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Another Type of Fuzzy Inner Product Space

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Abstract
In this paper, we generalize the definition of fuzzy inner product space that is introduced by Lorena Popa and Lavinia Sida on a complex linear space. Certain properties of the generalized fuzzy inner product function are shown. Furthermore, we prove that this fuzzy inner product produces a Nadaban-Dzitac fuzzy norm. Finally, the concept of orthogonality is given and some of its properties are proven.

Keywords: Fuzzy inner product, Fuzzy inner product space, Fuzzy norm.

نوع آخر من فضاء الضرب الداخلي الضبابي

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الخلاصة
في هذا البحث قمنا بتعميم تعريف الفضاء الضرب الداخلي الضبابي المقدم من قبل Lorena Popa و Lavinia Sidaعلى فضاء خطي معقد . تم اثبات خصائص معينة لدالة الضرب الداخلي الضبابي. علاوة على ذلك ، نثبت أن هذا الضرب الداخلي الضبابي يولد معياراً ضبابياً من نوع Nadaban–Dzitac. أخيرًا ، تم تقديم مفهوم التعامد وإثبات بعض خصائصه.

1. Introduction

The groundwork of fuzzy functional analysis was set by Katsaras' research articles [1,2]. In addition, he was the first to propose the idea of a fuzzy norm. Mathematicians have taken a great interest in this topic. Several mathematicians, such as C. Felbin [3], Shih-Chuan Cheng, and John N. Mordeson [4], and others afterward presented fuzzy normed linear spaces definitions in other ways. A considerable number of papers on fuzzy normed linear spaces have been published, for example, Bag and Samanta [5] defined fuzzy normed linear spaces in a different way and they studied some basic results on finite-dimensional fuzzy normed linear spaces in a general t-norm setting. Rana A et al.[6] presented a new direction between the subject of domain theory and fuzzy normed spaces to introduce the so-called fuzzy domain normed spaces. Raghad [7] introduced some types of fuzzy convergence sequences of operators that are defined on a standard fuzzy normed space and investigated some properties and relationships between these concepts. Authors in [8-10] proved some results about the

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best proximity point theorem and fixed point theorem in fuzzy normed spaces. J. R.Kider and M. N. Gheeab [11] introduced the definition of a general fuzzy normed space as a generalization of the notion of fuzzy normed space after that they investigated and proved the basic properties of this space. Kareem et al.[12] used the definition of fuzzy normed space to define a fuzzy bounded operator as an introduction to define the fuzzy norm of a fuzzy bounded linear operator. Also, they introduced different types of fuzzy convergence of operators.

Studies on fuzzy inner product spaces, on the other hand, are relatively new. R. Biswas[13] was the first who tried to give a meaningful definition of fuzzy inner product space. Two years later, Kohli and Kumar developed Biswas' concept of inner product space[14]. In [15], the authors discussed and presented the idea of fuzzy semi-inner-product space, as well as looked at its many properties. The concept of intuitionistic fuzzy inner product space was presented by Goudarzi et al [16] in 2009. A novel approach to a fuzzy Hilbert space was presented in 2010[17] by Hasankhani, Nazari, and Saheli. This notion is distinct from the others since the fuzzy inner product provides a new fuzzy norm of type Felbin. Authors in [18] proposed two new fuzzy inner product space notions and looked into some of their essential properties. Recently, L.Popa and L. Sida [19] came up with a good description for the notion of fuzzy Hilbert space. They examined the properties of a different approach to the concept of the fuzzy inner product.

In this paper, we start with L.Popa and L. Sida's definition [19] to give and present a new definition of the fuzzy inner product space. In fact, L.Popa and L. Sida's definition of fuzzy inner product space on a complex linear space is generalized. Several properties of the fuzzy inner product function are also established and demonstrated.

The following is how the paper is organized: In Section 2, we give and report some preliminary findings that are used throughout this research. The concept of fuzzy inner product space is presented in Section 3. Then, some properties of the fuzzy inner product function are also shown and the notion of orthogonality is presented and some of its properties are examined. Finally, the paper finished with a conclusion section.

2. Preliminaries

This section provides terms and results that will be utilized in this paper. First, basic terminologies in the fuzzy setting are recalled.
First, we define the triangular norm (t-norm) that is given by Schweizer and Sklar[20].

Definition 2.1 [20]: If a binary operation $\circledast: [0,1] \times [0,1] \to [0,1]$ fulfills the following conditions for all $\mathfrak{s}, p, v, q \in [0,1]$, then it is called a t-norm:
(i) $1 \circledast p = p$,
(ii) $p \circledast q = q \circledast p$,
(iii) $p \circledast (v \circledast q) = (p \circledast v) \circledast q$
(iv) If $p \leq q$ and $v \leq \mathfrak{s}$ then $p \circledast v \leq q \circledast \mathfrak{s}$,

Theorem 2.2 [20]: If $\circledast$ is a t-norm, then $p \circledast_{\circ} q \leq p \circledast q \leq p \circledast_1 q$, for all $p, q \in [0,1]$ where :
$$p \circledast_{\circ} q = \begin{cases} \min\{p,q\} , & \max\{p,q\} = 1 \\ 0 & otherwise \end{cases}$$
and $p \circledast_1 q = min\{p,q\}$.

Definition 2.3 [21]: Let L be a vector space over a field R. A fuzzy normed space is a triplet $(L, \eta, \circledast)$, where $\circledast$ is a t-norm and η is a fuzzy set on $L \times R$ that meets the following conditions for all $x, y \in L$:
(1) $\eta(x, 0) = 0$,
(2) $\eta(x, p) = 1, \forall p > 0$ if and only if $x = 0$,
(3) $\eta(\gamma x, p) = \eta(x, p/|\gamma|), \forall (0 \neq)\gamma \in R, p \geq 0$
(4) $\eta(x, p) \circledast \eta(y, q) \leq \eta(x + y, p + q), \forall p, q \geq 0$
(5) $\eta(x, .)$is left continuous for all $x \in L$, and $\lim_{p\to\infty} \eta(x, p) = 1$, $p \in R$.
Now the definition of fuzzy inner product space proposed by Lorena Popa and Lavinia Sida [19] is provided. Throughout this paper, the space of complex numbers will be denoted by C, while the set of all strictly positive real numbers will be denoted by R_+^*.

