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Periodic Solutions of the Forest Pest System Via Hopf Bifurcation and Averaging Theory

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Abstract

This work aims to analyse the dynamic behaviours of the forest pest system. We confirm the forest pest system in plane for limit cycles bifurcating existence from a Hopf bifurcation under certain conditions by using the first Lyapunov coefficient and the second-order of averaging theory. It is shown that all stationary points in this system have Hopf bifurcation points and provide an estimation of the bifurcating limit cycles.

Keywords: Limit cycles; Hopf bifurcation; Averaging theory; Forest pest system.

الحلول الدورية لنظام الآفات الحرجية عن طريق تشعب هوبف ونظرية المتوسط

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الخلاصة

الهدف من هذا البحث لتحليل الخصائص و الصفات الديناميكية لمنظومة أفة الغابة. وقد أكدنا ان هذه المنظومة في المستوى لها الدورات الحدودية بواسطة تفرع الهوبف تحت شروط معينه باستخدام معامل ليابانوف الأول والدرجة الثانية لنظرية المعدل. يتضح ان جميع نقاط الثابتة لهذه المنظومة لها نقاط تفرع هوبف و تعطى بصورة تخمينية تفرع الدارات الغائية.

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1. Introduction

Characterizing the existence of periodic solutions is a classic problem in the qualitative theory of real polynomial differential systems. The Hopf bifurcation theorem is the simplest requirement for a family of periodic solutions to bifurcate from a known family of stationary solutions in a dynamic system. However, in this study, we attempt to examine this phenomenon using Hopf bifurcation theory [1, 2]. Moreover, the method of averaging is another tool for studying the behaviour of non-linear planer differential systems, especially when investigating a periodic solution [3, 4].

Two simple age-structured forest pest system have been presented in [5, 6], which the insect pest attacks one of young or old trees. Thus, for the case where the pest feeds on undergrowth, it will be investigated that the Hopf bifurcation of the current system after some modifications and transforming it to the form below:

$$\begin{aligned} \dot{x} &= by - (y-1)^2\, x - a\, x = \mathrm{P}(x, y), \\ \dot{y} &= x - dy = Q(x, y)\,, \end{aligned} \tag{1}$$

where x is the young tree and y is the old tree, a, b and d are compound parameters. In [6] the authors studied the easiest models of mathematics for non-even-aged forests which can be affected by insect pests. Moreover, the authors in [6] used analytical methods such as the bifurcation theory and the numerical methods to study qualitative behaviours and dynamics of non-linear forest pest systems. For more information and details about the system, we refer to these references [7, 8, 9, 10, 11, 12, 13, 14] and references therein. Using the Hopf bifurcation theorem will be considered for finding the limit cycle (isolated closed orbits) of the forest pest system using the first Lyapunov coefficient and averaging theory of the first order and the second order.

The rest of this paper falls into these sections. In Section 2, basic definitions and results that are needed for this paper are introduced. In Section 3, the local stability of stationary points is discussed, as well as we prove that the system (1) has no limit cycles for some particular cases. In Section 4, we study Hopf bifurcations by using the first Lyapunov coefficient and first and second order of averaging theory, the direction of Hopf bifurcation and bifurcating periodic solutions stability are completely studied with numerical examples. Finally, conclusions of the paper are given.

2. Some basic definitions

2.1 The first Lyapunov coefficient

To begin our analytical investigation, we first recall some basic analytic facts form dynamical theory, for more details see [1].

Let C^n be a linear space that can be well-defined on the complex number field C. The scalar $\langle x, y\rangle$ for all $x, y \in \mathrm{C}^n$ satisfies the following properties:

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, where $\langle x, y\rangle = \bar{x}^T y = \sum_{i=1}^{n} \overline{x_i}\, y_i$,
2. $\langle x, \alpha y + \beta z \rangle = \alpha\langle x, y \rangle + \beta\langle x, z\rangle$, for each $\alpha, \beta \in \mathrm{C}$, and $x, y, z \in \mathrm{C}^n$,
3. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

If one introduces the norm $||x|| = \sqrt{\langle x, x \rangle}$ in C^n, then the space C^n becomes Hilbert space.

Now, a review of the projection method is described in [1, 2] for the calculation of the first Lyapunov coefficient associated with Hopf bifurcation.

