapping
$\theta_e: \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, such that

- $\theta_e(\varepsilon, g) = \varepsilon \cdot g = g, \forall g \in \mathbb{G}$ and ε identity element of $\mathbb{G}$.
- $\theta_e\big(g, \theta_e(g_1, g_2)\big) = \theta_e(g, g_1 g_2) = g \cdot (g_1 g_2) = (g g_1) \cdot g_2$

$= \theta_e(g g_1, g_2), \forall\, g, g_1, g_2 \in \mathbb{G}$.

We say that θ_e is soft bounded action. Thus, each soft bornological group $\mathbb{G}$ acts on itself by translation.

Definition 4.4: If $\mathbb{X}$ is a $\mathbb{G}$- soft bornological set and $\tilde{B}$ is a soft bounded subset of $\mathbb{X}$. Then a subset $\tilde{B}$ of $\mathbb{X}$ is called *an invariant soft bornological set* under the actions of soft bornological group $\mathbb{G}$ if $\mathbb{G} \cdot \tilde{B} = \tilde{B}$.

Example 4.5: Suppose that $(\mathbb{Z}, \tilde{\beta}_{fin})$ is a soft finite bornological group and $\mathbb{X} = (\mathbb{R}, \tilde{\beta}_{|\cdot|})$ is a soft usual bornological set. Let the soft bornological group $\mathbb{Z}$ act on the soft bornological set $\mathbb{R}$ by $\theta_e(z, r) = z \tilde{+} r, \quad for\ all\ \ z \in \mathbb{Z}, r \in \mathbb{R}$.

Then $\mathbb{R}$ is $\mathbb{Z}$- soft bornological set and $\mathbb{Q}$ the set of all rational numbers as a subset of $\mathbb{R}$ is an invariant soft bornological set such that

$$\mathbb{Z} \tilde{\cdot} \mathbb{Q} = \{z \tilde{+} q : z \in \mathbb{Z}, q \in \mathbb{Q}\} = \mathbb{Q}.$$

Proposition 4.6: Let $(\mathbb{X}, \tilde{\beta}_1)$ be a soft bornological set and $(\mathbb{Y}, \tilde{\beta}_2)$ be a soft bornological set. If $(\mathbb{X}, \tilde{\beta}_1)$ and $(\mathbb{Y}, \tilde{\beta}_2)$ are two $\mathbb{G}$- soft bornological sets, then $(\mathbb{X} \times \mathbb{Y}, \tilde{\beta}_1 \times \tilde{\beta}_2)$ is $\mathbb{G}$- soft bornological set.

Proof: Suppose $(\mathbb{G}, \tilde{\beta})$ is a soft bornological group. Consider $\mathbb{G}$- soft bornological set $(\mathbb{X}, \tilde{\beta}_1)$ with the soft (bounded) action

$$\theta_e : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$$

and $\mathbb{G}$- soft bornological set $(\mathbb{Y}, \tilde{\beta}_2)$ with the soft (bounded) action

$$\phi_e : \mathbb{G} \times \mathbb{Y} \to \mathbb{Y}$$

Then, we are a soft (bounded) action denoted as

$$\varphi_e : \mathbb{G} \times (\mathbb{X} \times \mathbb{Y}) \to (\mathbb{X} \times \mathbb{Y}).$$

That means, $\big(g, (x, y)\big) \mapsto \ \varphi_e\big(g, (x, y)\big) = \big(\theta_e(g, x), \phi_e(g, y)\big)$.

- $\varphi_e\big(g_1, \varphi_e(g_2, (x, y))\big) = \varphi_e(g_1,\ g_2.(x, y))$

$$= g_1.\big(g_2.(x, y)\big)$$
$$= g_1 g_2.(x, y)$$
$$= \varphi_e\big(g_1 g_2, (x, y)\big).\ \forall\, g_1, g_2 \in \mathbb{G}, (x, y) \in \mathbb{X} \times \mathbb{Y}.$$

- $\varphi_e\big(\varepsilon, (x, y)\big) = \varepsilon.(x, y) = (x, y)$.

It easy to say that φ_e is a soft (bounded) action of $(\mathbb{G}, \tilde{\beta})$ on $(\mathbb{X} \times \mathbb{Y}, \tilde{\beta}_1 \times \tilde{\beta}_2)$.

Proposition 4.7: For the $\mathbb{G}$-soft bornological set, if $x \in \mathbb{X}, g \in \mathbb{G}$, and $y = \theta_e(g, x)$, then $x = \theta_e(g^{-1}, y)$. Moreover, if $x \neq x'$ then $\theta_e(g, x) \neq \theta_e(g, x')$.