Definition 2.4 [19]: A fuzzy set $\widetilde{\mathfrak{T}}$ in $L \times L \times C$ is said to be a fuzzy inner product on L where L linear space over C if $\widetilde{\mathfrak{T}}$ satisfies:
(1) $\widetilde{\mathfrak{T}}(x, x, \tau) = 0, \forall x \in L, \forall \tau \in C \setminus R_+^*$
(2) $\widetilde{\mathfrak{T}}(x, y, p) = 1, \forall x \in L, \forall p \in R_+^*$ if and only if $x = 0$.
(3) $\widetilde{\mathfrak{T}}(\gamma x, y, \tau) = \widetilde{\mathfrak{T}}\left(x, y, \frac{\tau}{|\gamma|}\right), \forall x, y \in L, \forall \tau \in C, \forall \gamma \in C$.
(4) $\widetilde{\mathfrak{T}}(x, y, \tau) = \widetilde{\mathfrak{T}}(y, x, \bar{\tau}), \forall x, y \in L, \forall \tau \in C$.
(5) $\widetilde{\mathfrak{T}}(x + y, z, \tau + s) \geq \min\{\widetilde{\mathfrak{T}}(x, z, \tau), \widetilde{\mathfrak{T}}(y, z, s)\} \forall x, y, z \in L, \forall \tau, s \in C$.
(6) $\widetilde{\mathfrak{T}}(x, x, .) : R_+ \to [0, 1], \forall x \in L$ is left continuous and $\lim \widetilde{\mathfrak{T}}(x, x, \tau) = 1$ as $\tau \to \infty$
(7) $\widetilde{\mathfrak{T}}(x, y, \tau s) \geq \min\{\widetilde{\mathfrak{T}}(x, x, \tau^2), \widetilde{\mathfrak{T}}(y, y, s^2)\} \forall x, y \in L, \forall \tau, s \in R_+^*$.
The $(L, \widetilde{\mathfrak{T}})$ pair will be referred to as fuzzy inner product space.

Example 2.5 [19]: Let $<.,.> : L \times L \to C$ is an inner product, where L is a linear space over C. Then $\widetilde{\mathfrak{T}}: L \times L \times C \to [0,1]$,

$$\widetilde{\mathfrak{T}}(x, y, \tau) = \begin{cases} \frac{\tau}{\tau + |<x, y>|} & if \ \tau \in R_+^* \\ 0 & if \ \tau \in C \setminus R_+^* \end{cases}$$

is a fuzzy inner product on L for all $x, y \in L$.

3. Fuzzy Inner Product Space

In this part, the concept of the fuzzy inner product space on a complex linear space is introduced and several fuzzy inner product function properties are proved. In addition, the concept of orthogonality is presented and some of its properties are proven.

Definition 3.1: Let $\circledast$ be a continuous t-norm and let L be a linear space over C. If a fuzzy set $\widetilde{\mathfrak{T}}$ in $L \times L \times C$ satisfies the following conditions then it is termed a fuzzy inner product on L:
(F$\widetilde{\mathfrak{T}}$1) $\widetilde{\mathfrak{T}}(x, x, \tau) = 0, \forall x \in L, \forall \tau \in C \setminus R_+^*$, where R_+^* the set of all strict positive real numbers.
(F$\widetilde{\mathfrak{T}}$2)$\widetilde{\mathfrak{T}}(x, x, p) = 1, \forall x \in L, \forall p \in R_+^* \leftrightarrow x = 0$.
(F$\widetilde{\mathfrak{T}}$3) $\widetilde{\mathfrak{T}}(\gamma x, y, \tau) = \widetilde{\mathfrak{T}}\left(x, y, \frac{\tau}{|\gamma|}\right), \forall x, y \in L, \forall \tau \in C, \forall \gamma \in C$.
(F$\widetilde{\mathfrak{T}}$4) $\widetilde{\mathfrak{T}}(x, y, \tau) = \widetilde{\mathfrak{T}}(y, x, \bar{\tau}), \forall x, y \in L, \forall \tau \in C$.
(F$\widetilde{\mathfrak{T}}$5) $\widetilde{\mathfrak{T}}(x + y, z, \tau + s) \geq \widetilde{\mathfrak{T}}(x, z, \tau) \circledast \widetilde{\mathfrak{T}}(y, z, s), \forall x, y, z \in L, \forall \tau, s \in C$.
(F$\widetilde{\mathfrak{T}}$6) $\widetilde{\mathfrak{T}}(x, x, .) : R_+ \to [0, 1], \forall x \in L$ is left continuous and $\lim \widetilde{\mathfrak{T}}(x, x, \tau) = 1$as $\tau \to \infty$
(F$\widetilde{\mathfrak{T}}$7) $\widetilde{\mathfrak{T}}(x, y, \tau s) \geq \widetilde{\mathfrak{T}}(x, x, \tau^2) \circledast \widetilde{\mathfrak{T}}(y, y, s^2)$, $\forall x, y \in L, \forall \tau, s \in R_+^*$.
The triple $(L, \widetilde{\mathfrak{T}}, \circledast)$ will be referred to be a fuzzy inner product space briefly (F$\widetilde{\mathfrak{T}}$-space).

Example 3.2: Let $(L,<.,.>)$be an ordinary inner product space and $\circledast$ be a t-norm with the property that $p \circledast q = \min\{p,q\}, p, q \in [0,1]$. A mapping $\widetilde{\mathfrak{T}}: L \times L \times C \to [0,1]$ is defined as follows :

$$\widetilde{\mathfrak{T}}(x, y, \tau) = \begin{cases} \tau^{1/2}/(\tau^{1/2} + |<x, y>|^{1/2}) & if\ \tau \in R_+^* \\ 0 & if\ \tau \in C \setminus R_+^*. \end{cases}$$

Then $(L, \widetilde{\mathfrak{T}}, \circledast)$ is a F$\widetilde{\mathfrak{T}}$-space.