Assume that the following continuous-time dynamical system

$$\dot{x} = A\, x + \mathrm{N}(x), \quad A = (a_{ij})_{n\times n}\,, \quad x \in \mathbb{R}^n, \tag{2}$$

where $N(x) = O(||x||^2)$ is smooth function. Suppose that $\mathrm{N}(x)$ is written as

$$\mathrm{N}(x) = \frac{1}{2}\mathrm{B}(x, x) + \frac{1}{6}\mathrm{C}(x, x, x) + O(||x||^4), \tag{3}$$

that $\mathrm{B}(x,y)$ and $\mathrm{C}(x,y,z)$ are bilinear and trilinear functions. In coordinates, we have

$$\mathrm{B}_i(x,y)=\sum_{j,k=1}^{n}\frac{\partial^2 \mathrm{N}_i(\xi)}{\partial\xi_j\partial\xi_k}\bigg|_{\xi=0} x_j y_k,\quad \mathrm{C}_i(x,y,z)=\sum_{j,k,l=1}^{n}\frac{\partial^3 \mathrm{N}_i(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l}\bigg|_{\xi=0} x_j y_k z_l. \tag{4}$$

Suppose that A contains two complex eigenvalues on the imaginary axis: $\lambda_{1,2}=\pm i\omega(\omega>0)$, and these eigenvalues are the only eigenvalues with $Re(\lambda)=0$. Suppose that $q\,\epsilon\,\mathrm{C}^n$ be a complex eigenvector corresponding to $\lambda_1=i\omega$:

$$Aq=i\omega\, q,\qquad A\,\bar{q}=-i\omega\bar{q}. \tag{5}$$

Include the adjoint eigenvector as well $\mathrm{p}\,\epsilon\,\mathrm{C}^n$ admitting the properties:

$$A^{\mathrm{T}}p=-i\omega p,\qquad A^{\mathrm{T}}\bar{p}=i\omega\bar{p}, \tag{6}$$

and satisfying the normalization $\langle\, p,\mathrm{q}\,\rangle=1$. The first Lyapunov coefficient at the origin is defined as

$$\ell_1(0)=\frac{1}{2\omega^2}Re(i\,g20\,g11+\omega g21), \tag{7}$$

where

$$g20=\langle\, p,\mathrm{B}(q,q)\,\rangle,\ g11=\langle\, p,\mathrm{B}(q,\bar{q})\,\rangle, g21=\langle\, p,C(q,q,\bar{q})\,\rangle.$$

We know $\ell_1(0)<0\ (\ell_1(0)>0)$, the Hopf bifurcation is supercritical (subcritical), respectively.

2.2 Averaging theory of first and second order

A summary of essential results concerning the averaging theory that is needed for proving the existence of periodic solutions for the system is presented (1). We can see [3] for further reading on averaging theory.

Theorem 1. Consider the differential equation

$$\dot{x}=\varepsilon f_{0.1}+\varepsilon^2 f_{0.2}+\varepsilon^3 M(t,x,\varepsilon), \tag{8}$$

where $f_{0.1}, f_{0.2}:\mathbb{R}\times\mathrm{U}\to\mathbb{R}^n, M:\mathbb{R}\times\mathrm{U}\times(-\varepsilon_{\mathrm{f}},\varepsilon_{\mathrm{f}})\to\mathbb{R}^n$ are continuous functions, T-periodic in $t\,(T$ is independent of $\varepsilon)$ and $\mathrm{U}\subset\mathbb{R}^n$ is an open subset. Suppose that the following hypotheses

a. $f_{0.1}(t,.)\in\mathrm{C}^1(\mathrm{U})$ for all $t\in\mathbb{R}$, $f_{0.1}, f_{0.2}\, and\, M$ are locally Lipschitz with respect to x. The function M is twice differentiable with respect to x.

b. Define $\mathcal{F}_{0i}:\mathrm{U}\to\mathbb{R}^n$ for $i=1{,}2$ by

$$\mathcal{F}_{01}=\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} f_{0.1}(s,\mathrm{z})ds,$$

$$\mathcal{F}_{02}=\frac{1}{\mathrm{T}}\int_0^{\mathrm{T}}\left[f_{0.2}(s,\mathrm{z})+D_{\mathrm{z}}\,f_{0.1}(s,\mathrm{z})\int_0^s f_{0.1}(t,\mathrm{z})\,dt\right]ds,$$

where $D_{\mathrm{z}}f_{0.1}$ is the Jacobian determinant matrix of components of $f_{0.1}$ with respect to z.

c. For V bounded and an open set in U, for $\varepsilon\in(-\varepsilon_{\mathrm{f}},\varepsilon_{\mathrm{f}})\backslash\{0\}$ there is $r_\varepsilon\in V$ such that $\mathcal{F}_{01}+\varepsilon\mathcal{F}_{02}=0$ and $d_{\mathrm{B}}(\mathcal{F}_{01}+\varepsilon\mathcal{F}_{02})\neq 0$.

Then, for $|\varepsilon|>0$ is sufficiently small, there exists a T-periodic solution $\phi(t,\varepsilon)$ of the system such that $\phi(t,\varepsilon)\to r_\varepsilon$ as $\varepsilon\to 0$. Moreover, the stability of the periodic solution $\phi(t,\varepsilon)$ is given by the stability of the stationary point r.

