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On the Existence and Oscillatory Solutions of Multiple Delay Differential Equation

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Abstract

In this paper, we introduce new conditions to prove that the existence and boundedness of the solution by convergent sequences and convergent series. The theorem of Krasnoselskii, Lebesgue's dominated convergence theorem and fixed point theorem are used to get some sufficient conditions for the existence of solutions. Furthermore, we get sufficient conditions to guarantee the oscillatory property for all solutions in this class of equations. An illustrative example is included as an application to the main results.

Keywords: Existence of Nonocillatory Solutions, Oscillatory Property, Multiple Delay Differential Equation, Banach Space.

الوجود وتذبذب الحلول لمعادلة تفاضلية تباطؤية متعددة

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الخلاصة

في هذه الورقة ، قدمنا شروطًا جديدة لإثبات أن الحل موجود ومقيدة بمتواليات متقاربة ومتسلسلات متقاربة. استخدمنا نظريات نقطة Krasnoselskii الثابتة وتقارب Lebesgue's المهيمنة للحصول على شروط كافية جديدة لوجود الحلول. أضافتا الى ذلك، حصلنا على الشروط الكافية لضمان خاصية التذبذب لجبيع الحلول لهذا النوع من المعادلات. تم تضمين مثال توضيحي كتطبيق للنتائج الرئيسية.

1. Introduction

The consideration of theory of differential equations (DEs) includes several fields of study such as the existence of solutions [1,2], numerical solutions [3,4], and the finding the qualitative properties [5,6].

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In recent years, the study of solutions of delay differential equations and their properties such as oscillation, and asymptotic behavior has been increased due to its essential applications and widespread in real -world fields. In fact, researchers have faced the new models of this type of equations in the applied fields because of its great development in the fields of technology and various sciences. In addition, the delay differential equations (DDEs) have a great influence in modeling several scientific problems such as technical, physical, or biological models, as in studies [7-9].

In [10], S. S. Santra have considered the existence of positive solution and oscillatory property to the type of nonlinear neutral (NDDE):

$\frac{d}{dt}(x(t)+ r(t)\,x(t-\alpha)) + q(t)H(x(t-\upsilon)) = f(t).$

In [11], S. Pinelas and S. S. Santra have studied nonlinear NDDE with several delays:

$$\frac{d}{dt}\big(x(t)+ r(t)x(t-\tau)\big) + \sum_{\xi=1}^{\eta} b_\xi(t)G_\xi\left(x\left(t-\upsilon_\xi\right)\right) = 0$$

H. Xiao and B. Zheng in [12] have obtained sufficient conditions for existence to multiple periodic solutions to non-autonomous DDE:

$$\frac{d}{dt}x(t) = -ax(t-1)[1+x(t)]$$

H. Ahmad, S. W. Yao and others in [13] have studied the oscillatory of all solutions to the second order nonlinear NDDE with applications:

$\frac{d}{dt}\left(\psi(t)\left(\frac{d}{dt}x(t)\right)^{c}\right) + \sum_{\xi=1}^{\eta} b_\xi(t)x^{a_\xi}\left(\tau_\xi(t)\right) = 0,$ for $t \geqslant t_0$

S. H. Saker, M. Elabbasy and T. S. Hassan in [14] considered nonoscillatory properties to nonlinear NDDEs with several positive and negative coefficients of the form:

$$\frac{d}{dt}\big(x(t)+ \psi(t)x(t-\alpha)\big) + \sum_{\zeta=1}^{\mu} a_\zeta(t)G_\zeta\left(x\left(t-\tau_\zeta(t)\right)\right) - \sum_{\xi=1}^{\eta} b_\xi(t)G_\xi\left(x\left(t-\upsilon_\xi\right)\right) = 0.$$

In this paper, we focus on the existence and oscillatory solution to the following non-linear NDEs with Multiple delays:

$$\frac{d}{dt}x(t) = -\sum_{\xi=1}^{\eta} a_\xi(t)g_\xi\left(x\left(\tau_\xi(t)\right)\right) + \frac{d}{dt}\sum_{\xi=1}^{\eta} b_\xi(t)G_\xi\left(t, x\left(\tau_\xi(t)\right)\right) \qquad (1.1)$$

Throughout this work, we will consider the following hypotheses:

(i) $C(H_1, H_2)$ denotes to the set for all functions that are continuous; $f: H_1 \to H_2$ with the supremum norm $\|.\|$.

(ii) We suppose that $a_\xi,\ b_\xi \in C(\Re^+, \Re^+), (\xi = 1,2,\dots,\eta)$, and the functions $\tau_\xi : \Re^+ \to \Re^+$ are differentiable with $\tau_\xi(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

(iii) The functions $g_\xi(x)$ and $G_\xi(t,x)$ are continuous and satisfy Lipschitz condition in x. That is, there are positive constants M_ξ $(\xi = 1,2,\dots,\eta)$, such that

$\left|G_\xi(t,x) - G_\xi(t,\gamma)\right| \leqslant M_\xi|x-\gamma| \quad \xi = 1,2,\dots,\eta,$

The solution $x(t)$ satisfies Eq.(1.1) for $t \geqslant t_1$. We say that solution $x(t)$ is a non-oscillatory solution if it is eventually negative or eventually positive, so that there exists $t_* \geqslant t_0$, such that $x(t) \succ 0$ or $x(t) \prec 0$ for all $t \geqslant t_*$. Otherwise, the solution is said to be oscillatory [7].

Definition 1.1: Let $x(\mathrm{t})$ be a function, $x(\mathrm{t})$ is said to be relatively bounded of below or above if there exists a function $\gamma(\mathrm{t})$ and constant $\varkappa$: $\ni \gamma(\mathrm{t}) \preccurlyeq x(\mathrm{t}) \preccurlyeq \varkappa$ $\left(\varkappa \preccurlyeq x(\mathrm{t}) \preccurlyeq \gamma(\mathrm{t})\right)$.
The following lemma and theorem are needed in the next section to the main results.

Lemma 1.1: [14] (Theorem to Krasnoselskii of Fixed Point).
Let X be a Banach space, Ʊ is closed convex bounded set in X, if S_1, S_2: Ʊ ⟶ X, $\ni S_1x + S_2\gamma \in$ Ʊ, for all $x, \gamma \in$ Ʊ. If S_1 is a mapping with contractive feature and S_2 is a completely continuous mapping, then $S_1x + S_2\gamma = x$ is a solution onƱ.

Theorem 1.2 [15] (Lebesgue's Dominated Convergence Theorem)
Let $\{\wp_n\}$ be a sequence to measurable functions onᴇ, and q be integrable function on ᴇ with dominates $\{\wp_n\}$ on ᴇ such that $|\wp_n(x)| \preccurlyeq q(x)$ on ᴇ, for all n. If $\{\wp_n\} \to \{\wp\}$ pointwise almost everywhere on E, then $\wp$ is integrable on ᴇ with:
$lim_{n\to\infty} \int_{\mathrm{E}} \wp_n = \int_{\mathrm{E}} \wp$, ᴇ is a measurable finite set.

