DOI: 10.24996/ijs.2023.64.6.35

ISSN: 0067-2904

The Calculations of Wiener μ-invariant on the Corona Graph

Kavi B. Rasool[1*], Payman A. Rasheed[2]
[1] Department of Mathimatics, Faculty of Scinece, Zakho University, Duhok, Iraq
[2] Department of Mathimatics, College of Basic Education, University of Salahaddin, Erbil, Iraq

Received: 11/3/2022 Accepted: 8/11/2022 Published: 30/6/2023

Abstract.
A topological index is a real number that relates to a graph that must be a structural invariant. In this paper, we first define a new graph, which is a concept from the coronavirus, called a corona graph. We give some theoretical results for the Wiener and the hyper Wiener index of a graph. Moreover, we calculate some topological indices degree-based on a corona graph. In addition, we are introducing a new topological index SK_4, which is inspired by the definition of the SK_1 index.

Keywords: Winer, Hyper Winer, Corona graph, Zagreb indices, Gourava indices.

حساب قيم μ – لمؤشر وينر للبيان كورونا

كفي رسول*[1]، بيمان رشيد[2]
[1]قسم الرياضيات، كلية العلوم ، جامعة زاخو، دهوك، العراق
[2]قسم الرياضيات ، كلية التربية، جامعة صلاح الدين، اربيل ، العراق

الخلاصة
أن المؤشرات التبولوجية هي اعداد حقيقية مرتبطة بالبيانات ذات تركيبات ثابتة. يكمن الهدف من هذا البحث في تعريف بيان جديد لها شكل معين اسميناها بيان الكورونا تيمنا بفايروس الكورونا. لقد وجدنا نتائج نظرية جديدة لمؤشرات وينر, هايبر وينر ومؤشرات تبولوجية اخرى لبيان الكرونا الجديد وقمنا بحساب المؤشرات التبلوجية لاساس درجة الرأس. بالإضافة ان وجدنا مؤشر تبلوجي جديد سميناها بمؤشر SK_4 بالاعتماد على تعريف المؤشر SK_1 التبلوجي المعرف سابقا.

1. Introduction

The topological index of a chemical molecule is a number that can be used to characterize the molecule and forecast specific physiochemical parameters such as boiling point, molecular weight, density, refractive index, and so on [1, 2]. Topological indices are divided into several groups. Some of them are distance-based topological indices, degree-based topological indices, and counting-related polynomials and indices of graphs. The degree-based topological indices are the most essential in chemical graph theory of all the topological indices.

In this work, all graphs are simple and finite undirected, linked, loop less, and without numerous edges. Consider the graph $G = (V, E)$ with p vertices and q edges. The number of vertices adjacent to a vertex $u \in V\ (G)$ is denoted by $d_{gr}(u)$. A distance $d(u, v)$ between two distinct vertices u, v of a graph G is the smallest length of a path between u and v in G. The

*Email: kavi.rasool@uoz.edu.krd

diameter of G, denoted by Di (G), is defined as the greatest distance between any two vertices of G, which is given as follows: $Di\ (G)\ =\ max\{\ d(u,v), \forall\ (u,v) \in\ V(G)\}$. Finally, we define $D_G(\lambda)$ as the number of unordered pairs of vertices in G that are exactly λ distances for non-negative integers $\lambda = 1, 2, \dots, Di(G)$. The Wiener index, named after scientist Harold Wiener, is one of the most well-known topological descriptors, which is the oldest and most thoroughly examined use of topological indices chemistry in the study of paraffin boiling points, and the topological index is called a Wiener index. The wiener index [3] of a connected graph is defined as the total distance between all vertex pairs. The wiener index has been intensively investigated over the last quarter-century because it corresponds strongly with several physicochemical aspects of organic molecules. Zadeh and his co-authors [4] proposed a new topological index called the Wiener-type invariant of G in order to generalize the standard Wiener index. Gutman et al. introduced the first and second Zagreb indices in [5], which initially comes out in a topological formula for the total energy of conjugated molecules. These indexes are branching indices. The Zagreb indices have been used in research of QSPR and QSAR [6, 7]. For historical context, computational methodologies, and mathematical aspects of the Zagreb indices, we recommend reading [8, 9, 10, 11]. Inspired by the formulation of the Zagreb indices and their wide range of applications, Kulli published the first and second Gourava index of a molecular graph [12]. Also, Kulli proposed the multiplicative first and second status Gourava indices of a graph, which were inspired by the multiplicative Gourava indices [13]. New degree-based topological indices were proposed by Shigehalli and Kanabur, namely SK, SK_1 and SK_2 indexes of a molecular graph G [14]. The SK, SK_1 and SK_2 interval-weighted graph indices are then defined [15].

In this paper, we deal with a new graph called a coronagraph. The coronagraph is a graph obtained from cycle C_t with vertices v_i, $i\ = 1$ to t, and then by attaching u_j end vertices, $j = 1$ to s to each of the vertices of C_t and denoted by $\mathcal{CO}_t^s$, where $t\ \geq\ 3$ and $s \geq\ 1$. A corona graph has the same number of vertices and edges as shown in Figure1. Our goal is to compute the previous topological indices for the corona graph $\mathcal{CO}_t^s$.

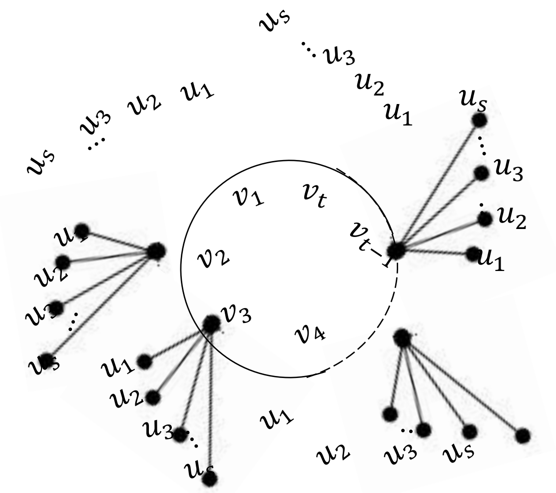


Figure 1: The Corona Graph $\mathcal{CO}_t^s$

2. Definitions and Results

Definition 2.1: [3] The wiener index (or wiener number) of G is the total W(G) of distances between all pairs of vertices in the graph G, named after chemist H. Wiener, who originally analyzed it in 1947 for acyclic topologies:
$Wi(G)\ =\ \sum_{\lambda\geq1}^{Di} D_G(\lambda)\lambda$

where $D_G(\lambda)$ is the number of pairs of vertices of G that are at a distance λ.