The term $d_{\mathrm{B}}(\mathcal{F}_{01}+\varepsilon\mathcal{F}_{02})\neq 0$ denotes the Brouwer degree of the function $\mathcal{F}_{01}+\varepsilon\mathcal{F}_{02}:V\to\mathbb{R}^n$ at its stationary point r which is not zero. Inequality is true when a sufficient condition of function's Jacobian $(\mathcal{F}_{01}+\varepsilon\mathcal{F}_{02})$ at r_ε is not to be zero.

If $\mathcal{F}_{01} \neq 0$, then the zeros of $\mathcal{F}_{01} + \varepsilon\mathcal{F}_{02}$ are mainly the zeros of $\mathcal{F}_{01}$ for ε sufficiently small. In this situation, we have the method of averaging of the first order. If $\mathcal{F}_{01}$ is identical to zero and $\mathcal{F}_{02} \neq 0$, then the zeros of $\mathcal{F}_{01} + \varepsilon\mathcal{F}_{02}$ are the zeros of $\mathcal{F}_{02}$ for ε sufficiently small. In this case, we have the method of averaging of the second order.
For other applications of averaging theory to the study limit cycles for systems, see for instance [3, 15].

3. Dynamical Analysis of the Forest Pest System and non-existence of Limit Cycles

3.1 Stability analysis and stationary points of the forest pest system

The stability analysis and persistence of system (1) are investigated. That is simple to get the system has only one isolated stationary point $E_0(0,0)$, if $d \neq 0, d(b-ad) < 0$; and if $d \neq 0$, $d(b-ad) > 0$, it has three isolated stationary points $E_0(0,0)$ and $E_{1,2}(x_0, y_0)$, where $x_0 = d \mp \sqrt{d(b-ad)}$, $y_0 = \frac{x_0}{d}$. The analysis of the corresponding linearized system is concentrated on determining the local stability of these stationary points. The Jacobian matrix of system (1) at the point $E(x, y)$ is computed as:

Case 1: Stationary point at $E_0(0,0)$
The system (1) at $E_0(0,0)$, the Jacobian matrix is
$J_0 = \begin{bmatrix} -(a+1) & b \\ 1 & -d \end{bmatrix}$.
The characteristic equation of J_0 is

$$\lambda^2 - \lambda T + D = 0, \tag{9}$$

where
$T = tr(J_0) = -(a+d+1)$ and $D = \det(J_0) = d(a+1) - b$.
The Jacobian of system (1) has the corresponding eigenvalues, linearized at (0, 0), they are:
$\lambda_{1,2} = \left(-(a+d+1) \mp \sqrt{(a-d+1)^2 + 4b}\right)/2$.
According to Eq. (9), we have the following conclusions:

I. If $d(a+1) < b$, then the stationary point $E_0(0,0)$ is saddle point.
II. If $-\frac{(a-d+1)^2}{4} < b < d(a+1)$ and $-(a+d) > 1$, then the stationary point $E_0(0,0)$ is unstable node point.
III. If $-\frac{(a-d+1)^2}{4} < b < d(a+1)$ and $-(a+d) < 1$, then the stationary point $E_0(0,0)$ is stable node point.
IV. If $-\frac{(a-d+1)^2}{4} > b$ and $-(a+d) > 1$, then the stationary point $E_0(0,0)$ is unstable focus point.
V. If $-\frac{(a-d+1)^2}{4} > b$ and $-(a+d) < 1$, then the stationary point $E_0(0,0)$ is stable focus point.
VI. If $b = -\frac{(a-d+1)^2}{4}$ and $a \neq -(d+1)$, then the $E_0(0,0)$ is either unstable improper node if $(a+d+1) < 0$ or stable improper node if $(a+d+1) > 0$.

Remark1: Since the arguments of the stationary points E_1 and E_2 are very similar, we only use the stationary point E_1 throughout this paper.

Case 2. Stationary point at E_1

Now, we move the stationary point E_1 of system (1) to the origin under the following transformation,

$$\begin{cases} x_1 = x - x_0, \\ y_1 = y - y_0, \end{cases}$$

this transforms the system (1) into the variant below:

$$\dot{x}_1 = -\frac{b}{d}x_1 + \left(2ad - b - 2\sqrt{d(b-ad)}\right)y_1 - \frac{2\sqrt{d(b-ad)}}{d}x_1y_1 - x_1y_1^2 - \left(d + \sqrt{d(b-ad)}\right)y_1^2, \quad (10)$$
$$\dot{y}_1 = x_1 - dy_1.$$

The Jacobian matrix for system (10) at E_1 is given by

$$J_{E_1} = \begin{bmatrix} -\frac{b}{d} & 2ad - b - 2\sqrt{d(b-ad)} \\ 1 & -d \end{bmatrix}.$$

The characteristic equation of J_{E_1} is

$$\lambda^2 + \left(d + \frac{b}{d}\right)\lambda + 2\left(b - ad + \sqrt{d(b-ad)}\right) = 0. \quad (11)$$

According to Routh-Hurwitz criteria [16] all roots of eq.(11) have negative real part if and only if $\left(d + \frac{b}{d}\right) > 0$ and $\left(b - ad + \sqrt{d(b-ad)}\right) > 0$.For the system (10) if $\left(d + \frac{b}{d}\right) > 0$ and $\left(b - ad + \sqrt{d(b-ad)}\right) > 0$, then E_1 is asymptotically stable.