2. Sufficient Conditions for Existence:

In this section, we introduce new sufficient conditions to ensure the existence and bounded of solution by two positive functions u and v on $[\mathrm{t}_1, \infty)$ of Eq.(1.1), $\mathrm{t}_1 \succcurlyeq \mathrm{t}_0$. The existence to positive bounded solution is studied, while existence of eventually negative solution can be found similarly.
Throughout this section, we suppose the following conditions hold in the included results:
A1. $N_1 \prec a_\xi(\mathrm{t}), b_\xi(\mathrm{t}) \preccurlyeq N_2$, where N_1 and $N_2 \neq 0$, are constants, $\xi = 1, 2, 3, \dots, \eta$.
A2. $R_1 x(\mathrm{t}) \preccurlyeq g_\xi\left(x\left(\tau_\xi(\mathrm{t})\right)\right) \preccurlyeq R_2 x(\mathrm{t})$, where R_1 and $R_2 \neq 0$, are constants, $\xi = 1, 2, 3, \dots, \eta$.
A3. $C_1 x(\mathrm{t}) \preccurlyeq G_\xi\left(\mathrm{t}, x\left(\tau_\xi(\mathrm{t})\right)\right) \preccurlyeq C_2 x(\mathrm{t})$, where C_1 and $C_2 \neq 0$ are constants, $\xi = 1, 2, 3, \dots, \eta$.

Theorem 2.1
Assume that A1- A3 hold, and the bounded functions $u, v \in C^1(\mathbb{N}, [0, \infty))$, such that $\mathrm{t}_1 \succcurlyeq \mathrm{t}_0 + \rho$:

$$u(\mathrm{t}) \preccurlyeq u(\mathrm{t}_1),\ \mathrm{t}_0 \preccurlyeq \mathrm{t} \preccurlyeq \mathrm{t}_1 \tag{2.1}$$

$$\frac{1}{R_2}\left(C_2 \sum_{\xi=1}^{\eta} v\left(\tau_\xi(\mathrm{t})\right) - \frac{1}{N_2} v(\mathrm{t})\right) \preccurlyeq \int_t^\infty \sum_{\xi=1}^{\eta} v\left(\tau_\xi(s)\right) ds$$

$$\int_t^\infty \sum_{\xi=1}^{\eta} u\left(\tau_\xi(s)\right) ds \preccurlyeq \frac{1}{R_1}\left(C_1 \sum_{\xi=1}^{\eta} u\left(\tau_\xi(\mathrm{t})\right) - \frac{1}{N_1} u(\mathrm{t})\right), \mathrm{t} \succcurlyeq \mathrm{t}_1, \tag{2.2}$$

$$\int_{t_0}^\infty \sum_{\xi=1}^{\eta} v\left(\tau_\xi(s)\right) ds \prec \infty \tag{2.3}$$

Then the Eq.(1.1) has a bounded solution by positive functions u and v.

Proof

Let $I(\mathrm{t}) = \int_t^{\infty} \sum_{\xi=1}^{\eta} v\left(\tau_\xi(s)\right) ds$ and then the condition (2.3) implies that

$$\lim_{t\to\infty} I(\mathrm{t}) = \lim_{t\to\infty} \int_t^{\infty} \sum_{\xi=1}^{\eta} v\left(\tau_\xi(s)\right) ds ds = 0. \qquad (2.4)$$

Let $(C([\mathrm{t}_0,\infty),\Re), \|.\|)$ such that $\|x\| = \sup_{\mathrm{t}\geq \mathrm{t}_0}|x(\mathrm{t})| \Longrightarrow C([\mathrm{t}_0,\infty),\Re)$ is the Banach space.

Let $\mho \subset C([t_0,\infty),\Re)$ which is defined as follows:

$$\mho = \{x(\mathrm{t}): x(t) \in \mathrm{C}([\mathrm{t}_0,\infty),\Re) \text{ with } u(\mathrm{t}) \leqslant x(\mathrm{t}) \leqslant v(\mathrm{t}), \mathrm{t} \geqslant \mathrm{t}_0\} \qquad (2.5)$$

Such that $\mho$ is closed and convex set.

The mappings S_1 and S_2: $\mho \to C([\mathrm{t}_0,\infty),\Re)$ are defined as follows:

$$(\mathrm{S}_1 x)(\mathrm{t}) = \begin{cases} \sum_{\xi=1}^{\eta} b_i(\mathrm{t}) G_\xi\left(t, x\left(\tau_\xi(\mathrm{t})\right)\right), & \mathrm{t} \geqslant \mathrm{t}_1, \\ (\mathrm{S}_1 x)(\mathrm{t}_1), & \mathrm{t}_0 \leqslant \mathrm{t} \leqslant \mathrm{t}_1, \end{cases}$$

$$(\mathrm{S}_2 x)(\mathrm{t}) = \begin{cases} -\int_t^{\infty} \sum_{\xi=1}^{\eta} a_\xi(s) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds & , \mathrm{t} \geqslant \mathrm{t}_1, \\ (\mathrm{S}_2 x)(\mathrm{t}_1) - u(\mathrm{t}_1) + v(\mathrm{t}), & \mathrm{t}_0 \leqslant \mathrm{t} \leqslant \mathrm{t}_1, \end{cases} \qquad (2.6)$$

Where S_1 and S_2 satisfy eq. (1.1)

For all $x, \gamma \in \mho$ and $\mathrm{t} \geqslant \mathrm{t}_1$, then:

$$(\mathrm{S}_1 x)(\mathrm{t}) + (\mathrm{S}_2 \gamma)(\mathrm{t}) = \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t}) G_\xi\left(\mathrm{t}, x\left(\tau_\xi(\mathrm{t})\right)\right) - \int_{\mathrm{t}}^{\infty} \sum_{\xi=1}^{\eta} a_\xi(s) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds$$

$$\leqslant N_2 C_2 \sum_{\xi=1}^{\eta} x\left(\tau_\xi(\mathrm{t})\right) - N_2 R_2 \int_t^{\infty} \sum_{\xi=1}^{\eta} x\left(\tau_\xi(s)\right) ds$$

$$\leqslant N_2 C_2 \sum_{\xi=1}^{\eta} v\left(\tau_\xi(\mathrm{t})\right) - N_2 R_2 \int_{\mathrm{t}}^{\infty} \sum_{\xi=1}^{\eta} v\left(\tau_\xi(s)\right) ds$$

$$\leqslant N_2 C_2 \sum_{\xi=1}^{\eta} v\left(\tau_\xi(\mathrm{t})\right) - N_2 R_2 \frac{1}{R_2}\left(C_2 \sum_{\xi=1}^{\eta} v\left(\tau_\xi(\mathrm{t})\right) - \frac{1}{N_2} v(\mathrm{t})\right) = v(\mathrm{t}).$$

For all $\mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_1]$, we have

$$(S_1 x)(\mathrm{t}) + (S_2 \gamma)(\mathrm{t}) = (S_1 x)(\mathrm{t}_1) + (S_2 \gamma)(\mathrm{t}_1) - u(\mathrm{t}_1) + v(\mathrm{t})$$
$$\leqslant v(\mathrm{t}_1) - u(\mathrm{t}_1) + v(\mathrm{t}) \leqslant u(\mathrm{t}_1) - u(\mathrm{t}_1) + v(\mathrm{t}) \leqslant v(\mathrm{t}).$$

So that for all $\mathrm{t} \geqslant \mathrm{t}_1$, this yields:

$$(S_1 x)(\mathrm{t}) + (S_2 \gamma)(\mathrm{t}) = \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t}) G_\xi\left(\mathrm{t}, x\left(\tau_\xi(\mathrm{t})\right)\right) - \int_{\mathrm{t}}^{\infty} \sum_{\xi=1}^{\eta} a_\xi(s) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds$$

$$\geqslant N_1 C_1 \sum_{\xi=1}^{\eta} x\left(\tau_\xi(\mathrm{t})\right) - N_1 R_1 \int_{\mathrm{t}}^{\infty} \sum_{\xi=1}^{\eta} x\left(\tau_\xi(s)\right) ds$$

$$\geqslant N_1 C_1 \sum_{\xi=1}^{\eta} u\left(\tau_\xi(\mathrm{t})\right) - N_1 R_1 \int_{\mathrm{t}}^{\infty} \sum_{\xi=1}^{\eta} u\left(\tau_\xi(s)\right) ds$$

$$\geqslant N_1 C_1 \sum_{\xi=1}^{\eta} u\left(\tau_\xi(\mathrm{t})\right) - N_1 R_1 \frac{1}{R_1}\left(C_1 \sum_{\xi=1}^{\eta} u\left(\tau_\xi(\mathrm{t})\right) - \frac{1}{N_1} u(\mathrm{t})\right) = u(\mathrm{t})$$