The hyper-wiener index WWi is one of the recently conceived distance-based graph invariants which is used as a structure descriptor for predicting physicochemical properties of organic compounds (often those significant for pharmacology, agriculture and environment protection). Milan Randi'c proposed the hyper-Wiener index of acyclic networks in 1993. Then, as a generalization of the Wiener index, Klein et al. [16] expanded Randi'c's notion to all connected graphs. Its definition is as follows:
$WWi(G) = (Wi_1(G) + Wi_2(G))$.
Let $Wi_\mu(G) = \sum_{\lambda\geq 1}^{Di} D_G(\lambda)\lambda^\mu$ be the wiener μ-invariant associated with a real number λ, where $\mu = 1$ is the actual wiener index $Wi(G)$ and if $\mu \geq 2$ is the wiener μ-invariant index.

Firstly, we give some results about the wiener and hyper-wiener indices $Wi(G)$ and $WWi(G)$ of general graph G in the following theorems.

Theorem 2.2: Let G be a connected finite simple graph with p vertices, and q edges, then
1- $Wi(G) = p^2 - p - q + D_G(3) + 2D_G(4) + \cdots + (Di-1)D_G(Di)$ if $Di(G) \geq 2$.
2- $Wi(G) = p^2 - p - q$, for $Di(G) = 2$.

Proof: From the definition of wiener μ -invariant $Wi_\mu(G) = \sum_{\lambda\geq 1}^{Di} D_G(\lambda)\lambda^\mu$, for $\mu = 1$, we get
$Wi(G) = Wi_1(G) = \sum_{\lambda\geq 1}^{Di} D_G(\lambda)\lambda^1$
$= D_G(1) + 2D_G(2) + 3D_G(3) + \cdots + DiD_G(Di)$ …..
(1)
We have $D_G(1) + D_G(2) + D_G(3) + \cdots + D_G(Di) = \binom{p}{2} = \frac{p(p-1)}{2}$, since $D_G(1) = q$, for $\mu = 1$, we get
$D_G(2) = \frac{p(p-1)}{2} - q - D_G(3) - \cdots - D_G(\text{Di})$ by substituting in equation (1)
$Wi(G) = q + 2(\frac{p(p-1)}{2} - q - D_G(3) - \cdots - D_G(\text{Di})) + 3D_G(3) + \cdots + DiD_G(Di)$
$= p^2 - p - q + D_G(3) + 2D_G(4) + \cdots + (Di-1)D_G(Di)$.
For part2, we put Di(G) = 2 then we get the result.

Theorem 2.3: The hyper-wiener index is defined as follows for a finite connected graph G of order p and size q:
$WWi(G) = [3p^2 - 3p - 4q + 6D_G(3) + 14D_G(4) + \cdots + (Di^2 + Di - 5)D_G(Di)]$

Proof: By the formula and definition of hyper-wiener, we have $WWi(G) = (Wi_1(G) + Wi_2(G))$.
The first part $Wi_1(G)$ of the formula is the Wiener index itself and proved in Theorem 2.2, and to calculate the second part $Wi_2(G)$ of the formula, we apply

$Wi_2(G) = \sum_{\lambda\geq 1}^{Di} D_G(\lambda)\lambda^2$
$= D_G(1)(1)^2 + D_G(2)(2)^2 + D_G(4)(4)^2 + \cdots + Di^2 D_G(Di)$ … (1)
It is clear that $D_G(1) = q$, and $D_G(1) + D_G(2) + D_G(3) + D_G(4) + \cdots + D_G(Di) = \binom{p}{2} = \frac{p(p-1)}{2}$
Then $D_G(2) = \frac{p(p-1)}{2} - q - D_G(3) - D_G(4) - \cdots - D_G(\text{Di})$ by substituting in (1)

$Wi_2(G) = \ D_G(1) + 4[\ \frac{p(p-1)}{2} - q - D_G(3) - \cdots - \ D_G(\mathrm{Di})] + \ \ 9D_G(3) + \ \ 16D_G(4) + \cdots$
$+ Di^2 D_G(Di)$
$= q + 2p^2 - 2p - \ 4q - 4D_G(3) - 4D_G(4) + \cdots - \ 4D_G(\mathrm{Di}) + 9D_G(3) +$
$16D_G(4) + \cdots + \qquad Di^2 D_G(Di)$
$Wi_2(G) = 2p^2 \ - 2p \ - 3q + 5D_G(3) + 12D_G(4) + \cdots + (Di^2 - 4)D_G(Di)$.
Then
$WWi(G) = \ [p^2 - p - q + D_G(3) + 2D_G(4) + \cdots + (Di-1)D_G(Di) + 2p^2 - 2p - 3q +$
$5(3) + 12D_G(4) + \cdots + (Di^2 - 4)D_G(Di),$ by simplifying, we get

$WWi(G) = \ [3p^2 - 3p - 4q + 6D_G(3) + 14D_G(4) + \cdots + (Di^2 + Di - 5)D_G(Di)].$

Now we compute wiener, wiener μ-invariant $Wi_\mu(G)$ and hyper-wiener indices of our graph and evaluate some distance-based topological indices of the coronagraph. And the diameter of the coronagraph is determined by the cycle C_t, where t is odd or even, as follows:

$$\mathrm{Di}(\mathcal{CO}_t^s) = \begin{cases} \frac{t+4}{2} & \text{if } t \text{ is even} \\ \frac{t+3}{2} & \text{if } t \text{ is odd.} \end{cases}$$

Then we discuss the distance between any two vertices of the graph $\mathcal{CO}_t^s$, we have distinct distances up to the vertices be it an end vertex or attaching vertex of the graph. The distances are between 1 and the diameter of the graph. Firstly, we solve the following example of a special case of the graph $\mathcal{CO}_t^s$, when $t = 5$ and $s = 2$.