3.2 Non-existence limit cycles

First, the Bendixson-Dulac criteria are used to investigate the non-existence of limit cycles in system (1). For more details see [17].

Proposition 2. (i) If $a + d = 0$, or $a, d > 0$, then the system (1) has no limit cycles.
(ii) If $b > 0$, then the system (1) has no limit cycles in a region $D = \{(x, y): xy \neq 0\}$.

Proof: We find that the divergence of system (1) is
$\text{div}(\mathrm{P}, Q) = \frac{\partial \mathrm{P}}{\partial x} + \frac{\partial Q}{\partial y} = -a - d - (y-1)^2$.
If $a + d = 0$ or $a, d > 0$, we obtain that sign $\text{div}(\mathrm{P}, Q) < 0$, by Bendixson's criterion [18]. Then, it cannot be limit cycles of the system (1) contained within plane.

(ii) Construct the Dulac function as follows $Ɓ(x, y) = \frac{1}{xy}$, then we have

$$\text{div}(Ɓ\mathrm{P}, ƁQ) = \frac{\partial(Ɓ\mathrm{P})}{\partial x} + \frac{\partial(ƁQ)}{\partial y} = -\frac{x^2 + by^2}{x^2y^2} < 0.$$

Also, if $b > 0$, we have sign $\text{div}(\mathrm{P}, Q) < 0$,so that by the Dulac Theorem [18] the system (1) has no limit cycles in D. $\square$

4. Hopf Bifurcation of forest pest system

4.1 Hopf bifurcation analysis by the first Lyapunov coefficient

We now show that the system (1) has limit cycles arising from Hopf bifurcation.

Proposition 3: The forest pest system (1) at the origin stationary point with eigenvalues $\pm i\omega$, $\omega \in \mathbb{R}^+$if and only if $d = d_h = -(a+1)$ and $b = -(a+1)^2 - \omega^2$, where $(a+1)^2 + b < 0$. Also, in the Eq. (9), which satisfy $\left(\frac{dRe(\lambda(d))}{d(d)}\right)_{d=d_h} = \frac{1}{2} \neq 0$, then the forest pest system (1) displays a Hopf bifurcation.

Proof: At the origin point the characteristic polynomial of the linear part of the forest pest system (1) is

$$P(\lambda) = \lambda^2 + (a + d + 1)\lambda + d(a + 1) - b.$$

In order to have a Hopf stationary point, we impose that $P(\lambda) = \lambda^2 + \omega^2$,so we obtain the system

$a + d + 1 = 0, \quad \omega^2 = d(a + 1) - b.$

When $d = d_h = -(a + 1)$, the Eq. (9) at the point $E_0(0,0)$ can be rewritten into

$$\lambda^2 - ((a + 1)^2 + b) = 0. \tag{12}$$

Clearly, the Eq. (12) has a pair of purely imaginary conjugate roots, namely $\lambda_{1,2} = \mp i\omega$, when $\omega = \sqrt{-(a + 1)^2 - b}$, where $((a + 1)^2 + b < 0)$.

Let $\lambda = \lambda(d)$, we define the following relation from the characteristic Eq. (9)

$$f(\lambda(d), d) = \lambda(d)^2 - \lambda(d)T + D = 0. \tag{13}$$

Differentiation of (13) with respect to d yields,

$$\frac{\partial f}{\partial \lambda}\frac{d\lambda}{d(d)} + \frac{\partial f}{\partial d} = 0.$$

We can obtain

$$\frac{d\lambda(d)}{d(d)} = -\frac{\partial f}{\partial d}\left(\frac{\partial f}{\partial \lambda}\right)^{-1} = -\frac{\lambda + a + 1}{2\lambda + a + d + 1}. \tag{14}$$

Taking the root $\lambda(d) = i\omega$, evaluating $d = d_h$, and substituting it into (14), we have

$$\left(\frac{dRe(\lambda(d))}{d(d)}\right)_{d=d_h} = -\frac{1}{2} \neq 0\,. \tag{15}$$

Obviously, the first two conditions of Hopf bifurcation are satisfied so that the Hopf bifurcation theorem holds. Therefore, by Guckenheimer and Holmes [19], we know that system (1) undergoes a Hopf bifurcation in stationary point at $E_0(0,0)$ when $d = d_h$. □

Theorem 4: If the conditions of Proposition 3 hold and $a^2 + 2a + b + 1 < 0$, then the first Lyapunov coefficient of system (1) at stationary point $E_0(0,0)$ is given by

$\ell_1(0) = \frac{a^2-2a+b-3}{2\sqrt{(-a^2-2a-b-1)^3}}$,

when $a \geq -1$, we have $\ell_1(0) < 0$, so the Hopf bifurcation at $E_0(0,0)$ is supercritical. Whereas, when $a^2 - 2a + b > 3$, we have $\ell_1(0) > 0$, therefore the Hopf bifurcation at $E_0(0,0)$ is subcritical.