For all $\mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_1]$ and from Eq. (2.2), we obtain

$$(S_1x)(t) + (S_2\gamma)(t) = (S_1x)(t_1) + (S_2\gamma)(t_1) - v(t_1) + u(t)$$
$$\geqslant u(t_1) - u(t_1) + v(t) = v(t) \geqslant u(t) \tag{2.7}$$

So that $S_1x + S_2\gamma \in \mho$ For all $x, \gamma \in \mho$.Now, we have to prove that S_1 is contraction mapping on$\mho$. For all $x, \gamma \in \mho$ and $t \geqslant t_1$,we get

$$\|S_1x - S_1\gamma\| = \sup_{t \geq t_1} |(S_1x)(t) - (S_1\gamma)(t)|$$
$$= \sup_{t \geqslant t_1} \left| \sum_{\xi=1}^{\eta} b_\xi(t) G_\xi\left(t, x\left(\tau_\xi(t)\right)\right) - \sum_{\xi=1}^{\eta} b_\xi(t)\, G_\xi\left(t, \gamma\left(\tau_\xi(t)\right)\right) \right|$$
$$\leqslant \sup_{t \geqslant t_1} \left| \sum_{\xi=1}^{\eta} b_\xi(t)\, [G_\xi\left(t, x\left(\tau_\xi(t)\right)\right) - \sum_{\xi=1}^{\eta} G_\xi\left(t, \gamma\left(\tau_\xi(t)\right)\right)] \right|$$
$$\leqslant \sup_{t \geqslant t_1} N_2 \left| \sum_{\xi=1}^{\eta} [G_\xi\left(t, x\left(\tau_\xi(t)\right)\right) - \sum_{\xi=1}^{\eta} G_\xi\left(t, \gamma\left(\tau_\xi(t)\right)\right)] \right|$$
$$\leqslant \sup_{t \geqslant t_1} N_2 \sum_{\xi=1}^{\eta} \left| [G_\xi\left(t, x\left(\tau_\xi(t)\right)\right) - G_\xi\left(t, \gamma\left(\tau_\xi(t)\right)\right)] \right|$$
$$\leqslant \sup_{t \geqslant t_1} N_2 \sum_{\xi=1}^{\eta} \left| G_\xi\left(t, x\left(\tau_\xi(t)\right)\right) - G_\xi\left(t, \gamma\left(\tau_\xi(t)\right)\right) \right|$$
$$\leqslant \sup_{t \geqslant t_1} N_2 (\left| G_1\left(t, x(\tau_1(t))\right) - G_1\left(t, \gamma(\tau_1(t))\right) \right| + \left| G_2\left(t, x(\tau_2(t))\right) - G_2\left(t, \gamma(\tau_2(t))\right) \right|$$
$$+ \cdots \left| G_\eta\left(t, x\left(\tau_\eta(t)\right)\right) - G_\eta\left(t, \gamma\left(\tau_\eta(t)\right)\right) \right|$$
$$\leqslant \sup_{t \geqslant t_1} N_2 (M_1|x - \gamma| + M_2|x - \gamma| + \cdots + M_\eta|x - \gamma|)$$
$$\leqslant \sup_{t \geqslant t_1} c(M_1 + M_2 + \cdots + M_n)|x - \gamma|$$
$$\leqslant M\|x - \gamma\| \tag{2.8}$$

Where , $M = N_2(M_1 + M_2 + \cdots + M_n)$

Also for $t \in [t_0, t_1]$.

$$\|S_1x - S_1\gamma\| = \sup_{t_0 \leqslant t \leqslant t_1} |(S_1x)(t) - (S_1\gamma)(t)|$$
$$= \sup_{t_0 \leqslant t \leqslant t_1} |(S_1x)(t_1) - (S_1\gamma)(t_1)|$$
$$= \left| \sum_{\xi=1}^{\eta} b_\xi(t_1) G_\xi\left(t_1, x\left(\tau_\xi(t_1)\right)\right) - \sum_{\xi=1}^{\eta} b_\xi(t_1) G_\xi\left(t_1, \gamma\left(\tau_\xi(t_1)\right)\right) \right|$$
$$\leqslant N_2 \left| \sum_{\xi=1}^{\eta} [G_\xi\left(t_1, x\left(\tau_\xi(t_1)\right)\right) - \sum_{\xi=1}^{\eta} G_\xi\left(t_1, \gamma\left(\tau_\xi(t_1)\right)\right)] \right|$$
$$\leqslant \sup_{t_0 \leqslant t \leqslant t_1} N_2 \sum_{\xi=1}^{\eta} \left| [G_\xi\left(t_1, x\left(\tau_\xi(t_1)\right)\right) - G_\xi\left(t_1, \gamma\left(\tau_\xi(t_1)\right)\right)] \right|$$
$$\leqslant \sup_{t \geqslant t_1} N_2 (\left| G_1\left(t, x(\tau_1(t))\right) - G_1\left(t, \gamma(\tau_1(t))\right) \right| + \left| G_2\left(t, x(\tau_2(t))\right) - G_2\left(t, \gamma(\tau_2(t))\right) \right|$$
$$+ \cdots \left| G_\eta\left(t, x\left(\tau_\eta(t)\right)\right) - G_\eta\left(t, \gamma\left(\tau_\eta(t)\right)\right) \right|$$
$$\leqslant \sup_{t \geqslant t_1} N_2 (M_1|x - \gamma| + M_2|x - \gamma| + \cdots + M_\eta|x - \gamma|)$$
$$\leqslant \sup_{t \geqslant t_1} N_2 (M_1 + M_2 + \cdots + M_\eta)|x - \gamma|$$

$$\leqslant M\|x-\gamma\| \tag{2.9}$$

Where , $M = N_2(M_1 + M_2 + \cdots + M_\eta)$

$$= M\|x-\gamma\|$$

This implies that

$$\|S_1 x - S_1 \gamma\| \leqslant M\|x-\gamma\| \tag{2.10}$$

Thus, S_1 is mapping with contractive property on Ʊ. Now, we have to prove that S_2 has completely property to continuous mapping. First of all, we need to show that S_2 is continuous mapping.