Example1: Let $\mathcal{CO}_5^2$ be a graph of order 15 and size 15 with 10 end vertices, it is a special case of the Figure1, then the distance between vertices are 1, 2, 3, 4 (Di($\mathcal{CO}_5^2$)) which is defined as follows:

Table1: The distance of the vertices of 〖CO〗_5^2

$d(u,v)$, $u,v \in V(\mathcal{CO}_t^s)$	No. of pair of vertices $D_G(\lambda)$
1	15
2	30
3	40
4	20

To find the wiener index of this graph, we have

$$Wi(\mathcal{CO}_5^2) = \sum_{\lambda=1}^{Di} D_G(\lambda)\lambda = \ 15(1) + \ 30\ (2) + 40(3) + 20(4) \ = 275.$$

Since this graph Co_t^s is obtained by cycle graph, where the cycle graph is a simple graph with t vertices and q edges that forms a cycle of length t, all of the vertices in a cycle graph have a degree of two. To begin working on our graph, we must first determine a wiener index of the cycle graph, which depends on t (the number of vertices) of the cycle.

Proposition 2.5: Let C_t be a cyclic graph of order$t$, then the wiener index of the cycleC_t, which are defined as follows:

$$Wi(C_t) = \begin{cases} \dfrac{t(t^2-1)}{8} & for\ t\ is\ odd \\ \dfrac{t^3}{8} & for\ t\ is\ even. \end{cases}$$

Proof: The diameter of the cycle depends on the order t, which is odd or even two cases:
Case1: When t is odd

$Wi(C_t) = \sum_{\lambda=1}^{Di} D_G(\lambda)\lambda = t(1) + t(2) + \cdots \ldots .. + t\frac{(t-1)}{2}$,

where $Di(C_t) = \frac{t-1}{2}$

$$= t\left(1 + 2 + 3 + \cdots + \frac{(t-1)}{2}\right) = \frac{t(t^2-1)}{8}$$

Case2: When t is even

$Wi(C_t) = \sum_{\lambda=1}^{Di} D_G(\lambda)\lambda = t(1) + t(2) + \cdots + t\frac{(t-2)}{2} + \frac{t}{2}(\frac{t}{2})$

where $Di(C_t) = \frac{t}{2}$

$$= t(1 + 2 + 3 + \cdots .. + \frac{(t-2)}{2}) + \frac{t^2}{4}$$

$$= t\frac{(t^2-2t)}{8} + \frac{t^2}{4} = \frac{t^3}{8}$$

The minimum (smallest) distance $d(u,v)$ for the graph Co_t^s is 1 and the maximum distance is $Di(C_t) + 2$ and the other distance between them, for example for $t = 11$, $s = 2$ in the graph $\mathcal{CO}_t^s$, Di($\mathcal{CO}_{11}^2$) $= \frac{(t-1)}{2} + 2 = 7$, as shown in Table 2 bellow:

Table 2: The distance of the vertices of $\mathcal{CO}_{11}^2$

$d(u,v)$, $u,v \in V(\mathcal{CO}_t^s)$	No. of pair of vertices $D_G(\lambda)$	
1	$t(s+1)$	33
2	$t(s-1) + t(s-2) + \cdots + 2ts + t$	66
3	$ts^2 + 2ts + t$	99
4	$ts^2 + 2ts + t$	99
5	$ts^2 + 2ts + t$	99
6	$ts^2 + 2ts$	88
7	ts^2	44

Then in general for t is odd and $s \geq 1$, we have the following distance in Table 3:

Table 3: The distance between all vertices of the graph 〖CO〗_t^s, where t is odd.

$d(u,v)$, $u,v \in V(\mathcal{CO}_t^s)$	No. of pair of vertices $D_G(\lambda)$	
1	ts	$+\sum_1^{Di} d(v_i,v_j)$, where v_i and v_j are the vertices of the cycle part of the graph $\mathcal{CO}_t^s$. i,j =1, 2, …..t, i<j.
2	$t(s-1) + t(s-2) + \cdots + 2ts$	
$3 \leq d(u,v) \leq \frac{t+1}{2}$	ts^2+2ts	
$\frac{t+3}{2}$	ts^2	

For t is even and $s \geq 1$, we have the following distance in Table 4:

Table 4: The distance between all vertices of the graph 〖CO〗_t^s, where t is even.

$d(u,v)$, $u,v \in V(\mathcal{CO}_t^s)$	No. of pair of vertices $D_G(\lambda)$	
1		
2	$t(s-1)+t(s-2)+\cdots+\ 2ts$	
$3 \le d(u,v) \le \frac{t+1}{2}$	ts^2+2ts	$+\sum_1^{Di} d(v_i,v_j)$, where v_i and v_j are the vertices of the cycle part of the graph $\mathcal{CO}_t^s$. i,j =1, 2, …..t, i<j.
$\frac{t+2}{2}$	ts^2+ts	
$\frac{t+4}{2}$	$\frac{t}{2}s^2$	

To calculate the wiener index of the coronagraph, first, we find the wiener of the cycle graph because it is a part of the coronagraph since the wiener of the corona graph depends on the diameter of the cycle.

Theorem 2.6: The wiener index of the graph Co_t^s is defined as follows:
$W(\mathcal{CO}_t^s) = \ (1+s)^2\frac{t(t^2-1)}{8} + st^2 + 2s^2\binom{t}{2} + 2t\binom{s}{2}$, where t is the order of the cycle part and is odd.