Proof:

(F$\widetilde{\mathfrak{T}}$1) $\forall x \in L, \forall \tau \in C \setminus R_+^*$, $\widetilde{\mathfrak{T}}(x, x, \tau) = 0$ is self-evident from the definition of $\widetilde{\mathfrak{T}}$.

(F$\widetilde{\mathfrak{T}}$2) $\forall x \in L, \forall \tau \in R_+^*$, $\widetilde{\mathfrak{T}}(x, x, \tau) = 1$ if and only if $\tau^{1/2} + |<x, x>|^{1/2} = \tau^{1/2}$ if and only if $|<x, x>|^{\frac{1}{2}} = 0 \Leftrightarrow |<x, x>| = 0$ if and if only $x = 0$.

(F$\widetilde{\mathfrak{T}}$3) $\widetilde{\mathfrak{T}}(\gamma x, y, \tau) = \widetilde{\mathfrak{T}}\left(x, y, \frac{\tau}{|\gamma|}\right), \forall x, y \in L, \forall \tau \in C, \forall \gamma \in C$ is evident for $\tau \in C \setminus R_+^*$.

If $\tau \in R_+^*$, then:

$$\widetilde{\mathfrak{T}}(\gamma x, y, \tau) = \frac{\tau^{1/2}}{\tau^{1/2} + |<\gamma x, y>|^{1/2}}$$
$$= \tau^{1/2}/(\tau^{1/2} + |\gamma|^{1/2}|<x, y>|^{1/2}$$
$$= (\tau/|\gamma|)^{1/2}/((\tau/|\gamma|)^{1/2} + |<x, y>|^{1/2})$$
$$= \widetilde{\mathfrak{T}}\left(x, y, \frac{\tau}{|\gamma|}\right)$$

(F$\widetilde{\mathfrak{T}}$4) $\widetilde{\mathfrak{T}}(x, y, \tau) = \widetilde{\mathfrak{T}}(y, x, \bar{\tau}), \forall x, y \in L, \forall \tau \in C$ is evident for $\tau \in C \setminus R_+^*$.

If $\tau \in R_+^*$, then $\tau = \bar{\tau}$ and

$\widetilde{\mathfrak{T}}(x, y, \tau) = \tau^{1/2}/(\tau^{1/2} + |<x, y>|^{1/2}) = \tau^{1/2}/(\tau^{1/2} + |<y, x>|^{1/2}) = \widetilde{\mathfrak{T}}(y, x, \bar{\tau})$.

(F$\widetilde{\mathfrak{T}}$5) $\widetilde{\mathfrak{T}}(x + y, z, \tau + \mathfrak{s}) \geq \widetilde{\mathfrak{T}}(x, z, \tau) \circledast \widetilde{\mathfrak{T}}(y, z, \mathfrak{s}), \forall x, y, z \in L, \forall \tau, \mathfrak{s} \in C$.

The result is evident if at least one of τ and $\mathfrak{s}$ is from $C \setminus R_+^*$.

If $\tau, \mathfrak{s} \in R_+^*$,

Suppose that $\widetilde{\mathfrak{T}}(x, z, \tau) = \min\{\widetilde{\mathfrak{T}}(x, z, \tau), \widetilde{\mathfrak{T}}(y, z, \mathfrak{s})\}$. Then

$\tau^{1/2}/(\tau^{1/2} + |<x, z>|^{1/2}) \leq \mathfrak{s}^{1/2}/(\mathfrak{s}^{1/2} + |<y, z>|^{1/2})$

$\Rightarrow (\tau^{1/2} + |<x, z>|^{1/2})/\tau^{1/2} \geq (\mathfrak{s}^{1/2} + |<y, z>|^{1/2})/\mathfrak{s}^{1/2}$

$\Rightarrow 1 + (|<x, z>|^{1/2}/\tau^{1/2}) \geq 1 + (|<y, z>|^{1/2}/\mathfrak{s}^{1/2})$

$\Rightarrow |<x, z>|^{1/2}/\tau^{1/2} \geq |<y, z>|^{1/2}/\mathfrak{s}^{1/2}$

$\Rightarrow \tau^{1/2}|<y, z>|^{1/2} \leq \mathfrak{s}^{1/2}|<x, z>|^{1/2}$

$\Rightarrow \tau|<y, z>| \leq \mathfrak{s}|<x, z>|$. So

$$\tau|<x + y, z>| \leq \tau|<x, z>| + \tau|<y, z>|$$
$$\leq \tau|<x, z>| + \mathfrak{s}|<x, z>|$$
$$\leq (\tau + \mathfrak{s})|<x, z>|$$

Then $(\tau^{1/2}|<x + y, z>|^{1/2}) \leq (\tau + \mathfrak{s})^{1/2}|<x, z>|^{1/2}$

$\Rightarrow \tau^{1/2}/(\tau^{1/2} + |<x, z>|^{1/2}) \leq (\tau + \mathfrak{s})^{1/2}/((\tau + \mathfrak{s})^{1/2} + |<x + y, z>|^{1/2})$

$\Rightarrow \widetilde{\mathfrak{T}}(x, z, \tau) \leq \widetilde{\mathfrak{T}}(x + y, z, \tau + \mathfrak{s})$.

Hence, by assumption $\widetilde{\mathfrak{T}}(x, z, \tau) = \min\{\widetilde{\mathfrak{T}}(x, z, \tau), \widetilde{\mathfrak{T}}(y, z, \mathfrak{s})\}$ and by t-norm $[p \circledast q = \min\{p,q\}, \forall p, q \in [0,1]]$, we obtain:

$\widetilde{\mathfrak{T}}(x, z, \tau) \circledast \widetilde{\mathfrak{T}}(y, z, \mathfrak{s}) = \min\{\widetilde{\mathfrak{T}}(x, z, \tau), \widetilde{\mathfrak{T}}(y, z, \mathfrak{s})\} = \widetilde{\mathfrak{T}}(x, z, \tau) \leq \widetilde{\mathfrak{T}}(x + y, z, \tau + \mathfrak{s})$

$\Rightarrow \widetilde{\mathfrak{T}}(x + y, z, \tau + \mathfrak{s}) \geq \widetilde{\mathfrak{T}}(x, z, \tau) \circledast \widetilde{\mathfrak{T}}(y, z, \mathfrak{s}), \forall x, y, z \in L, \forall \tau, \mathfrak{s} \in C$.