Proof: The Jacobian matrix A for system (1) at $E_0(0,0)$ when $d = -(a + 1)$, we can write in the form

$$A = \begin{bmatrix} -(a + 1) & b \\ 1 & a + 1 \end{bmatrix}. \tag{16}$$

Suppose that $q \in \mathbb{C}^2$ is an eigenvector of matrix A corresponding to the eigenvalues. Also, let $p \in \mathbb{C}^2$ be an eigenvector of the transposed matrix A^T corresponding to conjugate eigenvalues. We derive the four vectors via tedious calculations,

$$q = \begin{pmatrix} -(a + 1) + i\,\omega \\ 1 \end{pmatrix},\quad \bar{q} = \begin{pmatrix} -(a + 1) - i\,\omega \\ 1 \end{pmatrix},$$
$$\mathrm{p} = \frac{1}{-2i\omega}\begin{pmatrix} 1 \\ (a + 1) - i\,\omega \end{pmatrix},\quad \bar{\mathrm{p}} = \frac{1}{2i\omega}\begin{pmatrix} 1 \\ (a + 1) + i\,\omega \end{pmatrix}, \tag{17}$$

Where $\bar{q}$ is the conjugate vector of q. Which satisfies

$$Aq = i\omega \mathrm{q}, \qquad A^T p = -i\omega p, \qquad \text{and} \quad ,\langle\, p, q\,\rangle = 1\,. \tag{18}$$

In system (1) there will be bilinear and trilinear functions. Then, the $B(\xi, \eta)$,and $C(\xi, \eta, \zeta)$ define planar vectors

$\xi = (\xi_1, \xi_2)^{\mathrm{T}} \in \mathbb{R}^2$, $\eta = (\eta_1, \eta_2)^{\mathrm{T}} \in \mathbb{R}^2$, and $\zeta = (\zeta_1, \zeta_2)^{\mathrm{T}} \in \mathbb{R}^2$, the values

$$\mathrm{B}(\xi,\eta) = \begin{pmatrix} 2(\xi_1\eta_2 + \eta_1\xi_2) \\ 0 \end{pmatrix},\ C(\xi,\eta,\zeta) = \begin{pmatrix} -2(\xi_1\eta_2\zeta_2 + \xi_2\eta_2\zeta_1 + \xi_2\eta_1\zeta_2) \\ 0 \end{pmatrix}. \qquad (19)$$

From (16), (17) and (19), the straightforward and tedious calculation yields

$g20 = \langle p, \mathrm{B}(q,q) \rangle = \frac{2(\omega+i(a+1))}{\omega}$, $g11 = < p, \mathrm{B}(q,\bar{q}) > = \frac{2i(a+1)}{\omega}$,

$$g21 = \langle p, C(q,q,\bar{q}) \rangle = -\frac{\omega + 3i(1+a)}{\omega}.$$

The substitution of $g20, g11$ and $g21$ into the first Lyapunov coefficient $\ell_1(0)$ in Eq. (7), we obtain

$\ell_1(0) = \frac{1}{2\omega^2} Re(i\, g20 g11 + \omega g21) = \frac{a^2-2a+b-3}{2\sqrt{(-a^2-2a-b-1)^3}}$.

From the first Lyapunov coefficient $\ell_1(0)$, since $a^2 + 2a + 1 + b < 0$, then $a^2 - 2a + b - 3 < -4a - 4$.

Then, if $a \geq -1$, we have $\ell_1(0) < 0$, the Hopf bifurcation is supercritical. Although, when $a^2 - 2a + b > 3$, we have $\ell_1(0) > 0$, the Hopf bifurcation is subcritical. □

From Theorem 4, we should be noted that there are possible results where the first Lyapunov coefficient will not provide outcomes, this means that $\ell_1(0) = 0$, the Hopf bifurcation is degenerate, when

(I) $b = -a^2 + 2a + 3$.

In the previous case the higher-order Lyapunov's coefficient would be necessary to describe the existence of a periodic solution rising from stationary point $E_0(0,0)$.

The Hopf bifurcation at E_1 occurs, and the stability of E_1 depends on the value of the first Lyapunov coefficient ℓ_1. We have the next Proposition.