Proof: Based on Table 3 of the distance, $Wi(\mathcal{CO}_t^s) = \sum_{\lambda=1}^{Di} D_G(\lambda)\lambda + Wi(C_t)$

From Proposition 2.5, we have $Wi(C_t) = \frac{t(t^2-1)}{8}$ for t is odd, then

$$Wi(\mathcal{CO}_t^s) = (1)ts + (2)([t(s-1)+t(s-2)+\cdots+t] + 2ts) + (3)(ts^2+2ts) + \cdots + \binom{\frac{t+1}{2}}{}(ts^2+2ts) + \binom{\frac{t+3}{2}}{}ts^2 + \frac{t(t^2-1)}{8}$$

$$= ts + \left(2t\binom{s}{2} + 2ts\right) + \sum_{\lambda\ge3}^{\frac{t+1}{2}}(ts^2+2ts) + \left(\frac{t+3}{2}\right)(ts^2) + \frac{t(t^2-1)}{8}$$

After some calculations, we get

$$Wi(\mathcal{CO}_t^s) = (s+1)^2\frac{t(t^2-1)}{8} + t^2s + 2s^2\binom{t}{2} + 2t\binom{s}{2}$$

Theorem 2.7: The wiener index of the graph $\mathcal{CO}_t^s$, is defined as follows:
$Wi(\mathcal{CO}_t^s) = \ (1+s)^2\frac{t^3}{3} + t^2s + 2s^2\binom{t}{2} + 2t\binom{s}{2}$, where t is the order of the cycle part and is even.

Proof: The wiener index $Wi(G) = Wi(\mathcal{CO}_t^s) = \sum_{\lambda=1}^{Di} D_G(\lambda)\lambda$
Then based on Table4 the wiener of the coronagraph Co_t^s is

$$Wi(\mathcal{CO}_t^s) = \sum_{\lambda=1}^{Di} D_G(\lambda)\lambda$$

$$Wi(\mathcal{CO}_t^s) = \sum_{\lambda=1}^{Di} D_G(\lambda)\lambda + Wi(C_t)$$

$$= (1)ts + (2)([t(s-1)+t(s-2)+\cdots+t] + 2ts) + (3)(ts^2+2ts) + \cdots + \left(\frac{t}{2}\right)(ts^2+2ts)$$

$$+ \left(\frac{t+2}{2}\right)(ts^2+ts) + \left(\frac{t+4}{2}\right)\left(\frac{t}{2}s^2\right) + \frac{t^3}{8}$$

where $Wi(C_t) = \frac{t^3}{8}$, t is even mentioned in Proposition 2.5

$$= ts + (2t\binom{s}{2} + 2ts) + \sum_{\lambda\geq 3}^{\frac{t}{2}}(ts^2+2ts) + \left(\frac{t+2}{2}\right)(ts^2+\ 2ts) + \left(\frac{t+4}{2}\right)(\frac{t}{2}s^2) + \frac{t^3}{8}$$

After simplifying, we get

$Wi(CO_t^s) = (s+1)^2\frac{t^3}{8} + t^2s + 2s^2\binom{t}{2} + 2t\binom{s}{2}$.

Now after finding the wiener index of the coronagraph, we study the wiener-μ invariant and hyper wiener of this graph by the following theorem.

Theorem 2.8: The hyper-wiener index is of the connected graph Co_t^s, which is defined as follows:

$WWi(CO_t^s) = (s+1)^2\frac{t(t^2-1)}{8} + t^2s + 2s^2\binom{t}{2} + 2t\binom{s}{2} + ts\ + 4(\frac{ts^2-ts}{2})\ + ts^{\ 2}(\frac{t^3+12t^2+47t-60}{24})\ + 2ts(\frac{t^3+6t^2+11t-18}{24})\ + t(\frac{t^3-t}{24})$.

if t is odd

$WWi(CO_t^s) = (s+1)^2\frac{t^3}{8} + t^2s + 2s^2\binom{t}{2} + 2t\binom{s}{2} + ts\ + 4\left(\frac{ts^2-ts}{2}\right) + ts^{\ 2}\left(\frac{t^3+9t^2+26t-96}{24}\right) +$ 2ts$(\frac{t^3+3t^2+2t-24}{24})$ + t$(\frac{t^3-3t^2+2t}{24})$+ $(\frac{t^3}{8})$+ ts$^2(\frac{t^2+8t+16}{8})$ + ts$(\frac{t^2+4t+4}{4})$. if t is even

Proof: In general, hyper wiener $WWi(G) = (Wi_1(G) + Wi_2(G))$.

where $Wi_1(G)= \sum_{\lambda\geq 1}^{Di} D_G(\lambda)\lambda^1$ for $\mu = 1$ is wiener index itself and it is found on the theorem above, to calculate the second part $Wi_2(G)$ for $\mu = 2$, so the diameter of the corona graph depends on the number of vertices of the cycle part, we have to discuss two cases when it is odd or even.

Case1: For t is odd,

$Wi_2(G) = \sum_{\lambda\geq 1}^{Di} D_G(\lambda)\lambda^2$

$= D_G(1)(1)^2 + D_G(2)(2)^2 + D_G(3)(3)^2 + \cdots + D_G(Di-1)(Di-1)^2 + D_G(Di)(Di)^2$.