(F$\widetilde{\mathfrak{T}}$6) $\forall x \in L$, $\widetilde{\mathfrak{T}}(x, x, .) : R_+ \to [0, 1]$, is left continuous and $\lim \widetilde{\mathfrak{T}}(x, x, \tau) = 1$ as $\tau \to \infty$

$\lim_{\tau\to\infty} \widetilde{\mathfrak{T}}(x,x,\tau) = \lim_{\tau\to\infty} \tau^{1/2}/(\tau^{1/2} + |< x,x >|^{1/2})$

$= \lim_{\tau\to\infty} \tau^{1/2}/(\tau^{1/2}(1 + |< x,x >|^{1/2}/\tau^{1/2})) = 1.$

(F$\widetilde{\mathfrak{T}}$7) $\widetilde{\mathfrak{T}}(x, y, \tau\mathfrak{s}) \geq \widetilde{\mathfrak{T}}(x,x,\tau^2) \circledast \widetilde{\mathfrak{T}}(y,y,\mathfrak{s}^2)$, $\forall x, y \in L, \forall \tau, \mathfrak{s} \in R_+^*$.
The result is evident if at least one of τ and $\mathfrak{s}$ is from $C \setminus R_+^*$.
If $\tau, \mathfrak{s} \in R_+^*$,
Suppose that $\widetilde{\mathfrak{T}}(x,x,\tau^2) \leq \widetilde{\mathfrak{T}}(y,y,\mathfrak{s}^2)$. Then

$\tau/(\tau + |< x,x >|^{1/2}) \leq \mathfrak{s}/(\mathfrak{s} + |< y,y >|^{1/2})$

$(\tau + |< x,x >|^{1/2})/\tau \geq (\mathfrak{s} + |< y,y >|^{1/2}/\mathfrak{s})$

$(|< x,x >|^{1/2}/\tau) + 1 \geq (|< y,y >|^{1/2}/\mathfrak{s}) + 1$

$|< x,x >|^{1/2}/\tau \geq |< y,y >|^{1/2}/\mathfrak{s}$

$\mathfrak{s}|< x,x >|^{1/2} \geq \tau|< y,y >|^{1/2}$ and by using Cauchy Schwarze inequality obtain:

$\tau|< x,y >| \leq \tau\sqrt{|< x,x >|}\sqrt{|< y,y >|}$

$\leq \mathfrak{s}\sqrt{|< x,x >|}\sqrt{|< x,x >|} = \mathfrak{s}|< x,x >|$

$\Rightarrow \tau^2|< x,y >| \leq \tau\mathfrak{s}|< x,x >|$

$\Rightarrow \tau|< x,y >|^{1/2} \leq (\tau\mathfrak{s})^{1/2}|< x,x >|^{1/2}$

$\Rightarrow \tau^{3/2}\mathfrak{s}^{1/2} + \tau|< x,y >|^{1/2} \leq \tau^{3/2}\mathfrak{s}^{1/2} + (\tau\mathfrak{s})^{1/2}|< x,x >|^{1/2}$

$\Rightarrow \tau[(\tau\mathfrak{s})^{1/2} + |< x,y >|^{1/2}] \leq (\tau\mathfrak{s})^{1/2}[\tau + |< x,x >|^{1/2}]$

$\Rightarrow \tau/(\tau + |< x,x >|^{1/2}) \leq (\tau\mathfrak{s})^{1/2}/((\tau\mathfrak{s})^{1/2} + |< x,y >|^{1/2})$

$\Rightarrow \widetilde{\mathfrak{T}}(x,x,\tau^2) \leq \widetilde{\mathfrak{T}}(x,y,\tau\mathfrak{s})$

Since $\widetilde{\mathfrak{T}}(x,x,\tau^2) \circledast \widetilde{\mathfrak{T}}(y,y,\mathfrak{s}^2) = \min\{\widetilde{\mathfrak{T}}(x,x,\tau^2), \widetilde{\mathfrak{T}}(y,y,\mathfrak{s}^2)\}$

$\Rightarrow \widetilde{\mathfrak{T}}(x,x,\tau^2) \circledast \widetilde{\mathfrak{T}}(y,y,\mathfrak{s}^2) = \widetilde{\mathfrak{T}}(x,x,\tau^2) \leq \widetilde{\mathfrak{T}}(x,y,\tau\mathfrak{s})$.

$\Rightarrow \widetilde{\mathfrak{T}}(x,y,\tau\mathfrak{s}) \geq \widetilde{\mathfrak{T}}(x,x,\tau^2) \circledast \widetilde{\mathfrak{T}}(y,y,\mathfrak{s}^2)$, $\forall x, y \in L, \forall \tau, \mathfrak{s} \in R_+^*$.

Example 3.3: Let $(L, <.,.>)$ be an ordinary inner product space and let $\circledast$ be a t-norm such that $p \circledast q = \min\{p,q\}, p,q \in [0,1]$. A mapping $\widetilde{\mathfrak{T}}: L \times L \times C \to [0,1]$ is defined as follows:

$$\widetilde{\mathfrak{T}}(x,y,\tau) = \begin{cases} 0 & if\ \tau \leq |< x,y >| \\ 1 & if\ \tau > |< x,y >| \\ 0 & if\ \tau \in C \setminus R_+^* \end{cases}$$

Then $\widetilde{\mathfrak{T}}$ is a F$\widetilde{\mathfrak{T}}$ on L.