Proposition 5: The forest pest system (10) at $E_1(0,0)$ stationary point with eigenvalues $\pm i\omega$, $\omega \in \mathbb{R}^+$,if and only if $b = b_h = -d^2$, $\omega^2 = 2\left(-b - ad + \sqrt{d(b-ad)}\right)$, such that $\left(2\left(-d(a+d) + \sqrt{-d^2(a+d)}\right)\right) > 0, and\ (a+d) < 0$. Also, in the Eq. (11), which satisfy $\left(\frac{dRe(\lambda_+(b))}{d(b)}\right)_{b=b_h} = \frac{-1}{2d} \neq 0$, then the forest pest system (1) displays a Hopf bifurcation.

Proof: At the E_1 point the characteristic polynomial of the linear part of the forest pest system (1) is

$P(\lambda) = \lambda^2 + \left(d + \frac{b}{d}\right)\lambda + 2(b - ad + \sqrt{d(b-ad)})$.

In order to have a Hopf stationary point, we impose that $P(\lambda) = \lambda^2 + \omega^2$,we obtain the system

$\left(d + \frac{b}{d}\right) = 0, \quad \omega^2 = 2(b - ad + \sqrt{d(b-ad)})$.

When $b = b_h$, the Eq. (11) at the point E_1 can be rewritten into

$$\lambda^2 + (2(-d(a+d) + \sqrt{-d^2(a+d)})) = 0.$$

Clearly, Eq. (11) contains two purely imaginary conjugate roots, $\lambda_{1,2} = \mp i\omega$, when $\omega = \sqrt{2(-d(a+d) + \sqrt{-d^2(a+d)})}, \left(2\left(-d(a+d) + \sqrt{-d^2(a+d)}\right) > 0\right)$.

Let $\lambda = \lambda(b)$, define the relation from the characteristic Eq. (11)

$$f(\lambda(b), b) = \lambda(b)^2 + \left(\left(d + \frac{b}{d}\right)\right)\lambda(b) + 2(b - ad + \sqrt{-d(b-ad)}) = 0. \qquad (20)$$

Differentiation of (20) with respect to b yields,

$$\frac{\partial f}{\partial \lambda}\frac{d\lambda}{d(b)} + \frac{\partial f}{\partial b} = 0.$$

We can obtain

$$\frac{d\lambda(b)}{d(b)} = -\frac{\partial f}{\partial b}\left(\frac{\partial f}{\partial \lambda}\right)^{-1} = -\frac{\lambda\sqrt{d(b-ad)} + 2d\sqrt{d(b-ad)} + d^2}{(2d\lambda + d^2 + b)\sqrt{d(b-ad)}}. \qquad (21)$$

Taking the root $\lambda(b) = \lambda_+(b) = i\omega$, evaluating $b = b_h$, and substituting it into (21), we have

$$\left(\frac{dRe(\lambda_+(b))}{d(b)}\right)_{b=b_h} = \frac{-1}{2d} \neq 0\,. \qquad (22)$$

Obviously, the first two conditions of Hopf bifurcation are fulfilled, and the Hopf bifurcation theorem holds. Therefore, by Guckenheimer and Holmes [19], we know that system (1) undergoes a Hopf bifurcation at stationary point at E_1 when $b = b_h$. ▭

Theorem 6: If the conditions of Proposition 5 hold, then the first Lyapunov coefficient of system at stationary point E_1 satisfying $b = b_h$ is given by

$\ell_1(0) = \frac{-2d(a+d)}{\omega^3} \neq 0$,

when $d < 0$, we have $\ell_1(0) < 0$, so the Hopf bifurcation at E_1 is nondegenerate and supercritical. Although, $d > 0$, we have $\ell_1(0) > 0$, consequently the Hopf bifurcation at E_1 is nondegenerate and subcritical.

Proof: The Jacobian matrix A for system (10) at E_1 when $b = -d^2$, we can write in the form

$$A = \begin{bmatrix} d & 2ad + d - \sqrt{-d^2(a+d)} \\ 1 & -d \end{bmatrix}. \qquad (23)$$

Suppose that $q \epsilon \mathbb{C}^2$ is an eigenvector of matrix A corresponding to eigenvalues. Also, let $p \epsilon \mathbb{C}^2$ be an eigenvector of the transposed matrix A^T corresponding to conjugate eigenvalues. We derive the four vectors via tedious calculations,

$$q = \begin{pmatrix} d + i\,\omega \\ 1 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} d - i\,\omega \\ 1 \end{pmatrix}, \quad \mathrm{p} = \frac{1}{-2i\omega}\begin{pmatrix} 1 \\ -d - i\,\omega \end{pmatrix}, \ \bar{\mathrm{p}} = \frac{1}{2i\omega}\begin{pmatrix} 1 \\ -d + i\,\omega \end{pmatrix}, \qquad (24)$$

Where $\bar{q}$ is the conjugate vector of q. Which satisfies

$$Aq = i\omega \mathrm{q}, \qquad A^T\mathrm{p} = -i\omega\mathrm{p}, \qquad \text{and} \quad ,\langle p, q \rangle = 1\,. \qquad (25)$$