But, $D_G(1)$ is the number of vertices of distance one it is equal to the size of the graph, and it is equal t(s+1) in this graph, then

$$Wi_2(CO_t^s) = t(s+1) + [t(s-1) + t(s-2) + \cdots + 2ts\ + t](2)^2 + (ts^2+2ts+t)(3)^2 + (ts^2+2ts+t)(4)^2 + \cdots + (ts^2+2ts + t)(\ Di-2)^2 + (ts^2+2ts)\ (Di-1)^2 + (ts^2)(Di)^2.$$

After simplifying this equation, we get

$Wi_2(CO_t^s) = \text{ts} + 4\,(t\sum_{i=1}^{s-1}(s-i)) + \text{ts}^2\sum_{\lambda=3}^{Di}\lambda^2 + 2\text{ts}\sum_{\lambda=2}^{(Di-1)}\lambda^2 + t\sum_{\lambda=1}^{(Di-2)}\lambda^2$

Substituting, $Di = \frac{t+3}{2}$

$$= \text{ts} + 4\,(t\sum_{i=1}^{s-1}(s-i)) + \text{ts}^2\sum_{\lambda=3}^{\frac{t+3}{2}}\lambda^2 + 2ts\sum_{\lambda=2}^{\frac{t+1}{2}}\lambda^2 + t\sum_{\lambda=1}^{\frac{t-1}{2}}\lambda^2,$$

Set $\sum_{\lambda=2}^{\frac{t+1}{2}}\lambda^2 = \sum_{\lambda=1}^{\frac{t-1}{2}}\lambda^2 + \frac{(t-1)(t+3)}{4}$ and $\sum_{\lambda=3}^{\frac{t+3}{2}}\lambda^2 = \sum_{\lambda=1}^{\frac{t-1}{2}}\lambda^2 + 4(\sum_{\lambda=1}^{\frac{t-1}{2}}\lambda + \frac{t-1}{2})$

$Wi_2(CO_t^s) = ts\ + 4(\frac{ts^2-ts}{2})\ + ts^2\,(\frac{t^3+12t^2+47t-60}{24})\ + 2ts(\frac{t^3+6t^2+11t-18}{24})\ + t(\frac{t^3-t}{24})$

Case2: For t is even,

$Wi_2(G) = \sum_{\lambda\geq 1}^{Di} D_G(\lambda)\lambda^2$

$= D_G(1)(1)^2 + D_G(2)(2)^2 + D_G(3)(3)^2 + \cdots + D_G(Di-1)(Di-1)^2 + D_G(Di)(Di)^2$

$= ts + 4(t\sum_{i=1}^{s-i}(s-i)) + ts^2\sum_{\lambda=3}^{(Di-1)}\lambda^2 + 2ts\sum_{\lambda=2}^{(Di-2)}\lambda^2 + t\sum_{\lambda=1}^{(Di-3)}\lambda^2 + \frac{t}{2}(Di-2)^2 + \frac{t}{2}s^2(Di)^2 + ts(Di-1)^2.$

Where Di is a diameter for t even and $Di = \frac{t+4}{2}$, $Di-1 = \frac{t+2}{2}$, $Di-2 = \frac{t}{2}$, $Di-3 = \frac{t-2}{2}$. Put in the above equation, we get

$$= ts + 4(t\sum_{i=1}^{s-i}(s-i)) + ts^2\sum_{\lambda=3}^{\frac{t+2}{2}}\lambda^2 + 2ts\sum_{\lambda=2}^{\frac{t}{2}}\lambda^2 + t\sum_{\lambda=1}^{\frac{t-2}{2}}\lambda^2 + \frac{t}{2}(\frac{t}{2})^2 + \frac{t}{2}s^2(\frac{t+4}{2})^2 + ts(\frac{t+2}{2})^2$$

Then the sum, $\sum_{\lambda=3}^{\frac{t+2}{2}}\lambda^2 = \sum_{\lambda=1}^{\frac{t-2}{2}}\lambda^2 + 4(\sum_{\lambda=1}^{\frac{t-2}{2}}\lambda + \frac{t-2}{2})$ and $\sum_{\lambda=2}^{\frac{t}{2}}\lambda^2 = \sum_{\lambda=1}^{\frac{t-2}{2}}\lambda^2 + \frac{(t-2)(t+2)}{2}$

After some calculations, we get

$Wi_2(\mathcal{CO}_t^s) = ts + 4(\frac{ts^2-ts}{2}) + ts^2 (\frac{t^3+9t^2+26t-96}{24}) + 2ts(\frac{t^3+3t^2+2t-24}{24}) + t(\frac{t^3-3t^2+2t}{24}) + (\frac{t^3}{8}) + ts^2(\frac{t^2+8t+16}{8}) + ts(\frac{t^2+4t+4}{4})$.

3. Some Degree-based Topological Indices

● The first and second Zagreb indices of G were first proposed by Gutman and Trinajesti'c [5]. They are defined as follows:

$M_1(G) = \sum_{uv\in E(G)}\left[d_{gr}(u) + d_{gr}(v)\right]$

$M_2(G) = \sum_{uv\in E(G)}\left[d_{gr}(u).d_{gr}(v)\right]$

● Shigehalli and Kanabur presented the three new indices [14], which are as follows:

$SK(G) = \sum_{uv\in E(G)}\frac{d_{gr}(u)+d_{gr}(v)}{2}$

$SK_1(G) = \sum_{uv\in E(G)}\frac{d_{gr}(u).\ d_{gr}(v)}{2}$

$SK_2(G) = \sum_{uv\in E(G)}\left(\frac{d_{gr}(u)+d_{gr}(v)}{2}\right)^2$

● Inspired by the definitions of the Zagreb indices and their vast applications, Kulli [12] was proposed the first Gourava index of a molecular graph as follows. For a graph G, the first and second Gourava indices are defined as:

$GO_1(G) = \sum_{uv\in E(G)}\left[\left(d_{gr}(u) + d_{gr}(v)\right) + (d_{gr}(u).d_{gr}(v))\right]$

$$GO_2(G) = \sum_{uv\in E(G)}\left[\left(d_{gr}(u) + d_{gr}(v)\right).(d_{gr}(u).d_{gr}(v))\right]$$

Results: [12, 17, 18] The first and second Zagreb indices SK, SK_1, SK_2 indices, and the first and second Gourava indices for the cycle graph ($C_t, t \geq 3$) are given by:

- $M_1(C_t) = M_2(C_t) = 4t$
- $SK(C_t) = SK_1(C_t) = \frac{1}{2}M_1(C_t)$

• $SK_2(C_t) = M_1(C_t)$
• $GO_1(C_t) = 2\ M_1(C_t)$
• $GO_2(C_t) = 4\ M_1(C_t)$

In this section, we study the M_1, M_2, SK, SK_1, SK_2, GO_1 and GO_2 of coronagraph.