Proof:
(F$\widetilde{\mathfrak{T}}$1) $\forall x \in L, \forall \tau \in C \setminus R_+^*$, $\widetilde{\mathfrak{T}}(x,x,\tau) = 0$ is evident from the definition of $\widetilde{\mathfrak{T}}$.
(F$\widetilde{\mathfrak{T}}$2) $\forall x \in L$, $if\ x = 0\ then < x,x \geq 0 \Rightarrow \tau > |< x,x >|, \forall \tau > 0$
$\Rightarrow \widetilde{\mathfrak{T}}(x,x,\tau) = 1.$
(F$\widetilde{\mathfrak{T}}$3) $\widetilde{\mathfrak{T}}(\gamma x, y, \tau) = \widetilde{\mathfrak{T}}\left(x, y, \frac{\tau}{|\gamma|}\right), \forall x, y \in L, \forall \tau \in C, \forall \gamma \in C$ is evident for $\tau \in C \setminus R_+^*$.
For $\tau \in R_+^*$, then the property follows from the fact that:
$|< \gamma x, y >| = |\gamma||< x,y >|$.
(F$\widetilde{\mathfrak{T}}$4) $\widetilde{\mathfrak{T}}(x,y,\tau) = \widetilde{\mathfrak{T}}(y,x,\bar{\tau}), \forall x, y \in L, \forall \tau \in C$ is evident for $\tau \in C \setminus R_+^*$.
For $\tau \in R_+^*$, the property follows from the fact that:
$|< x,y >| = |< y,x >|$.
(F$\widetilde{\mathfrak{T}}$5) we will show $\widetilde{\mathfrak{T}}(x + y, z, \tau + \mathfrak{s}) \geq \widetilde{\mathfrak{T}}(x,z,\tau) \circledast \widetilde{\mathfrak{T}}(y,z,\mathfrak{s}), \forall x, y, z \in L, \forall \tau, \mathfrak{s} \in C$.
Consider the following cases:
(1)If $\tau > |< x,z >|$ and $\mathfrak{s} > |< y,z >|$ then $\tau + \mathfrak{s} > |< x + y, z >|$. So,
$\widetilde{\mathfrak{T}}(x + y, z, \tau + \mathfrak{s}) = 1, \widetilde{\mathfrak{T}}(x,z,\tau) = 1, \widetilde{\mathfrak{T}}(y,z,\mathfrak{s}) = 1$ that is,

$\widetilde{\mathfrak{T}}(x + y, z, \tau + \mathfrak{s}) = 1 \geq \widetilde{\mathfrak{T}}(x, z, \tau) \circledast \widetilde{\mathfrak{T}}(y, z, \mathfrak{s})$.
(2) If any one of $\tau \leq |< x, z >|$, $\mathfrak{s} \leq |< y, z >|$ holds then
$\widetilde{\mathfrak{T}}(x, z, \tau) \circledast \widetilde{\mathfrak{T}}(y, z, \mathfrak{s}) = \min\{\widetilde{\mathfrak{T}}(x, z, \tau), \widetilde{\mathfrak{T}}(y, z, \mathfrak{s})\} = 0$ and obviously
$\widetilde{\mathfrak{T}}(x + y, z, \tau + \mathfrak{s}) = 1 \geq \widetilde{\mathfrak{T}}(x, z, \tau) \circledast \widetilde{\mathfrak{T}}(y, z, \mathfrak{s})$.
(F$\widetilde{\mathfrak{T}}$6) $\forall x \in L$, $\widetilde{\mathfrak{T}}(x, x, .): R_+ \to [0, 1]$ is left continuous and $\lim \widetilde{\mathfrak{T}}(x, x, \tau) = 1$ as $\tau \to \infty$ is clear.
(F$\widetilde{\mathfrak{T}}$7) To show that $\widetilde{\mathfrak{T}}(x, y, \tau\mathfrak{s}) \geq \widetilde{\mathfrak{T}}(x, x, \tau^2) \circledast \widetilde{\mathfrak{T}}(y, y, \mathfrak{s}^2)$, $\forall x, y \in L, \forall \tau, \mathfrak{s} \in R_+^*$. Observe that $\tau^2 > |< x, x >|$ and $\mathfrak{s}^2 > |< y, y >|$
$\Rightarrow \tau^2\mathfrak{s}^2 > |< x, x >||< y, y >| = \|x\|^2 \|y\|^2$.
$\Rightarrow \tau\mathfrak{s} > \|x\| \|y\| \geq |< x, y >|$.
So $\widetilde{\mathfrak{T}}(x, x, \tau) = 1$, $\widetilde{\mathfrak{T}}(y, y, \mathfrak{s}) = 1$, $\widetilde{\mathfrak{T}}(x, y, \tau\mathfrak{s}) = 1$, that is,
$\widetilde{\mathfrak{T}}(x, y, \tau\mathfrak{s}) = 1 \geq \widetilde{\mathfrak{T}}(x, x, \tau^2) \circledast \widetilde{\mathfrak{T}}(y, y, \mathfrak{s}^2)$.

Proposition3.4: If $\tau \in R_+^*$ and $x \in L$, then $\widetilde{\mathfrak{T}}(x, 0, \tau) = 1$.