Let $x_k = x_k(t) \in$ Ʊ. Since Ʊ is closed, thus $x_k(t)$ tends to $x(t)$ as $k \to \infty$, $x(t) \in$ Ʊ. For $t \geqslant t_1$ this yields:

$$\|(S_2 x_k)(t) - (S_2 x)(t)\| = \sup_{t\geqslant t_1} |(S_2 x_k)(t) - (S_2 x)(t)|$$

$$\leqslant \sup_{t\geqslant t_1} \left| -\int_t^\infty \sum_{\xi=1}^{\eta} a_\xi(t)\, g_\xi\left(x_k\left(\tau_\xi(s)\right)\right) ds + \int_t^\infty \sum_{\xi=1}^{\eta} a_\xi(t) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds \right|$$

$$\leqslant \sup_{t\geqslant t_1} N_2 \left| -\int_t^\infty \sum_{\xi=1}^{\eta} g_\xi\left(x_k\left(\tau_\xi(s)\right)\right) ds + \int_t^\infty \sum_{\xi=1}^{\eta} g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds \right|$$

$$\leqslant \sup_{t\geqslant t_1} N_2 \left| \int_t^\infty \sum_{\xi=1}^{\eta} [g_\xi\left(x_k\left(\tau_\xi(s)\right)\right) - g_\xi\left(x\left(\tau_\xi(s)\right)\right)] ds \right|$$

$$\leqslant \sup_{t\geqslant t_1} N_2 (\left| \int_t^\infty [g_1\left(x_k(\tau_1(s))\right) - g_1\left(x(\tau_1(s))\right)]\, ds \right|$$

$$+ \left| \int_t^\infty [g_2\left(x_k(\tau_2(s))\right) - g_2\left(x(\tau_2(s))\right)]\, ds \right| + \cdots$$

$$+ \int_t^\infty \left| [g_\eta\left(x_k\left(\tau_\eta(s)\right)\right) - g_\eta\left(x\left(\tau_\eta(s)\right)\right)] ds \right| \tag{2.11}$$

According to (2.3), and the bounded property of $g_\xi\left(x\left(\tau_\xi(t)\right)\right)$, we get

$$\int_{t_1}^\infty g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds \prec \infty. \tag{2.12}$$

Since $\left|[g_\xi\left(x_k(\tau_1(s))\right) - g_\xi\left(x(\tau_1(s))\right)\right| \to 0$ as k tends to ∞, $\xi = 1, 2, 3, \ldots, \eta$. By Lebesgue's dominated convergence theorem, we get

$$\lim_{k\to\infty} \|(S_2 x_k)(t) - (S_2 x)(t)\| = 0 \tag{2.13}$$

It reduces that S_2 will be continuous mapping.

To prove that S_2Ʊ is a relatively compact, we must accentual that $\{S_2 x : x \in$ Ʊ$\}$ is uniformly bounded and equicontinuous on $[t_0, \infty]$, by Arzelã-Ascoli theorem [16]. From (2.5), we get $\{S_2 x : x \in$ Ʊ$\}$ is a uniformly bounded.

To secure that $\{S_2 x : x \in$ Ʊ$\}$ is equicontinuouson on $[t_0, \infty)$, let $x \in$ Ʊ and for any $\varepsilon \succ 0$, by (2.12), so that there exists $t_* \geqslant t_1$ large enough:

$$\int_{t_*}^\infty g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds \prec \frac{\varepsilon}{2\eta N_2}, \ , t \geqslant t_* \geqslant t_1 \tag{2.14}$$

Then, for any given $\varepsilon \succ 0$ and $x \in$ Ʊ, $T_2 \succ T_1 \geqslant t_*$, we have

$$\|(S_2x_k)(T_2)-(S_2x)(T_1)\| = \sup_{T_2 \succ T_1 \succcurlyeq t_*} |(S_2x_k)(T_2)-(S_2x)(T_1)|$$
$$\preccurlyeq |(S_2x_k)(T_2)| + |(S_2x)(T_1)|$$
$$\preccurlyeq \int_{T_2}^{\infty} \sum_{\xi=1}^{\eta} a_\xi(t)\, g_\xi\left(x_k\left(\tau_\xi(s)\right)\right) ds + \int_{T_1}^{\infty} \sum_{\xi=1}^{\eta} a_\xi(t) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds$$
$$N_2 \int_{T_2}^{\infty} \sum_{\xi=1}^{\eta} g_\xi\left(x_k\left(\tau_\xi(s)\right)\right) ds + N_2 \int_{T_1}^{\infty} \sum_{\xi=1}^{\eta} g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds$$

$$\prec N_2 \underbrace{\left(\frac{\varepsilon}{2\eta N_2}+\frac{\varepsilon}{2\eta N_2}+\cdots+\frac{\varepsilon}{2\eta N_2}\right)}_{\eta - times} + N_2 \underbrace{\left(\frac{\varepsilon}{2\eta N_2}+\frac{\varepsilon}{2\eta N_2}+\cdots+\frac{\varepsilon}{2\eta N_2}\right)}_{\eta - times} \preccurlyeq \varepsilon, \qquad (2.15)$$

For $x \in \mho$ and $t_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq t_*$, we get

$$\|(S_2x)(T_2)-(S_2x)(T_1)\| = \sup_{t_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq t_*} |(S_2x)(T_2)-(S_2x)(T_1)|$$
$$= \sup_{t_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq t_*} \left| \int_{T_2}^{t_*} \sum_{\xi=1}^{\eta} a_\xi(t) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds - \int_{T_1}^{t_*} \sum_{\xi=1}^{\eta} a_\xi(t) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds \right|$$
$$\preccurlyeq \sup_{t_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq t_*} N_2 \left| \int_{T_2}^{t_*} \sum_{\xi=1}^{\eta} g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds - \int_{T_1}^{t_*} \sum_{\xi=1}^{\eta} g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds \right|$$
$$= N_2 \int_{T_1}^{T_2} \sum_{\xi=1}^{\eta} g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds$$
$$= N_2 \int_{T_1}^{T_2} g_1\left(x(\tau_1(s))\right) ds + N_2 \int_{T_1}^{T_2} g_2\left(x(\tau_2(s))\right) ds + \cdots + N_2 \int_{T_1}^{T_2} g_\eta\left(x\left(\tau_\eta(s)\right)\right) ds$$
$$\preccurlyeq N_2 \eta M_2 (T_2 - T_1).$$

Thus there exists $\delta_1 = \frac{\varepsilon}{\eta M_2}$, such that

$$|(S_2x)(T_2)-(S_2x)(T_1)| \prec \varepsilon, \text{if} \;\; 0 \prec T_2 - T_1 \prec \delta_1 \qquad (2.16)$$

Finally, let $V(t) = \frac{v(t)}{a(t)}$, then for any $x \in \mho$, $t_0 \preccurlyeq T_1 \prec T_2 \preccurlyeq t_1$, by mean value theorem there exists $k_1 \in (T_1, T_2)$ and $\delta_2 = \frac{\varepsilon}{V'(k_1)} \succ 0$ such that

$$|(S_2x)(T_2)-(S_2x)(T_1)| = \left|\left(\frac{v}{a}\right)(T_2) - \left(\frac{v}{a}\right)(T_1)\right|$$
$$= |V(T_2) - V(T_1)|$$
$$= |V'(k_1)(T_2 - T_1)|$$
$$= |V'(k_1)|(T_2 - T_1) \prec \varepsilon,$$

if $0 \prec T_2 - T_1 \prec \delta_2$. (2.17)

Hence $S_2\mho$ is a compact relatively set. By using lemma (1.1), it reduces that Eq. (1.1) has solution which is bounded relatively from below.

Next theorem is generalizing of theorem (2.1). We will show that the solution of Eq. (1.1) exists and bounded by convergent series $\sum_{\xi=1}^{\eta} u_\xi$ (t) and $\sum_{\xi=1}^{\eta} v_\xi$ (t).