Theorem 3.1: Let Co_t^s be the coronagraph, where t is the number of vertices of a cycle and s be the number of end vertices that are adjacent to each vertex of a cycle for $t \geq 3$, $s \geq 1$. Then the first and second Zagreb index for the corona graph Co_t^s are defined as the following:
1. $M_1(\mathcal{CO}_t^s) = ts^2 + 5ts + 4t$
2. $M_2(\mathcal{CO}_t^s) = 2ts^2 + 6ts + 4t$

Proof: Let C_t be the cycle graph on t vertices. The coronagraph $\mathcal{CO}_t^s$ is shown in Figure 1. Furthermore, corona graph $\mathcal{CO}_t^s$ consist of $t(s+1)$ vertices and also $t(s+1)$ edges. $E(\mathcal{CO}_t^s)$ is divided into two edge divisions based on the degrees of end vertices. The ts edges u_jv_i are contained in the first edge partition $E_1(\mathcal{CO}_t^s)$, where $d_{gr}(u_j) = 1$ and $d_{gr}(v_i) = s + 2$. The second edge partition $E_2(\mathcal{CO}_t^s)$ has t edges v_iv_{i+1} with $d_{gr}(v_i) = d_{gr}(v_{i+1}) = s + 2$. Then we have the following using expressions:

1- $M_1(G) = \sum_{uv\in E(G)}\left[d_{gr}(u) + d_{gr}(v)\right]$

$$M_1(\mathcal{CO}_t^s) = \sum_{u_jv_i\in E_1(\mathcal{CO}_t^s)}\left(d_{gr}(u_j) + d_{gr}(v_i)\right) + \sum_{v_iv_{i+1}\in E_2(\mathcal{CO}_t^s)}\left(d_{gr}(v_i) + d_{gr}(v_{i+1})\right)$$
$$= |E_1(\mathcal{CO}_t^s)|\,(d_{gr}(u_j) + d_{gr}(v_i)) + |E_2(\mathcal{CO}_t^s)|\,(d_{gr}(v_i) + d_{gr}(v_{i+1}))$$
$$= ts\,[1 + (s+2)] + t\,[(s+2) + (s+2)]$$

We get the following result after some simple simplifications.

$$M_1(\mathcal{CO}_t^s) = ts^2 + 5ts + 4t.$$

2- $M_2(G) = \sum_{uv\in E(G)}\left[d_{gr}(u).d_{gr}(v)\right]$

$$M_2(\mathcal{CO}_t^s) = \sum_{u_jv_i\in E_1(\mathcal{CO}_t^s)}\left(d_{gr}(u_j)\,.\,d_{gr}(v_i)\right) + \sum_{v_iv_{i+1}\in E_2(\mathcal{CO}_t^s)}\left(d_{gr}(v_i)\,.\,d_{gr}(v_{i+1})\right)$$
$$= |E_1(\mathcal{CO}_t^s)|\,(d_g(u_j)\,.\,d_g(v_i)) + |E_2(\mathcal{CO}_t^s)|\,(d_g(v_i)\,.\,d_g(v_{i+1}))$$
$$= ts\,[1\,.(s+2)] + t\,[(s+2)\,.(s+2)]$$

We get the following result after some simple simplifications.

$$M_2(\mathcal{CO}_t^s) = 2ts^2 + 6ts + 4t$$

Theorem 3.2: Let $\mathcal{CO}_t^s$ be the corona graph, where t is the number of vertices of a cycle and s be the number of end vertices that are adjacent to each vertex of a cycle, and $t \geq 3$, $s \geq 1$. Then the SK, SK_1 and SK_2 for the corona graph $\mathcal{CO}_t^s$ are defined as the following:
1. $SK\ (\mathcal{CO}_t^s) = \frac{1}{2} M_1(\mathcal{CO}_t^s)$
2. $SK_1\ (\mathcal{CO}_t^s) = \frac{1}{2} M_2(\mathcal{CO}_t^s)$
3. $SK_2\ (\mathcal{CO}_t^s) = \frac{1}{4}\left[(s+4)\ M_1(\mathcal{CO}_t^s) + ts(s+1)\right]$

Proof: Let C_t be the cycle graph on t vertices. The coronagraph $\mathcal{CO}_t^s$ is shown in Figure 1. Furthermore, corona graph $\mathcal{CO}_t^s$ consist of $t(s+1)$ vertices and also $t(s+1)$ edges. $E(\mathcal{CO}_t^s)$ is divided into two edge divisions based on the degrees of end vertices. The ts edges u_jv_i are contained in the first edge partition $E_1(\mathcal{CO}_t^s)$, where $d_{gr}(u_j) = 1$ and $d_{gr}(v_i) = s +$

2. The second edge partition $E_2(\mathcal{CO}_t^s)$, has t edges v_iv_{i+1}, with $d_{gr}(v_i) = d_{gr}(v_{i+1}) = s + 2$. We now have the following using expressions:

1- $SK(G) = \sum_{uv\in E(G)} \frac{d_{gr}(u)+d_{gr}(v)}{2}$

$$SK(\mathcal{CO}_t^s) = \sum_{u_jv_i\in E_1(\mathcal{CO}_t^s)} \left(\frac{d_{gr}(u_j)+ d_{gr}(v_i)}{2}\right) + \sum_{v_iv_{i+1}\in E_2(\mathcal{CO}_t^s)} \left(\frac{d_{gr}(v_i)+ d_{gr}(v_{i+1})}{2}\right)$$
$$= |E_1(\mathcal{CO}_t^s)| \left(\frac{d_{gr}(u_j)+ d_{gr}(v_i)}{2}\right) + |E_2(\mathcal{CO}_t^s)| \left(\frac{d_{gr}(v_i)+ d_{gr}(v_{i+1})}{2}\right)$$
$$= ts\left(\frac{1 + (s+2)}{2}\right) + t\left(\frac{(s+2) + (s+2)}{2}\right)$$