Proof: The following is derived from conditions (F$\widetilde{\mathfrak{T}}$3) and (F$\widetilde{\mathfrak{T}}$6):
$\widetilde{\mathfrak{T}}(x, 0, \tau) = \widetilde{\mathfrak{T}}(x, 0, 2n\tau)$
$= \widetilde{\mathfrak{T}}(x, x - x, n\tau + n\tau)$
$\geq \widetilde{\mathfrak{T}}(x, x, n\tau) \circledast \widetilde{\mathfrak{T}}(x, -x, n\tau)$
$\geq \widetilde{\mathfrak{T}}(x, x, n\tau) \circledast \widetilde{\mathfrak{T}}\left(x, x, \frac{n\tau}{|-1|}\right) \geq \widetilde{\mathfrak{T}}(x, x, n\tau) \circledast \widetilde{\mathfrak{T}}(x, x, n\tau)$
Now by Theorem2.2 if $\max\{\widetilde{\mathfrak{T}}(x, x, n\tau), \widetilde{\mathfrak{T}}(x, x, n\tau)\} = 1$ then
$\widetilde{\mathfrak{T}}(x, 0, \tau) \geq \widetilde{\mathfrak{T}}(x, x, n\tau) \circledast \widetilde{\mathfrak{T}}(x, x, n\tau) \geq \min\{\widetilde{\mathfrak{T}}(x, x, n\tau), \widetilde{\mathfrak{T}}(x, x, n\tau)\} = \widetilde{\mathfrak{T}}(x, x, n\tau) = 1$
and $\widetilde{\mathfrak{T}}(x, x, n\tau) \to 1$ as $n \to \infty$ then $\widetilde{\mathfrak{T}}(x, 0, \tau) \geq 1$, but $\widetilde{\mathfrak{T}}(x, 0, \tau) \leq 1 \Rightarrow \widetilde{\mathfrak{T}}(x, 0, \tau) = 1$
Otherwise, $\widetilde{\mathfrak{T}}(x, 0, \tau) \geq \widetilde{\mathfrak{T}}(x, x, n\tau) \circledast \widetilde{\mathfrak{T}}(x, x, n\tau) = 1$.
Hence, $\widetilde{\mathfrak{T}}(x, 0, \tau) \geq 1$ and then the proposition holds.

Proposition3.5: For $\tau \in R_+^*$ and $y \in L$, we have
$\widetilde{\mathfrak{T}}(0, y, \tau) = 1$.

Proof: The following is derived from conditions (F$\widetilde{\mathfrak{T}}$3) and (F$\widetilde{\mathfrak{T}}$6):
$\widetilde{\mathfrak{T}}(0, y, \tau) = \widetilde{\mathfrak{T}}(y - y, y, 2n\tau)$
$\geq \widetilde{\mathfrak{T}}(y, y, n\tau) \circledast \widetilde{\mathfrak{T}}(-y, y, n\tau)$
$\geq \widetilde{\mathfrak{T}}(y, y, n\tau) \circledast \widetilde{\mathfrak{T}}\left(y, y, \frac{n\tau}{|-1|}\right)$
$\geq \widetilde{\mathfrak{T}}(y, y, n\tau) \circledast \widetilde{\mathfrak{T}}(y, y, n\tau)$
Again by Theorem2.2 if $\max\{\widetilde{\mathfrak{T}}(y, y, n\tau), \widetilde{\mathfrak{T}}(y, y, n\tau)\} = 1$,then
$\widetilde{\mathfrak{T}}(0, y, \tau) \geq \widetilde{\mathfrak{T}}(y, y, n\tau) \circledast \widetilde{\mathfrak{T}}(y, y, n\tau)$
$\geq \min\{\widetilde{\mathfrak{T}}(y, y, n\tau), \widetilde{\mathfrak{T}}(y, y, n\tau)\}$
$= \widetilde{\mathfrak{T}}(y, y, n\tau) = 1$
and $\widetilde{\mathfrak{T}}(y, y, n\tau) \to 1$ as $n \to \infty$ then $\widetilde{\mathfrak{T}}(0, y, \tau) \geq 1$, but $\widetilde{\mathfrak{T}}(0, y, \tau) \leq 1 \Rightarrow \widetilde{\mathfrak{T}}(0, y, \tau) = 1$.
Otherwise, $\widetilde{\mathfrak{T}}(0, y, \tau) \geq \widetilde{\mathfrak{T}}(y, y, n\tau) \circledast \widetilde{\mathfrak{T}}(y, y, n\tau) = 1$.
Hence, $\widetilde{\mathfrak{T}}(0, y, \tau) \geq 1$ and then the proposition holds.

Proposition3.6: For all $x, y \in L$, $\widetilde{\mathfrak{T}}(x, y, .): R_+ \to [0,1]$ is a monotonic non-decreasing function.

Proof: Let $\tau, \mathfrak{s} \in R_+$ where $\mathfrak{s} \leq \tau$. Then there exists k such that $= \mathfrak{s} + k$.

$\widetilde{\mathfrak{T}}(x, y, \tau) = \widetilde{\mathfrak{T}}(x + 0, y, s + k)$
$\geq \widetilde{\mathfrak{T}}(x, y, s) \circledast \widetilde{\mathfrak{T}}(0, y, k)$ (by F$\widetilde{\mathfrak{T}}$5)
$\geq \widetilde{\mathfrak{T}}(x, y, s) \circledast 1 = \widetilde{\mathfrak{T}}(x, y, s)$ (by Proposition3.5)
Thus, $\widetilde{\mathfrak{T}}(x, y, s) \leq \widetilde{\mathfrak{T}}(x, y, \tau)$ for $s \leq \tau$.

Proposition3.7: For all x, y and $z \in L$ and for all $\tau \in C$, the following holds
$\widetilde{\mathfrak{T}}(x, y, \tau) \geq \widetilde{\mathfrak{T}}(x, y - z, \tau) \circledast \widetilde{\mathfrak{T}}(x, y + z, \tau)$ **.**

Proof: $\widetilde{\mathfrak{T}}(x, y, \tau) = \widetilde{\mathfrak{T}}(x, 2y, 2\tau)$
$= \widetilde{\mathfrak{T}}(x, y + z + y - z, \tau + \tau) \geq \widetilde{\mathfrak{T}}(x, y + z, \tau) \circledast \widetilde{\mathfrak{T}}(x, y - z, \tau)$.
Hence, $\widetilde{\mathfrak{T}}(x, y, \tau) \geq \widetilde{\mathfrak{T}}(x, y - z, \tau) \circledast \widetilde{\mathfrak{T}}(x, y + z, \tau)$.

Theorem 3.8: Let $(L, \widetilde{\mathfrak{T}}, \circledast)$ be a F$\widetilde{\mathfrak{T}}$-space where $p \circledast q = \min\{p, q\}, for\ all\ p, q \in [0,1]$. Then $\eta: L \times [0, \infty) \to [0,1]$ is a fuzzy norm on L which is defined by $\eta(x, \tau) = \widetilde{\mathfrak{T}}(x, x, \tau^2)$.