Theorem 2.2

Suppose that A1- A3, (2.3) hold, and there is convergent series $\sum_{\xi=1}^{\eta} u_\xi(\mathrm{t}), \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}) \in (\mathbb{N}, [0,\infty)), \mathrm{t}_1 \geqslant \mathrm{t}_0 + \rho$ such that

$$v(\mathrm{t}) \geqslant v(\mathrm{t}_1) \tag{2.18}$$

$$\int_{\mathrm{t}}^{\infty} u\left(\left(\tau_\xi(s)\right)\right) ds \leqslant \frac{C_1N_1 - \frac{1}{K_1}}{R_1N_1} \sum_{\xi=1}^{\eta} u_\xi\left(\tau_\xi(\mathrm{t})\right), \mathrm{t} \geqslant \mathrm{t}_1 \tag{2.19}$$

$$\int_{\mathrm{t}}^{\infty} v\left(\left(\tau_\xi(s)\right)\right) ds \geqslant \frac{C_2N_2 - \frac{1}{K_2}}{R_2N_2} \sum_{\xi=1}^{\eta} v_\xi\left(\tau_\xi(\mathrm{t})\right), \mathrm{t} \geqslant \mathrm{t}_1 \ .$$

Then, the Eq.(1.1) has a bounded solution by convergent series $\sum_{\xi=1}^{\eta} u_\xi(\mathrm{t}), \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}) \in C^1$.

Proof

Let $(C([\mathrm{t}_0,\infty),\mathfrak{R}), \|.\|)$ such that $\|x\| = \sup_{\mathrm{t}\geqslant \mathrm{t}_0}|x(\mathrm{t})| \Longrightarrow \mathrm{C}([\mathrm{t}_0,\infty),\mathfrak{R})$ is Banach space, let $\mho \subset C([\mathrm{t}_0,\infty), \mathfrak{R})$ which is defined as follows:

$$\mho = \{x(\mathrm{t}): x(t) \in \mathrm{C}([\mathrm{t}_0,\infty),\mathfrak{R}): \ u(\mathrm{t}) \leqslant x(\mathrm{t}) \leqslant v(\mathrm{t}), \quad \mathrm{t} \geqslant \mathrm{t}_0,$$
$$K_1x(\mathrm{t}) \leqslant x\left(\tau_i(\mathrm{t})\right) \leqslant K_2x(\mathrm{t}), \mathrm{t} \geqslant \mathrm{t}_0, \} \tag{2.20}$$

Such that $\mho$ is a closed and convex. The mappings S_1 and $S_2: \mho \to \mathrm{C}([\mathrm{t}_0,\infty),\mathfrak{R})$ are defined as:

$$(\mathrm{S}_1x)(\mathrm{t}) = \begin{cases} \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t}) G_\xi\left(t, x\left(\tau_\xi(\mathrm{t})\right)\right), & t \geqslant t_1, \\ (\mathrm{S}_1x)(\mathrm{t}_1), & t_0 \leqslant t \leqslant t_1, \end{cases}$$

$$(\mathrm{S}_2x)(\mathrm{t}) = \begin{cases} -\int_t^{\infty} \sum_{\xi=1}^{\eta} a_\xi(s) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds, & t \geqslant t_1, \\ (\mathrm{S}_2x)(\mathrm{t}_1) - \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}_1) + \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}), & t_0 \leqslant t \leqslant t_1, \end{cases} \tag{2.21}$$

We are going to prove for any $x, \gamma \in \mho$ such that $S_1x + S_2\gamma \in \mho$ and for all $x, \gamma \in \mho, \mathrm{t} \geqslant \mathrm{t}_1$, we have

$$(S_1x)(\mathrm{t}) + (S_2\gamma)(\mathrm{t}) = \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t}) G_\xi\left(\mathrm{t}, x\left(\tau_\xi(\mathrm{t})\right)\right) - \int_{\mathrm{t}}^{\infty} \sum_{\xi=1}^{\eta} a_\xi(s) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds$$
$$\leqslant N_2C_2 \sum_{\xi=1}^{\eta} x\left(\tau_\xi(\mathrm{t})\right) - N_2R_2 \int_{\mathrm{t}}^{\infty} \sum_{\xi=1}^{\eta} x\left(\tau_\xi(s)\right) ds$$
$$\leqslant N_2C_2 \sum_{\xi=1}^{\eta} K_2\, x_\xi(\mathrm{t}) - N_2R_2 \int_{\mathrm{t}}^{\infty} K_2 \sum_{\xi=1}^{\eta} x_\xi(s)\, ds$$
$$\leqslant N_2C_2 \sum_{\xi=1}^{\eta} K_2\, v_\xi(\mathrm{t}) - N_2R_2 \int_{\mathrm{t}}^{\infty} K_2 \sum_{\xi=1}^{\eta} v_\xi(s)\, ds$$

$$\leqslant N_2 C_2 K_2 \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}) - N_2 R_2 \frac{C_2 N_2 - \frac{1}{K_2}}{R_2 N_2} K_2 \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}) = \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}) \tag{2.22}$$

Let $\mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_1]$, using (2.22), we get:

$$(S_1 x)(\mathrm{t}) + (S_2 \gamma)(\mathrm{t}) = (S_1 x)(\mathrm{t}_1) + (S_2 \gamma)(\mathrm{t}_1) - \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}_1) + \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t})$$

$$\leqslant \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}_1) - \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}_1) + \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}) = \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t})$$

Moreover, $\forall\, \mathrm{t} \geqslant \mathrm{t}_1$, yield:

$$(S_1 x)(\mathrm{t}) + (S_2 \gamma)(\mathrm{t}) = \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t}) G_\xi\left(\mathrm{t}, x\left(\tau_\xi(\mathrm{t})\right)\right) - \int_{\mathrm{t}}^{\infty} \sum_{\xi=1}^{\eta} a_\xi(s) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds$$

$$\geqslant N_1 C_1 \sum_{\xi=1}^{\eta} x\left(\tau_\xi(\mathrm{t})\right) - N_1 R_1 \int_{\mathrm{t}}^{\infty} \sum_{\xi=1}^{\eta} x\left(\tau_\xi(s)\right) ds$$

$$\geqslant N_1 C_1 \sum_{\xi=1}^{\eta} K_1\, x_\xi(\mathrm{t}) - N_1 R_1 \int_{\mathrm{t}}^{\infty} K_1 \sum_{\xi=1}^{\eta} x_\xi(s)\, ds$$

$$\geqslant N_1 C_1 \sum_{\xi=1}^{\eta} K_1\, u_\xi(\mathrm{t}) - N_1 R_1 \int_{\mathrm{t}}^{\infty} K_1 \sum_{\xi=1}^{\eta} u_\xi(s)\, ds$$

$$\geqslant N_2 C_2 K_2 \sum_{\xi=1}^{\eta} u_\xi(\mathrm{t}) - N_2 R_2 \frac{C_2 N_2 - \frac{1}{K_2}}{R_2 N_2} K_2 \sum_{\xi=1}^{\eta} u_\xi(\mathrm{t}) = \sum_{\xi=1}^{\eta} u_\xi(\mathrm{t}) \tag{2.23}$$

Then for $\mathrm{t} \in [\mathrm{t}_0, \mathrm{t}_1]$, using (2.18) and (2.23), we obtain:

$$(S_1 x)(\mathrm{t}) + (S_2 \gamma)(\mathrm{t}) = (S_1 x)(\mathrm{t}_1) + (S_2 \gamma)(\mathrm{t}_1) - \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}_1) + \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t})$$

$$\geqslant \sum_{\xi=1}^{\eta} u_\xi(\mathrm{t}_1) - \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}_1) + \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t})$$

$$\geqslant \sum_{\xi=1}^{\eta} u_\xi(\mathrm{t}_1) - \sum_{\xi=1}^{\eta} u_\xi(\mathrm{t}_1) + \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}) = \sum_{\xi=1}^{\eta} v_\xi(\mathrm{t}) \geqslant \sum_{\xi=1}^{\eta} u_\xi(\mathrm{t})$$

Thus, $S_1 x + S_2 \gamma \in \mho, \forall\, x, \gamma \in \mho$. By using similarly steps in theorem (2.1), we conclude the result. By lemma (1.1) there exists $x_0 \in \mho, \ni S_1 x_0 + S_2 x_0 = x_0$. We realize that $x_0(t)$ is a one side relatively bounded solution of the Eq. (1.1).