After any easy simplification, we obtain

$$= \frac{1}{2}(ts^2 + 5ts + 4t)$$
$$SK\ (\mathcal{CO}_t^s) = \frac{1}{2} M_1(\mathcal{CO}_t^s)$$

2- $SK_1(G) = \sum_{uv\in E(G)} \frac{d_{gr}(u)\,.\,d_{gr}(v)}{2}$

$$SK_1(\mathcal{CO}_t^s) = \sum_{u_jv_i\in E_1(\mathcal{CO}_t^s)} \left(\frac{d_{gr}(u_j)\,.\, d_{gr}(v_i)}{2}\right) + \sum_{v_iv_{i+1}\in E_2(\mathcal{CO}_t^s)} \left(\frac{d_{gr}(v_i)\,.\, d_{gr}(v_{i+1})}{2}\right)$$
$$= |E_1(\mathcal{CO}_t^s)| \left(\frac{d_{gr}(u_j)\,.\, d_{gr}(v_i)}{2}\right) + |E_2(\mathcal{CO}_t^s)| \left(\frac{d_{gr}(v_i)\,.\, d_{gr}(v_{i+1})}{2}\right)$$
$$= ts\left(\frac{1\,.\,(s+2)}{2}\right) + t\left(\frac{(s+2)\,.\,(s+2)}{2}\right)$$

After any easy simplification, we obtain

$$= \frac{1}{2}(2ts^2 + 6ts + 4t)$$
$$SK_1\ (\mathcal{CO}_t^s) = \frac{1}{2} M_2(\mathcal{CO}_t^s)$$

3- $SK_2(G) = \sum_{uv\in E(G)} \left(\frac{d_{gr}(u)+d_{gr}(v)}{2}\right)^2$

$$SK_2(\mathcal{CO}_t^s) = \sum_{u_jv_i\in E_1(\mathcal{CO}_t^s)} \left(\frac{d_{gr}(u_j) + d_{gr}(v_i)}{2}\right)^2 + \sum_{v_iv_{i+1}\in E_2(\mathcal{CO}_t^s)} \left(\frac{d_{gr}(v_i) + d_{gr}(v_{i+1})}{2}\right)^2$$
$$= |E_1(\mathcal{CO}_t^s)| \left(\frac{d_{gr}(u_j) + d_{gr}(v_i)}{2}\right)^2 + |E_2(\mathcal{CO}_t^s)| \left(\frac{d_{gr}(v_i) + d_{gr}(v_{i+1})}{2}\right)^2$$
$$= ts\left(\frac{1 + (s+2)}{2}\right)^2 + t\left(\frac{(s+2)+(s+2)}{2}\right)^2$$

After any easy simplification, we obtain

$$SK_2\ (\mathcal{CO}_t^s) = \frac{1}{4}\,[(s + 4)\, M_1(\mathcal{CO}_t^s) + ts(s + 1)]$$

Theorem 3.3: Let $\mathcal{CO}_t^s$ be the corona graph, where t is the number of vertices of a cycle and s be the number of end vertices that are adjacent to each vertex of a cycle, and $t \geq 3$, $s \geq 1$. Then the GO_1 and GO_2 for the corona graph $\mathcal{CO}_t^s$ are defined as the following:
1. $GO_1(\mathcal{CO}_t^s) = M_1(\mathcal{CO}_t^s) + M_2(\mathcal{CO}_t^s)$
2. $GO_2(\mathcal{CO}_t^s) = (s + 2)\,(M_1(\mathcal{CO}_t^s) + M_2(\mathcal{CO}_t^s))$
we have the following expressions:

1- $GO_1(G) = \sum_{uv\in E(G)}\left[\left(d_{gr}(u)+d_{gr}(v)\right)+\left(d_{gr}(u)\,.\,d_{gr}(v)\right)\right]$

$GO_1(\mathcal{CO}_t^s) = \sum_{u_jv_i\in E_1(\mathcal{CO}_t^s)}\left[\left(d_{gr}(u_j)+d_{gr}(v_i)\right)+\left(d_{gr}(u_j)\,.\,d_{gr}(v_i)\right)\right]+$
$\sum_{v_iv_{i+1}\in E_2(\mathcal{CO}_t^s)}\left[\left(d_{gr}(v_i)+d_{gr}(v_{i+1})\right)+\left(d_{gr}(v_i)\,.\,d_{gr}(v_{i+1})\right)\right]$
$= |E_1(\mathcal{CO}_t^s)| + \left[\left(d_{gr}(u_j)+d_{gr}(v_i)\right)+\left(d_{gr}(u_j)\,.\,d_{gr}(v_i)\right)\right] + |E_2(\mathcal{CO}_t^s)|\left[\left(d_{gr}(v_i)+d_{gr}(v_{i+1})\right)+\left(d_{gr}(v_i)\,.\,d_{gr}(v_{i+1})\right)\right]$
$= ts\left[\left(1+(s+2)\right)+\left(1\,.(s+2)\right)\right]+t\left[\left((s+2)+(s+2)\right)+\left((s+2)\,.\,(s+2)\right)\right]$
$= ts\,(2s+5)+t\,[(2s+4)+(s^2+4s+4)]$
$= ts\,(2s+5)+t\,(s^2+6s+8).$

After any easy simplification, we obtain

$GO_1(\mathcal{CO}_t^s) = M_1(\mathcal{CO}_t^s)+M_2(\mathcal{CO}_t^s)$

2- $GO_2(G) = \sum_{uv\in E(G)}\left[\left(d_g(u)+d_g(v)\right).\left(d_g(u)\,.\,d_g(v)\right)\right]$