Proof: Let $y = \theta_e(g, x)$ such that

$$\theta_e(g^{-1}, y) = \theta_e\big(g^{-1}, \theta_e(g, x)\big) = \theta_e(g^{-1} g, x) = \theta_e(\varepsilon, x) = x.$$

Assume that $x \neq x'$, so $\theta_e(g, x) = \theta_e(g, x')$.

When g^{-1} is applied to both sides, it follows that

$$\theta_e\big(g^{-1}, \theta_e(g, x)\big) = \theta_e(g^{-1}, \theta_e(g, x'))$$
$$\theta_e(g^{-1} g, x) = \theta_e(g^{-1} g, x')$$
$$\theta_e(\varepsilon, x) = \theta_e(\varepsilon, x')$$
$$x = x'$$

This contradicts by the hypothesis $x \neq x'$. Hence, $\theta_e(g, x) \neq \theta_e(g, x')$.

Definition 4.8: A soft bornological group $(\mathbb{G}, \tilde{\beta})$ acts on itself by conjugation $(\theta_e)_g(h) = ghg^{-1}$, and for all $h \in \mathbb{G}$, the soft centralizer of $\mathbb{G}$ is a set defined by

$$C(e) = \{g \in \mathbb{G}: \theta_e(g, h) = \theta_e(h, g)\}.$$

Proposition 4.9: The soft centralizer $C(e)$, as stated previously, is a soft bornological subgroup of$\mathbb{G}$.

Proof: Consider

$$C_{\mathbb{G}}: E \to P(G)$$

It is easy to check that $C(e)$ is a subgroup of$\mathbb{G}$. $(C_{\mathbb{G}}, \tilde{\beta}_{C_{\mathbb{G}}})$ is a soft bornological subspace of $\mathbb{G}$. Then, the pair $(C_{\mathbb{G}}, \tilde{\beta}_{C_{\mathbb{G}}})$ is a soft bornological subgroup of $\mathbb{G}$.

Definition 4.10: Let the soft bornological group $\mathbb{G}$ act on itself by conjugation and let$\mathbb{H} \tilde{\subseteq} \mathbb{G}$. The soft stability subgroup is defined as follows:

$$N(e) = \{g \in \mathbb{G}: \theta_e(g, h) = h, h \in \mathbb{H}\}$$

It is called a soft normalizer of $\mathbb{H}$.
From this definition, we can get the following proposition:

Proposition 4.11: The soft normalizer $N(e)$ that is given above is the soft bornological subgroup of $\mathbb{G}$.

Proof: We define

$$N_{\mathbb{H}}: E \to P(G)$$

Then, it is simple to confirm that $N(e)$ is a subgroup of $\mathbb{G}$. $(N_{\mathbb{H}}, \tilde{\beta}_{N_{\mathbb{H}}})$ is a soft bornological subspace of $\mathbb{G}$. Hence, the pair $(N_{\mathbb{H}}, \tilde{\beta}_{N_{\mathbb{H}}})$ is a soft bornological subgroup of $\mathbb{G}$.

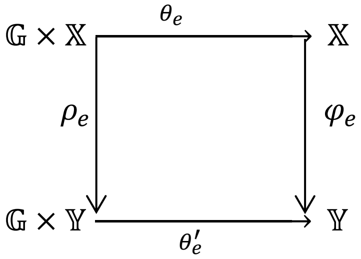
Definition 4.12: A soft (bounded) action of the soft bornological group $(\mathbb{G}, \tilde{\beta})$ on the soft bornological set $(\mathbb{X}, \tilde{\beta}')$ is said to be:

i. Transitive if for each pair $x, y \in \mathbb{X}$ there exists an element g in $\mathbb{G}$ such that $\theta_e(g, x) = x$.
ii. Effective (or faithful) if for each two disjoint elements $g, h \in \mathbb{G}$ there is an element $x \in \mathbb{X}$ such that $\theta_e(g, x) \neq \theta_e(h, x)$.
iii. Free if for given $g, h \in \mathbb{G}$, the existence of an element $x \in \mathbb{X}$ with $\theta_e(g, x) = \theta_e(h, x)$ implies $g = h$.