Proof:
(1) $\forall x \in L$, $\eta(x, 0) = \widetilde{\mathfrak{T}}(x, x, 0) = 0$ from F$\widetilde{\mathfrak{T}}$1.
(2) $[\eta(x, \tau) = 1, \forall \tau > 0] \Leftrightarrow [\widetilde{\mathfrak{T}}(x, x, \tau^2) = 1, \forall \tau > 0]$
$\Leftrightarrow x = 0$ from F$\widetilde{\mathfrak{T}}$2
(3) $\eta(\gamma x, \tau) = \widetilde{\mathfrak{T}}(\gamma x, \gamma x, \tau^2) = \widetilde{\mathfrak{T}}\left(x, \gamma x, \frac{\tau^2}{|\gamma|}\right) = \widetilde{\mathfrak{T}}\left(\gamma x, x, \frac{\overline{\tau^2}}{|\gamma|}\right) = \widetilde{\mathfrak{T}}\left(\gamma x, x, \frac{\tau^2}{|\gamma|}\right)$
$= \widetilde{\mathfrak{T}}\left(x, x, \frac{\tau^2}{|\gamma|^2}\right) = \widetilde{\mathfrak{T}}\left(x, x, (\frac{\tau}{|\gamma|})^2\right) = \eta\left(x, \frac{\tau}{|\gamma|}\right), \forall \tau \geq 0, \forall \gamma \in C$.
(4) $\eta(x + y, \tau + s) \geq \eta(x, \tau) \circledast \eta(y, s)$.
The above inequality is evident if $\tau = 0$ or $s = 0$. Now consider $\tau, s > 0$, then
$\eta(x + y, \tau + s) = \widetilde{\mathfrak{T}}(x + y, x + y, (\tau + s)^2)$
$=\widetilde{\mathfrak{T}}(x + y, x + y, \tau^2 + \tau s + \tau s + s^2)$
$\geq \widetilde{\mathfrak{T}}(x, x + y, \tau^2 + \tau s) \circledast \widetilde{\mathfrak{T}}(y, x + y, \tau s + s^2)$
$\geq \widetilde{\mathfrak{T}}(x, x, \tau^2) \circledast \widetilde{\mathfrak{T}}(x, y, \tau s) \circledast \widetilde{\mathfrak{T}}(y, x, \tau s) \circledast \widetilde{\mathfrak{T}}(y, y, s^2)$
$\geq \widetilde{\mathfrak{T}}(x, x, \tau^2) \circledast \widetilde{\mathfrak{T}}(y, y, s^2) \circledast \widetilde{\mathfrak{T}}(x, y, \tau s)$
$\geq \widetilde{\mathfrak{T}}(x, x, \tau^2) \circledast \widetilde{\mathfrak{T}}(y, y, s^2) \circledast \widetilde{\mathfrak{T}}(x, x, \tau^2) \circledast \widetilde{\mathfrak{T}}(y, y, s^2)$
and by t-norm $[p \circledast q = \min\{p, q\}$, for all $p, q \in [0,1]]$, we obtain
$\eta(x + y, \tau + s) \geq \widetilde{\mathfrak{T}}(x, x, \tau^2) \circledast \widetilde{\mathfrak{T}}(y, y, s^2) = \eta(x, \tau) \circledast \eta(y, s)$.
Hence, $\eta(x + y, \tau + s) \geq \eta(x, \tau) \circledast \eta(y, s)$.
(5) From F$\widetilde{\mathfrak{T}}$6 , we get that $\eta(x, .)$ is a non-decreasing function and $\lim \eta(x, \tau) = 1$ as $\tau \to \infty$.
Hence, $\eta(x, \tau) = \widetilde{\mathfrak{T}}(x, x, \tau^2)$ is a fuzzy norm on L.
The notion of orthogonality in a F$\widetilde{\mathfrak{T}}$-space has now been established, and various orthogonality features have been investigated.

Definition 3.9: Consider $(L, \widetilde{\mathfrak{T}}, \circledast)$ be a F$\widetilde{\mathfrak{T}}$-space. If $\widetilde{\mathfrak{T}}(x, y, \tau) = 1, \forall \tau \in R_+^*$, and $\widetilde{\mathfrak{T}}(x, y, \tau) = 0, \forall \tau \in C \setminus R_+^*$ then $x, y \in L$ is orthogonal, and it is indicated as $x \perp y$. For subset $G, \mathcal{W} \subseteq L$ we express $x \perp G$ if $x \perp g$ for each $g \in G$ and $G \perp \mathcal{W}$ if $g \perp w$ for each $g \in G$ and all $w \in \mathcal{W}$.

Example 3.10: Let $(L, <.,.>))$ be an ordinary inner product space**.** $G, \mathcal{W} \subseteq L$ and $G \perp \mathcal{W}$. Since $G \perp \mathcal{W}$ then $x \perp y$ for each $x \in G$ and $y \in \mathcal{W}$**.** Thus, $< x, y >= 0$.

Assume that $(L, \widetilde{\mathfrak{T}}, \circledast)$ be a F$\widetilde{\mathfrak{T}}$ −space such that $\widetilde{\mathfrak{T}} : L \times L \times C \to [0,1]$ is a fuzzy inner product on L which is defined by

$$\widetilde{\mathfrak{T}}(x, y, \tau) = \begin{cases} \dfrac{\tau}{\tau + |< x, y >|} & if\ \tau \in R_+^* \\ 0 & if\ \tau \in C \setminus R_+^* \end{cases}$$

Then $\widetilde{\mathfrak{T}}(x, y, \tau) = 1$ for all $x \in G$, $y \in \mathcal{W}$, $\tau \in R_+^*$, and $\widetilde{\mathfrak{T}}(x, y, \tau) = 0$ for all $\tau \in C \setminus R_+^*$. So $G \perp \mathcal{W}$ in a F$\widetilde{\mathfrak{T}}$-space $(L, \widetilde{\mathfrak{T}}, \circledast)$.