$GO_2(\mathcal{CO}_t^s) = \sum_{u_jv_i\in E_1(\mathcal{CO}_t^s)}\left[\left(d_{gr}(u_j)+d_{gr}(v_i)\right).\left(d_{gr}(u_j)\,.\,d_{gr}(v_i)\right)\right]+$
$\sum_{v_iv_{i+1}\in E_2(\mathcal{CO}_t^s)}\left[\left(d_{gr}(v_i)+d_{gr}(v_{i+1})\right).\left(d_{gr}(v_i)\,.\,d_{gr}(v_{i+1})\right)\right]$
$= |E_1(\mathcal{CO}_t^s)|\left[\left(d_{gr}(u_j)+d_{gr}(v_i)\right).\left(d_{gr}(u_j)\,.\,d_{gr}(v_i)\right)\right]$
$+ |E_1(\mathcal{CO}_t^s)|\left[\left(d_{gr}(v_i)+d_{gr}(v_{i+1})\right).\left(d_{gr}(v_i)\,.\,d_{gr}(v_{i+1})\right)\right]$
$= ts\,[(1+(s+2))\;.(1\,.(s+2))]+t[((s+2)+(s+2))\;.\;((s+2)\;.(s+2))]$
$= ts\,(s^2+5s+6)+t(2s^3+12s^2+24s+16).$

After any easy simplification, we obtain

$GO_2(\mathcal{CO}_t^s) = (s+2)\,(M_1(\mathcal{CO}_t^s)+M_2(\mathcal{CO}_t^s))$

4. New results of new topological index SK_4.

In this section, we define a new topological index whose definition is inspired by the definition of the SK_1 index. Then we applied to the corona graph.

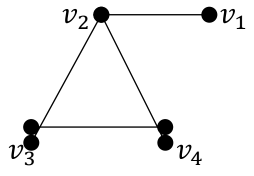
Definition 4.1: The SK_4 index of a graph $G = (V,E)$ is defined as:

$$SK_4\,(G) = \sum_{uv\in E(G)}\left(\frac{d_{gr}(u)\,.\,d_{gr}(v)}{2}\right)^2$$

where $d_{gr}(u)$ and $d_{gr}(v)$ be the degree of the vertex u and v, respectively.

Example 4.2: For the simple graph G, the SK_4 index of a graph G equal to 24.25 .

G:



To find the SK_4 index, then the degree of the vertices of the graph G are, $d_{gr}(v_1)=1$, $d_{gr}(v_2)=3$ and $d_{gr}(v_3)=d_{gr}(v_4)=2$, by the definition of the SK_4 index is:

$$SK_4\,(G) = \sum_{uv\in E(G)}\left(\frac{d_{gr}(u)\,.\,d_{gr}(v)}{2}\right)^2$$
$$= \left(\frac{1\,.\,3}{2}\right)^2+2\left(\frac{3\,.\,2}{2}\right)^2+\left(\frac{2\,.\,2}{2}\right)^2$$
$$= 24.25.$$

Lemma 4.3: Let C_t be a cycle with $t \geq 3$ vertices. Then $SK_4\ (C_t)$ is $M_1(C_t)$.
Proof: Let C_t be a cycle with $t \geq 3$ vertices. Then $SK_4\ (C_t) = t(\frac{2.2}{2})^2 = 4t = M_1(C_t)$.
Theorem 4.4: The SK_4 index for corona graph $\mathcal{CO}_t^s$ is given by:
$SK_4\ (\mathcal{CO}_t^s) = \frac{(s+2)^2}{4} M_1(\mathcal{CO}_t^s)$.

Proof: Let C_t be the cycle graph on t vertices. The corona graph $\mathcal{CO}_t^s$ is shown in Figure 1. Furthermore, corona graph $\mathcal{CO}_t^s$ consist of $t(s+1)$ vertices and also $t(s+1)$ edges. The edge set $E(\mathcal{CO}_t^s)$ is divided into two edge divisions based on the degrees of end vertices. The ts edges u_jv_i are contained in the first edge partition $E_1(\mathcal{CO}_t^s)$, where $d_{gr}(u_j) = 1$ and $d_{gr}(v_i) = s+2$. The second edge partition $E_2(\mathcal{CO}_t^s)$ has t edges v_iv_{i+1}, with $d_{gr}(v_i) = d_{gr}(v_{i+1}) = s+2$. Now by Definition 4.1, we have:

$$SK_4\ (\mathcal{CO}_t^s) = \sum_{u_jv_i \in E_1(\mathcal{CO}_t^s)} \left(\frac{d_{gr}(u_j)\ .\ d_{gr}(v_i)}{2}\right)^2 + \sum_{v_iv_{i+1} \in E_2(\mathcal{CO}_t^s)} \left(\frac{d_{gr}(v_i)\ .\ d_{gr}(v_{i+1})}{2}\right)^2$$
$$= |E_1(\mathcal{CO}_t^s)| \left(\frac{d_{gr}(u_j)\ .\ d_{gr}(v_i)}{2}\right)^2 + |E_1(\mathcal{CO}_t^s)| \left(\frac{d_{gr}(v_i)\ .\ d_{gr}(v_{i+1})}{2}\right)^2$$
$$= ts\left(\frac{1\ .\ (s+1)}{2}\right)^2 + t\left(\frac{(s+1)\ .\ (s+1)}{2}\right)^2$$

After any easy simplification, we obtain
$SK_4\ (\mathcal{CO}_t^s) = \frac{(s+2)^2}{4} M_1(\mathcal{CO}_t^s)$.

5. Conclusion

In this study, a new graph is defined and studied. The new graph is related to the coronavirus which is called a coronagraph. Some properties and important results have been investigated and discussed. A new topological index is also introduced that is inspired by the definition of the SK_1 index. Finally, many theoretical results are given.

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