3. Oscillation Criteria of multiple delay Differential Equation:

In the present section, we'll seek for oscillatory criteria to Eq. (1.1) and we use some basic lemmas:

Lemma 3.1 [5] If these assumptions hold

$\varphi, \vartheta, x, \tau, \varrho \in C[[\mathrm{t}_0, \infty), \Re]$, $\varphi(\mathrm{t}) \prec 0, \lim_{t\to\infty} \varphi(\mathrm{t})$ exists, $0 \prec \vartheta_1(\mathrm{t}) \preccurlyeq 1$, $\tau(\mathrm{t}) \prec \mathrm{t}$, $\varrho(\mathrm{t}) \succcurlyeq \mathrm{t}$, $\mathrm{t} \succcurlyeq \mathrm{t}_0$, $\lim_{t\to\infty} \tau(\mathrm{t}) = \infty$ and

$$x(\mathrm{t}) \preccurlyeq \varphi(\mathrm{t}) + \vartheta_1(\mathrm{t}) \max\{x(s): \tau(\mathrm{t}) \preccurlyeq s \preccurlyeq \varrho(\mathrm{t})\}, \qquad \mathrm{t} \succcurlyeq \mathrm{t}_0 \ . \tag{3.1}$$

Then $x(\mathrm{t})$ cannot be positive for $\mathrm{t} \succcurlyeq \mathrm{t}_1 \succcurlyeq \mathrm{t}_0$.

II. If these assumptions hold $\varphi, \vartheta, x, \tau, \varrho \in C[[\mathrm{t}_0, \infty); \Re]$, $\varphi(t) \succ 0, \lim_{t\to\infty} \varphi(\mathrm{t})$ exist, $0 \prec \vartheta_2(\mathrm{t}) \preccurlyeq 1, \tau(\mathrm{t}) \prec \mathrm{t}, \varrho(\mathrm{t}) \succcurlyeq \mathrm{t}$, $\mathrm{t} \succcurlyeq \mathrm{t}_0$, $\lim_{\mathrm{t}\to\infty} \tau(\mathrm{t}) = \infty$ and

$$x(\mathrm{t}) \succcurlyeq \varphi(\mathrm{t}) + \vartheta_2(\mathrm{t}) \min\{x(s): \tau(\mathrm{t}) \preccurlyeq s \preccurlyeq \varrho(\mathrm{t})\}, \qquad \mathrm{t} \succcurlyeq \mathrm{t}_0. \tag{3.2}$$

Then $x(\mathrm{t})$ cannot be negative for $\mathrm{t} \succcurlyeq \mathrm{t}_1 \succcurlyeq \mathrm{t}_0$

Lemma 3.2 [17]

Assume that $\upsilon, p \in C[\Re^+, \Re^+]$ are continuous functions such that $\upsilon(\mathrm{t}) \prec \mathrm{t}, \upsilon'(\mathrm{t}) \succcurlyeq 0$ for $\mathrm{t} \succcurlyeq \mathrm{t}_0$ with $\lim_{\mathrm{t}\to\infty} \upsilon(\mathrm{t}) = \infty$.

$$\text{If } \lim_{\mathrm{t}\to\infty} \inf \int_{\upsilon(\mathrm{t})}^{\mathrm{t}} p(s)\, ds \succ \frac{1}{e} \tag{3.3}$$

then the inequality $x'(\mathrm{t}) + p(\mathrm{t})x(\upsilon(\mathrm{t})) \preccurlyeq 0$ has no eventually positive solution.

Lemma 3.3

Assume that:

$$\Upsilon(\mathrm{t}) = x(\mathrm{t}) - \sum_{\xi=1}^{\eta} \int_{\mathrm{t}}^{\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))} a_\xi(s) g_\xi\left(x\left(\tau_\xi(s)\right)\right) ds - \sum_{\xi=1}^{n} b_\xi(\mathrm{t}) G_\xi\left(\mathrm{t}, x\left(\tau_\xi(\mathrm{t})\right)\right) \tag{3.4}$$

And the following assumptions hold:

H1: $\vartheta_2(\mathrm{t}) \preccurlyeq \frac{G_\xi\left(\mathrm{t}, x\left(\tau_\xi(\mathrm{t})\right)\right)}{x\left(\tau_\xi(\mathrm{t})\right)} \preccurlyeq \frac{g_\xi\left(x\left(\tau_\xi(\mathrm{t})\right)\right)}{x\left(\tau_\xi(\mathrm{t})\right)} \preccurlyeq \vartheta_1(\mathrm{t})$, $\rho(\mathrm{t}) = \max\{\tau_\xi(\mathrm{t})\}$

H2: $\lim_{\mathrm{t}\to\infty} sup \sum_{\xi=1}^{\eta} \left[\int_{\mathrm{t}}^{\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))} a_\xi(s)\, ds + b_\xi(\mathrm{t})\right] \preccurlyeq 1$

If $x(\mathrm{t})$ is eventually positive solution of Eq. (1.1) with$(\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t})))' \succcurlyeq 0$ then:

$\Upsilon(\mathrm{t})$ is positive non-increasing function.

Proof

Assume that a solution $x(\mathrm{t})$ is a non-oscillatory of the Eq.(1.1). So that let $x(\mathrm{t})$ be eventually positive solution, there is $\mathrm{t}_1 \succcurlyeq \mathrm{t}_0 + \rho \ \ni x(\mathrm{t}) \succ 0$ for $\mathrm{t} \succcurlyeq \mathrm{t}_1$.

$$\Upsilon'(\mathrm{t}) = x'(\mathrm{t}) - \sum_{\xi=1}^{\eta} \left[a_\xi(\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))) g_\xi\left(x\left((\upsilon_\xi(\mathrm{t})\right)\right)(\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t})))' - a_\xi(\mathrm{t})\, g_\xi\left(x\left(\tau_\xi(\mathrm{t})\right)\right)\right]$$
$$- \frac{d}{d\mathrm{t}} \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t})\, G_\xi\left(\mathrm{t}, x\left(\tau_\xi(\mathrm{t})\right)\right)$$

From Eq. (1.1), we obtain that

$$\Upsilon'(\mathrm{t}) = -\sum_{\xi=1}^{\eta} a_\xi(\mathrm{t}) g_\xi\left(x\left(\tau_\xi(\mathrm{t})\right)\right) + \frac{d}{dt}\sum_{\xi=1}^{\eta} b_\xi(\mathrm{t})\, G_\xi\left(t, x\left(\tau_\xi(\mathrm{t})\right)\right)$$
$$- \sum_{\xi=1}^{\eta} a_\xi(\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))) g_\xi\left(x\left((\upsilon_\xi(\mathrm{t})\right)\right)(\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t})))'$$
$$+ \sum_{\xi=1}^{\eta} a_\xi(\mathrm{t}) g_\xi\left(x\left(\tau_\xi(\mathrm{t})\right)\right) - \frac{d}{dt}\sum_{\xi=1}^{\eta} b_\xi(\mathrm{t}) G_\xi\left(t, x\left(\tau_\xi(\mathrm{t})\right)\right)$$

$$\Upsilon'(t) = -\sum_{\xi=1}^{\eta} a_\xi(\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t})))g_\xi\left(x\left((\upsilon_\xi(\mathrm{t}))\right)\right)(\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t})))' \preccurlyeq 0 \quad (3.5)$$

So, we conclude that $\Upsilon'(\mathrm{t})$ is non-increasing.
Let $\Upsilon(\mathrm{t})$ is positive. Otherwise there is a $\mathrm{t}_1 \succcurlyeq \mathrm{t}_0, \ni \Upsilon(\mathrm{t}) \preccurlyeq 0$ for $\mathrm{t}_2 \succcurlyeq \mathrm{t}_1$.