In the system (10) there will be bilinear and trilinear functions. Then, the $\mathrm{B}(\xi,\eta)$,and $C(\xi,\eta,\zeta)$ define planar vectors

$\xi = (\xi_1, \xi_2)^{\mathrm{T}} \in \mathbb{R}^2$, $\eta = (\eta_1, \eta_2)^{\mathrm{T}} \in \mathbb{R}^2$, and $\zeta = (\zeta_1, \zeta_2)^{\mathrm{T}} \in \mathbb{R}^2$, the values

$$\mathrm{B}(\xi,\eta) = \begin{pmatrix} -2\left(\frac{\sqrt{-d^2(a+d)}}{d}(d\,\xi_2\eta_2 + \eta_2\xi_1 + \eta_1\xi_2) + d\xi_2\eta_2\right) \\ 0 \end{pmatrix},$$
$$C(\xi,\eta,\zeta) = \begin{pmatrix} -2(\xi_1\eta_2\zeta_2 + \xi_2\eta_2\zeta_1 + \xi_2\eta_1\zeta_2) \\ 0 \end{pmatrix}. \qquad (26)$$

From (23), (24) and (26), the straightforward and tedious calculation yields

$g20 = \langle p, \mathrm{B}(q,q) \rangle = -\frac{2\sqrt{-d^2(a+d)}}{\mathrm{d}} + i\left(\frac{\left(d+3\sqrt{-d^2(a+d)}\right)}{\omega}\right)$,

$g11 = \langle p, \mathrm{B}(q,\bar{q}) \rangle = \frac{i\left(d+3\sqrt{-\mathrm{d}^2(a+d)}\right)}{\omega}$,

$$g21 = \langle p, C(q,q,\bar{q}) \rangle = -\frac{\omega - 3id}{\omega}.$$

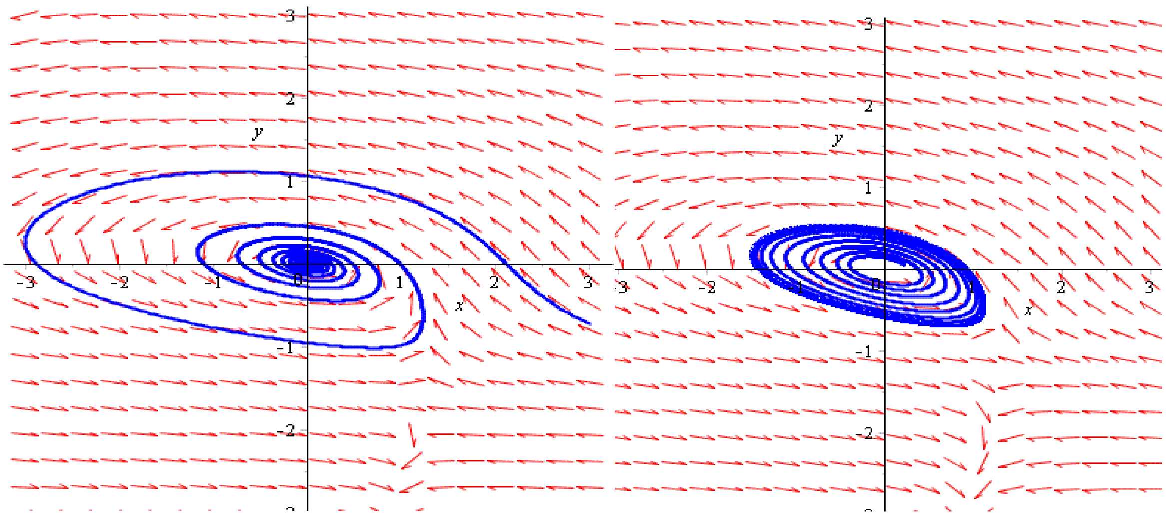
The substitution of $g20, g11$ and $g21$ in to the first Lyapunov coefficient $\ell_1(0)$ in Eq. (7), we obtain

$\ell_1(0) = \frac{1}{2\omega^2}Re(i\,g20g11 + \omega g21) = -\frac{2d(a+d)}{\omega^3}$.