Definition 4.13: Suppose $(\mathbb{X}, \tilde{\beta}_1), (\mathbb{Y}, \tilde{\beta}_2)$ are two soft bornological sets. Also, let $(\mathbb{X}, \tilde{\beta}_1)$ and $(\mathbb{Y}, \tilde{\beta}_2)$ are $\mathbb{G}$- soft bornological sets with the soft (bounded) action θ_e and θ'_e. Then, the map

$$\varphi_e: \mathbb{X} \to \mathbb{Y}$$

is defined by $\varphi_e\big(\theta_e(g, x)\big) = \theta'_e\big(g, \varphi_e(x)\big)$
is said to be $\mathbb{G}$- equivariant soft bounded map.
More shortly $\varphi_e(g.x) = g.\varphi_e$.
It is clear that φ_e is an equivariant map if and only if the diagram below is commutative



where $\rho_e = id_{\mathbb{G}} \times \varphi_e$ is the product of identity mapping $id_{\mathbb{G}}$ of $\mathbb{G}$ and the mapping φ_e. An equivariant map $\varphi_e: \mathbb{X} \to \mathbb{Y}$ which is also an isomorphism of soft bornological sets is called equivalence of $\mathbb{G}$- soft bornological sets.
In this case, we note that the inverse φ_e^{-1} of φ_e is also equivariant.
If $y = \varphi_e(x)$, then

$$\varphi_e^{-1}(g.y) = \varphi_e^{-1}\big(g.\varphi_e(x)\big) = \varphi_e^{-1}\varphi_e(g.x) = g.x = g.\varphi_e^{-1}(y).$$

Definition 4.14: If $(\mathbb{X}, \tilde{\beta})$ is $\mathbb{G}$- soft bornological set. Then, each all soft subset $\mathbb{H} \tilde{\subset} \mathbb{X}$, we define the fixed point set as follows:

$$Fix_{\mathbb{H}}(e) = \{g \in \mathbb{G}: \theta_e(g,x) = x, x \in \mathbb{H}\}$$

In particular, for each $x \in \mathbb{X}$, the fixed point set x is defined by

$$Stab_{\mathbb{X}}(e) = \{g \in \mathbb{G}: \theta_e(g,x) = x\}$$

At this point, we can give the next result.

Proposition 4.15: The sets $Fix_{\mathbb{H}}(e)$ and $Stab_{\mathbb{X}}(e)$ that are defined above are soft bornological subgroups of $\mathbb{G}$.

Proof: We have the map

$$Fix_{\mathbb{H}} : E \to P(G).$$

It is clear that $Fix_{\mathbb{H}}(e)$ is a subgroup of $\mathbb{G}$. Then, $(Fix_{\mathbb{H}},, \tilde{\beta}_{Fix\ \mathbb{H}})$ is soft bornological subgroup of $\mathbb{G}$. We analogously consider the map

$$Stab_{\mathbb{X}}: E \to P(G).$$

Which implies that $Stab_{\mathbb{X}}(e)$ is a subgroup of $\mathbb{G}$. Therefore, the pair $(Stab_{\mathbb{X}}, \tilde{\beta}_{Stab\ \mathbb{X}})$ is a soft bornological subgroup of $\mathbb{G}$.

Proposition 4.16: For $Fix_{\mathbb{H}}(e)$ and $Stab_{\mathbb{X}}(e)$ that are given above, the following equality holds:
$Fix_{\mathbb{H}}(e) = \tilde{\cap}_{x\in\mathbb{H}}\, Stab_{\mathbb{X}}(e)$.

Proof: Consider $g \in Fix_{\mathbb{H}}(e)$. Then, $\theta_e(g,x) = x$ for all $x \in \mathbb{H}$, which implies that $g \in Stab_{\mathbb{X}}(e)$. That is, $g \in \tilde{\cap}_{x\in\mathbb{H}}\, Stab_{\mathbb{X}}(e)$ and so $Fix_{\mathbb{H}}(e) \tilde{\subseteq} \tilde{\cap}_{x\in\mathbb{H}}\, Stab_{\mathbb{X}}(e)$ is obtained.
Conversely, if we take $g \in \tilde{\cap}_{x\in\mathbb{H}}\, Stab_{\mathbb{X}}(e)$. Then, $g \in Stab_{\mathbb{X}}(e)$ for all $x \in \mathbb{H}$ and so $\theta_e(g,x) = x$.
Thus, it is verified that $g \in Fix_{\mathbb{H}}(e)$. The proof is now completed.
Definition 4.17: If $(\mathbb{X}, \tilde{\beta}')$ is a $\mathbb{G}$- soft bornological set, and for each$x \in \mathbb{X}$. Then, the soft orbit of x under the action $\mathbb{G}$ is the subset

$$\mathbb{G}(x) = \{y \in \mathbb{X}: \ there\ exists\ g \in \mathbb{G}\ s.t, y = g.x\}.$$