$$x(\mathrm{t}) = \Upsilon(\mathrm{t}) + \sum_{\xi=1}^{\eta} \int_{t}^{\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))} a_\xi(s)g_\xi\left(x\left(\tau_\xi(s)\right)\right)ds + \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t})G_\xi\left(t, x\left(\tau_\xi(\mathrm{t})\right)\right)$$

$$\preccurlyeq \Upsilon(\mathrm{t}) + \sum_{\xi=1}^{\eta} \int_{\mathrm{t}}^{\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))} a_\xi(s)\vartheta_1(\mathrm{t})x\left(\tau_\xi(s)\right)ds + \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t})\vartheta_1(\mathrm{t})x\left(\tau_\xi(s)\right)$$

$$\preccurlyeq \Upsilon(\mathrm{t}) + \vartheta_1(\mathrm{t}) \max_{\rho(t)\preccurlyeq s\preccurlyeq t} x(s) \sum_{\xi=1}^{n}\left[\int_{\mathrm{t}}^{\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))} a_\xi(s)\,ds + b_\xi(\mathrm{t})\right]$$

$$\preccurlyeq \Upsilon(\mathrm{t}) + \vartheta_1(\mathrm{t}) \max\nolimits_{\rho(t)\preccurlyeq s\preccurlyeq t} x(s) \text{ for } \mathrm{t}_2 \succcurlyeq \mathrm{t}_1.$$

By using lemma (3.1-I), then $x(\mathrm{t})$ cannot be positive function on $[\mathrm{t}_3, \infty)$ which contradicts to $x(\mathrm{t}) \succ 0$.

Theorem 3.1

Assume that H1, H2 hold and $\Upsilon(\mathrm{t})$ is defined as in (3.4) in addition to the condition:

$$\liminf_{\mathrm{t}\to\infty} \int_{\upsilon_\xi(\mathrm{t})}^{\mathrm{t}} \left[\left[\sum_{\xi=1}^{\eta} a_\xi(\tau_\xi^{-1}(\upsilon_\xi(s)))\vartheta_2(\tau_\xi^{-1}(\upsilon_\xi(s)))(\tau_\xi^{-1}(\upsilon_\xi(s)))'\right]\left[1 + \sum_{\xi=1}^{\eta} \int_{\upsilon_\xi(s)}^{\tau_\xi^{-1}(\upsilon_\xi(\upsilon(s)))} a_\xi(u)\vartheta_2(u)\,du + \sum_{\xi=1}^{\eta} b_\xi\left(\upsilon_\xi(s)\right)\vartheta_2(\upsilon_\xi(s))\right]\right]ds > \frac{1}{e} \quad (3.6)$$

Let $\upsilon_\xi \in C(\Re^+, \Re^+), \upsilon_\xi(\mathrm{t}) \prec \mathrm{t}$ such that $\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t})) \succ \mathrm{t}$, $\upsilon(\mathrm{t}) \longrightarrow \infty$ as $\mathrm{t} \longrightarrow \infty$.
Then every solution of Eq. (1.1) oscillates.

Proof

Assume that a solution $x(\mathrm{t})$ is non-oscillatory of the Eq. (1.1). So, let $x(\mathrm{t})$ is eventually positive solution, there is $\mathrm{t}_1 \succcurlyeq \mathrm{t}_0 + \rho, \ni x(\mathrm{t}) \succ 0, \ \mathrm{t} \succcurlyeq \mathrm{t}_1$.
From (3.4): $x(\mathrm{t}) \succcurlyeq \Upsilon(\mathrm{t})$ then:

$$x(\mathrm{t}) = \Upsilon(\mathrm{t}) + \sum_{\xi=1}^{\eta} \int_{\mathrm{t}}^{\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))} a_\xi(s)g_\xi\left(x\left(\tau_\xi(s)\right)\right)ds + \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t})G_\xi\left(\mathrm{t}, x\left(\tau_\xi(\mathrm{t})\right)\right)$$

$$\succcurlyeq \Upsilon(\mathrm{t}) + \sum_{\xi=1}^{\eta} \int_{\mathrm{t}}^{\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))} a_\xi(s)\vartheta_2(s)x\left(\tau_\xi(s)\right)ds + \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t})\vartheta_2(\mathrm{t})x\left(\tau_\xi(\mathrm{t})\right)$$

$$\succcurlyeq \Upsilon(\mathrm{t}) + \sum_{\xi=1}^{\eta} \int_{\mathrm{t}}^{\tau_\xi^{-1}(\upsilon_\xi(\mathrm{t}))} a_\xi(s)\vartheta_2(s)\Upsilon\left(\tau_\xi(s)\right)ds + \sum_{\xi=1}^{\eta} b_\xi(\mathrm{t})\vartheta_2(\mathrm{t})\Upsilon\left(\tau_\xi(\mathrm{t})\right)$$

$$\geqslant \Upsilon(t) + \sum_{\xi=1}^{\eta} \Upsilon\left(\left(\upsilon_\xi(t)\right)\int_{t}^{\tau_\xi^{-1}(\upsilon_\xi(t))} a_\xi(s)\vartheta_2(s)\,ds + \sum_{\xi=1}^{\eta} b_\xi(t)\vartheta_2(t)\Upsilon\left(\tau_\xi(t)\right)$$

$$\geqslant \Upsilon(t) + \Upsilon(t)\sum_{\xi=1}^{\eta}\int_{t}^{\tau_\xi^{-1}(\upsilon_\xi(t))} a_\xi(s)\vartheta_2(s)\,ds + \Upsilon(t)\sum_{\xi=1}^{\eta} b_\xi(t)\vartheta_2(t)$$

$$= \Upsilon(t)\left[1 + \sum_{\xi=1}^{\eta}\int_{t}^{\tau_\xi^{-1}(\upsilon_\xi(t))} a_\xi(s)\vartheta_2(s)\,ds + \sum_{\xi=1}^{\eta} b_\xi(t)\vartheta_2(t)\right]$$

$$x(t) \geqslant \Upsilon(t)\left[1 + \sum_{\xi=1}^{\eta}\int_{t}^{\tau_\xi^{-1}(\upsilon_\xi(t))} a_\xi(s)\vartheta_2(s)\,ds + \sum_{\xi=1}^{\eta} b_\xi(t)\vartheta_2(t)\right]$$

$$x(\upsilon_\xi(t)) \geqslant \Upsilon\left(\upsilon_\xi(t)\right)\left[1 + \sum_{\xi=1}^{\eta}\int_{\upsilon_\xi(t)}^{\tau_\xi^{-1}\left(\upsilon_\xi(\upsilon_\xi(t))\right)} a_\xi(s)\vartheta_2(s)\,ds + \sum_{\xi=1}^{\eta} b_\xi\left(\upsilon_\xi(t)\right)\vartheta_2(\upsilon_\xi(t))\right]$$

From (3.5), we have:

$$\Upsilon'(t) + \sum_{\xi=1}^{\eta} a_\xi\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right) g_\xi\left(x\left(\left(\upsilon_\xi(t)\right)\right)\right)\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right)' \leqslant 0$$

$$\Upsilon'(t) + \sum_{\xi=1}^{\eta} a_\xi\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right)\vartheta_2\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right) x\left(\upsilon_\xi(t)\right)\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right)' \leqslant 0$$

$$\Upsilon'(t) + \left[\sum_{\xi=1}^{\eta} a_\xi\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right)\vartheta_2\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right)\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right)'\right]\left[1 + \sum_{\xi=1}^{\eta}\int_{\upsilon_\xi(t)}^{\tau_\xi^{-1}\left(\upsilon_\xi(\upsilon_\xi(t))\right)} a_\xi(s)\vartheta_2(s)\,ds + \sum_{\xi=1}^{\eta} b_\xi\left(\upsilon_\xi(t)\right)\vartheta_2(\upsilon_\xi(t))\right]\Upsilon\left(\upsilon_\xi(t)\right) \leqslant 0$$

By lemma (3.2), then the last inequality has no eventually positive solution.