Theorem 3.11: Suppose that $(L, \widetilde{\mathfrak{T}}, \circledast)$ be a F$\widetilde{\mathfrak{T}}$-space. The properties of orthogonality are as follows:

(i) $\forall x \in L, 0 \perp x$.
(ii) $\forall x, y \in L$, if $x \perp y$ then $y \perp x$.
(iii) $x = 0$ if $x \perp x$.
(iv) $\forall x, y \in L, \forall \alpha \in C$, if $x \perp y$ then $x \perp \alpha y$.
(v) If $x \perp x_i$ $\forall i = 1,2, \dots$ then $x \perp \sum x_i$

Proof(i):
From Proposition3.5, $\widetilde{\mathfrak{T}}(0, x, \tau) = 1, \forall x \in L, \forall \tau \in R_+^*$ and $\widetilde{\mathfrak{T}}(0, x, \tau) = 0$, $\forall \tau \in C \setminus R_+^*$ from F$\widetilde{\mathfrak{T}}$1. Hence, $0 \perp x\ \forall x \in L$.

Proof(ii): If $x \perp y \Rightarrow \widetilde{\mathfrak{T}}(x, y, \tau) = 1, for\ all\ \tau \in R_+^*$, and $\widetilde{\mathfrak{T}}(x, y, \tau) = 0, for\ all\ \tau \in C \setminus R_+^*$ and from condition F$\widetilde{\mathfrak{T}}$4,$\widetilde{\mathfrak{T}}(x, y, \tau) = \widetilde{\mathfrak{T}}(y, x, \bar{\tau}), \forall x, y \in L, \forall \tau \in C$.
If $\tau \in C \setminus R_+^*, \widetilde{\mathfrak{T}}(x, y, \tau) = 0 = \widetilde{\mathfrak{T}}(y, x, \tau), \forall x, y \in L$.
If $\tau \in R_+^*$, then $\tau = \bar{\tau}$ and $\widetilde{\mathfrak{T}}(x, y, \tau) = 1 = \widetilde{\mathfrak{T}}(y, x, \tau)\ \forall \tau \in R_+^*$.
$\Rightarrow \forall x, y \in L, y \perp x$.

Proof(iii): If $x \perp x$ then $\widetilde{\mathfrak{T}}(x, x, \tau) = 1, \forall \tau \in R_+^*$, and $\widetilde{\mathfrak{T}}(x, x, \tau) = 0, \forall \tau \in C \setminus R_+^*$ and from condition F$\widetilde{\mathfrak{T}}$2 obtain $x = 0$.

Proof(iv): If $x \perp y$ then $for\ all\ \tau \in R_+^*$, $\widetilde{\mathfrak{T}}(x, y, \tau) = 1$ and $\widetilde{\mathfrak{T}}(x, y, \tau) = 0, \forall \tau \in C \setminus R_+^*$.
Let $\tau \in R_+^*\ then\ \tau = \bar{\tau}$,

$$\widetilde{\mathfrak{T}}(x, \alpha y, \tau) = \widetilde{\mathfrak{T}}(\alpha y, x, \bar{\tau}) = \widetilde{\mathfrak{T}}(\alpha y, x, \tau) = \widetilde{\mathfrak{T}}\left(y, x, \frac{\tau}{|\alpha|}\right) = \widetilde{\mathfrak{T}}\left(x, y, \frac{\tau}{|\alpha|}\right) = 1$$

Let $\tau \in C \setminus R_+^*, \alpha \in C$, then $\widetilde{\mathfrak{T}}(x, \alpha y, \tau) = 0$ is evident from the condition F$\widetilde{\mathfrak{T}}$1.

Proof(v): If $x \perp x_i$ then $for\ all\ \tau \in R_+^*, \widetilde{\mathfrak{T}}(x, x_i, \tau) = 1$ and $\widetilde{\mathfrak{T}}(x, x_i, \tau) = 0, \forall \tau \in C \setminus R_+^*$.
Let $\tau \in R_+^*$,

$$\widetilde{\mathfrak{T}}\left(x, \sum_{i=1}^{n} x_i, \tau\right) = \widetilde{\mathfrak{T}}(x, x_1 + x_2 + \dots + x_n, \tau)$$
$$\geq \widetilde{\mathfrak{T}}(x, x_1, \tau_1) \circledast \widetilde{\mathfrak{T}}(x, x_2, \tau_2) \circledast \dots \circledast \widetilde{\mathfrak{T}}(x, x_n, \tau_n)$$

where $\tau = \tau_1 + \tau_2 + \dots + \tau_n$.

$$\geq 1 \circledast 1 \circledast \dots \circledast 1 = 1.$$

Hence, for all $x \in L$, we have $\widetilde{\mathfrak{T}}(x, \sum_{i=1}^{n} x_i, \tau) \geq 1$ and $\widetilde{\mathfrak{T}}(x, \sum_{i=1}^{n} x_i, \tau) \leq 1 \Rightarrow \widetilde{\mathfrak{T}}(x, \sum_{i=1}^{n} x_i, \tau) = 1$.
Now for $\in C \setminus R_+^*$, by (iv), $-x \perp x_i, \forall i = 1,2, \dots$
Hence, $\widetilde{\mathfrak{T}}(-x, \sum_{i=1}^{n} x_i, \tau) = \widetilde{\mathfrak{T}}\left(x, \sum_{i=1}^{n} x_i, \frac{\tau}{|-1|}\right)$

$$= \widetilde{\mathfrak{T}}(x, \sum_{i=1}^{n} x_i, \tau) = 0.$$

4. Conclusions

In this paper, a new type for the fuzzy inner product space on a complex linear space has been proposed. Some properties of inner product function are proven. Moreover, the concept of orthogonality has been introduced and some of its properties have been shown. This work can be considered the groundwork for further research on problems and various concepts within the theory of a fuzzy inner product space.

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