A generalization of this, for any soft bounded set $\tilde{B} \tilde{\subseteq} \mathbb{X}$, the union of all orbits of points of $\tilde{B}$ is

$$\mathbb{G}(\tilde{B}) = \{g \cdot b : g \in \mathbb{G}, b \in \tilde{B}\}.$$

For a soft bounded set$\tilde{B} \tilde{\subseteq} \mathbb{X}$, and subgroup $\mathbb{H}$ of $\mathbb{G}$, we put

$$\mathbb{H}(\tilde{B}) = \{h \cdot b: h \in \mathbb{H}, b \in \tilde{B}\}.$$

The action θ_e of $\mathbb{G}$ on $\mathbb{X}$ defines an equivalence relation as follows: For each $x, y \in \mathbb{X}, xRy$ if and only if there exists $g \in \mathbb{G}$ such that

$$\theta_e(g, x) = g \cdot x = y.$$

The equivalence classes with respect to this equivalence relation are the orbits of the elements of $\mathbb{X}$. Let $\mathbb{X}/R$ denote the set of orbits$\mathbb{G}(x)$, and $\pi_e: \mathbb{X} \to \mathbb{X}/R$ be the natural map taking x into its orbit$\mathbb{G}(x)$.

If we soft bornologize $\mathbb{X}/R$ by the soft quotient bornology, then the soft bornological set $\mathbb{X}/R$ is said to be the orbital soft bornological set of $\mathbb{X}$ (with respect to $\mathbb{G}$), or the soft quotient bornological set by $\mathbb{G}$ and is denoted by

$$\mathbb{X}/R = \{\mathbb{G}(x): x \in \mathbb{X}\}.$$

Equivalently, the soft bornological group $\mathbb{G}$ acts on a soft bornological set $\mathbb{X}$ and $\mathbb{Y} = \mathbb{X}/\mathbb{G}$ is the corresponding orbit set. Let $\mathbb{Y}$ carry the soft quotient bornological generated by the orbital projection

$$\pi_e: \mathbb{X} \to \mathbb{X}/\mathbb{G}.$$

That means a set $\tilde{B}_1 \tilde{\subset} \mathbb{Y}$ is a soft bounded set in $\mathbb{Y}$ if and only if it is the image of soft bounded set $\tilde{B}$ in $\mathbb{X}$, such that $\tilde{B}_1 = \pi_e(\tilde{B})$. The soft bornological set $\mathbb{X}/\mathbb{G}$ so, obtained is called the orbital soft bornological set or the orbit space of the $\mathbb{G}$- soft bornological set $\mathbb{X}$.

Conclusions:

The aim of this paper is to consider a soft bornological group such that the product and inverse map are soft bounded. We also explained a soft bornological action. When a soft bornological group acts on a soft bornological set, this process is called soft bounded action such that the effect of the soft bounded action is to partition a soft bornological set into a class of soft orbital.

References

[1] H. Hogbe-Nlend, *Bornologies and Functional Analysis, Netherlands: North-Holland Publishing Company*, 1977.

[2] Anwar N. Imran, Bornological Structures on Some Algebraic Systems, Universiti Purta Malaysia PHD Thesis, 2018.

[3] D. P. Pombo Jr., “Bornological Groups,” *Indian J. Math,* vol. 54, pp. 225-258, 2012.

[4] F. Bambozzi, “On a Generalization of Affinioid Varieties,” 2013.

[5] D. Molodtsov, Soft set theory-First results, Computers and Mathematics with Applications, 1999.

[6] P.K. Maji, R. Biswas ,and R. Roy, “An application of soft sets in a decision making problem,” *Computers and Mathematics with Applications,* vol. 44, pp. 1077-1083, 2002.

[7] P.K. Maji, R. Biswas,and R. Roy, “Soft set theory,” *Computers and Mathematics with Applications,* vol. 45, pp. 555-562, 2003.

[8] H. Aktas and N. Cagman, “Soft sets and soft groups,” *Information Sciences1,* vol. 77, pp. 2726-2735, 2007.

[9] I. A. Onyeozili and T.M. Gwary , “A study of The Fundamentals of Soft Set Theory,” *International Journal of Scientific Technology,* vol. 3, 2014.

[10] S. Ray and S Goldar, “Soft Set and Soft Groupsfrom Classical View Point,” *Journal of Indian Mathematical Society,* vol. 84, 2017.

[11] Abdal, F. Ashwaq , and Anwar N. Imran, “Soft Bornological Structures,” *Iraqi Journal of Science, Sumbitted* 2022.