Corollary 3.1

Let $\Upsilon(t)$ is defined as in (3.4) and the conditions H1 and H2 hold, in addition to the following conditions:

Let $\mu > 0$ such that:

$$0 \prec \mu \leqslant \sum_{\xi=1}^{\eta}\int_{\upsilon_i(t)}^{\tau_\xi^{-1}\left(\upsilon_\xi(\upsilon_\xi(t))\right)} a_\xi(s)\vartheta_2(s)\,ds + \sum_{\xi=1}^{\eta} b_\xi\left(\upsilon_\xi(t)\right)\vartheta_2\left(\upsilon_\xi(t)\right) \tag{3.7}$$

$$\sum_{\xi=1}^{\eta} a_\xi\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right)\vartheta_2\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right)\left(\tau_\xi^{-1}\left(\upsilon_\xi(t)\right)\right)' \geqslant \frac{1}{e\, min_{t \geqslant t_0}\left\{t - \upsilon_\xi(t)\right\}} \tag{3.8}$$

Then any solution of Eq. (1.1) oscillates.

Proof

It is easy to see that the condition (3.8) satisfies the following inequality:

$$\int_{\upsilon_\xi(t)}^{t}\left[\left[\sum_{\xi=1}^{\eta} a_\xi\left(\tau_\xi^{-1}\left(\upsilon_\xi(s)\right)\right)\vartheta_2\left(\tau_\xi^{-1}\left(\upsilon_\xi(s)\right)\right)\left(\tau_\xi^{-1}\left(\upsilon_\xi(s)\right)\right)'\right]\right]ds$$

$$\geqslant \frac{1}{e}\int_{\upsilon_\xi(t)}^{t}\frac{1}{\min_{t\geq t_0}\{s-\upsilon_\xi(s)\}}ds \geq \frac{t-\upsilon_\xi(t)}{\min_{t\geq t_0}\{t-\upsilon_\xi(t)\}} \geqslant \frac{1}{e}$$

Then the condition (3.6) holds, by theorem (3.1) every solution of Eq. (1.1) oscillates.

Example 3.1

Consider the following multiple DDE**:**

$$\frac{d}{dt}x(t) = -\sum_{\xi=1}^{2} a_\xi(t)g_\xi\left(x\left(\tau_\xi(t)\right)\right) + \frac{d}{dt}\sum_{\xi=1}^{2} b_\xi(t)G_\xi\left(t, x\left(\tau_\xi(t)\right)\right) \quad (3.9)$$

Solution

Let $\vartheta_1(t) = 1,\ \vartheta_2(t) = \frac{3}{4}$ with the condition:

$$\frac{3}{4} \preccurlyeq \frac{G_\xi\left(t, x\left(\tau_\xi(t)\right)\right)}{x\left(\tau_\xi(t)\right)} \preccurlyeq \frac{g_\xi\left(x\left(\tau_\xi(t)\right)\right)}{x\left(\tau_\xi(t)\right)} \preccurlyeq 1$$

Set $a_\xi(t) = \frac{1}{5} + e^{-t}, b_\xi(t) = \frac{1}{4} + e^{-t}, \tau_\xi(t) = t-2, \upsilon_\xi(t) = t-1,\ \xi = 1,2$

To satisfy the condition H2:

$$\lim_{t\to\infty}\sup\sum_{\xi=1}^{2}\left[\int_{t}^{\tau_\xi^{-1}(\upsilon_\xi(t))} a_\xi(s)\,ds + b_\xi(t)\right] = \lim_{t\to\infty}\sup\sum_{\xi=1}^{2}\left[\int_{t}^{t+1}\left(\frac{1}{5}+e^{-s}\right)ds + \frac{1}{4}+e^{-t}\right]$$

$$= \lim_{t\to\infty}\sup\sum_{\xi=1}^{2}\left[\frac{1}{5} - \left(e^{-(t+1)} - e^{-t}\right) + \frac{1}{4} + 2e^{-t}\right] = \frac{9}{10} \prec 1$$

To satisfy the condition (3.6):

$$\lim_{t\to\infty}\inf\int_{\upsilon_\xi(t)}^{t}\left[\left[\sum_{\xi=1}^{\eta} a_\xi\left(\tau_\xi^{-1}\left(\upsilon_\xi(s)\right)\right)\vartheta_2\left(\tau_\xi^{-1}\left(\upsilon_\xi(s)\right)\right)\left(\tau_\xi^{-1}\left(\upsilon_\xi(s)\right)\right)'\right]\right[1$$

$$+\sum_{\xi=1}^{\eta}\int_{\upsilon_\xi(s)}^{\tau_\xi^{-1}\left(\upsilon_\xi(\upsilon_\xi(s))\right)} a_\xi(u)\vartheta_2(u)\,du + \sum_{\xi=1}^{\eta} b_\xi\left(\upsilon_\xi(s)\right)\vartheta_2\left(\upsilon_\xi(s)\right)\left.\right]\left.\right]ds$$

$$= \liminf_{t\to\infty} \int_{t-1}^{t} \left[\left[\sum_{\xi=1}^{2} \frac{3}{4} (\frac{1}{5} + e^{-(s+1)}) \right] \left[1 + \sum_{\xi=1}^{2} \int_{s-1}^{s} \frac{3}{4} (\frac{1}{5} + e^{-u}) du + \sum_{\xi=1}^{2} \frac{3}{4} (\frac{1}{4} + e^{-(s-1)}) \right] \right] ds$$

$$= \liminf_{t\to\infty} \int_{t-1}^{t} \left[\left[\frac{3}{2} (\frac{1}{5} + e^{-(s+1)}) \right] \left[1 + \frac{3}{2} (\frac{1}{5} - e^{-s} + e^{-(s-1)}) + \frac{3}{2} (\frac{1}{4} + e^{-(s-1)}) \right] \right] ds$$

$$= \liminf_{t\to\infty} \int_{t-1}^{t} \left[\left[\frac{3}{10} + \frac{3}{2} e^{-(s+1)} \right] \left[\frac{67}{40} - \frac{1}{2} e^{-s} + e^{-s+1} \right] \right] ds$$

$$= \liminf_{t\to\infty} \int_{t-1}^{t} \left[\frac{201}{400} + \frac{67}{80} e^{-s-1} - \frac{3}{20} e^{-s} - \frac{1}{4} e^{-2s-1} + \frac{3}{10} e^{-s+1} + \frac{1}{2} e^{-2s} \right] ds$$

$$= 0.5025 \succ 0.36787 = \frac{1}{e}$$

By theorem (3.1) then every solution of the Eq. (3.9) has an oscillatory property**.**

4. Conclusions

Many researchers in the existence's field of bounded solutions have been focused on bounded solutions by a constant and others have been studied the existence of bounded solutions by functions, for more details see the references. The sufficient conditions have been found to ensure the existence of bounded solutions by using convergent sequences, as well as bounded by convergent series. The theorem of Krasnoselskii and Lebesgue's dominated convergence theorem are used. The sufficient conditions for oscillatory solution are more flexible and they are easy to apply in examples. The obtained conditions are more applicable and they are easier than others in a similar work.

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