From the above $\ell_1(0)$, must be $(a + d) < 0$ and $\omega > 0$.
Hence, when $d > 0$, we have $\ell_1(b_h) > 0$, so the Hopf bifurcation is nondegenerate and subcritical. while, when $d < 0$, we have $\ell_1(b_h) < 0$, so the Hopf bifurcation is nondegenerate and supercritical. $\square$

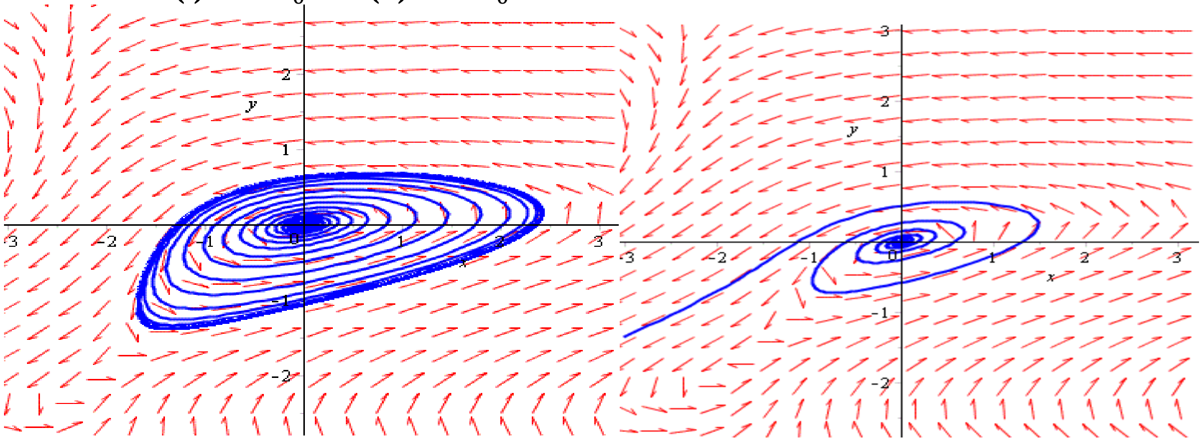
4.2 Numerical results on Hopf bifurcation

We give examples for the stationary point $E_0(0,0)$ about phase portrait forest pest system (1), respectively. Firstly, according to Theorem 4, for the Hopf bifurcation at the stationary point at $E_0(0,0)$, we fix $a = -0.2$ and $b = -5$, so that $d_0 = -0.9$ with initial conditions $x(0) = 0.01$ *and* $y(0) = 0.3$. Likewise, $\ell_1(0) = -0.4541365201 < 0$, then the Hopf bifurcation is supercritical. Here, we see that it is unstable when $d = -0.7 > -0.9 = d_0$. While, $d = -1.1 < -0.9 = d_0$. Therefore, stable limit cycles yield, the result is shown in Figure 1.
Secondly, according to Theorem 6 for the Hopf bifurcation at the stationary point at E_1 we fix $a = -2$ and $d = 1$, so that $b_0 = -1$ with initial conditions $x(0) = y(0) = 0.3$. Also, $\ell_1(0) = 0.25 > 0$, then the Hopf bifurcation is subcritical. Here, we see that it is stable when $= -1.3 < -1 = b_0$. While, $b = -0.8 > -1 = b_0$ unstable limit cycles yields, the result is shown in Figure 2.



(i) **(ii)**

Figure 1: Phase portraits of system (1) at the stationary point at $E_0(0,0)$, we fix $a = -0.2$, and $b = -5$. **(i)** $d > d_0$ and **(ii)** $d < d_0$.



(i) **(ii)**

Figure 2: Phase portraits of system (1) at the stationary point at E_1, we fix $a = -2$, and $d = 1$. **(i)** $b > b_0$ and **(ii)** $b < b_0$.

4.3 Hopf bifurcation analysis by Averaging theory

In the next results, we show that the averaging method can be used to find sufficient conditions on parameters of the system (1) has a limit cycles.

Theorem 7: Consider the forest pest system (1), if $d = -(a+1) + \varepsilon\, d_1 + \varepsilon^2 d_2$ and $b = -(a+1)^2 - \omega^2$ with $\omega > 0$, $\frac{d_2}{\omega^2+4a+4} < 0$, with $\varepsilon > 0$ is a sufficiently small parameter. Via averaging theory of second-order has one limit cycle bifurcating from the Hopf stationary point at the origin. The limit cycle is stable if $d_2 < 0$ and unstable if $d_2 > 0$.

Proof: Let the parameters for perturbations $(b,d) = (-(a+1)^2 - \omega^2, -(a+1) + \varepsilon\, d_1 + \varepsilon^2 d_2)$, then the forest pest system (1) becomes

$$\begin{aligned}\dot{x} &= -(a+1)\,x - ((a+1)^2 + \omega^2)y + 2\,xy - \,xy^2,\\ \dot{y} &= x + (a - \varepsilon d_1 - \varepsilon^2 d_2 + 1)\,y.\end{aligned} \tag{27}$$

Doing the rescaling of variable $(x,y) = (\varepsilon X, \varepsilon Y)$, the system (27) in the new variable (X,Y) is

$$\begin{aligned}\dot{X} &= -(a+1)X - ((a+1)^2 + \omega^2)Y + 2\varepsilon XY - \varepsilon^2 XY^2,\\ \dot{Y} &= X + (a - \varepsilon d_1 - \varepsilon^2 d_2 + 1)Y.\end{aligned} \tag{28}$$

Now, the linear can be written at the stationary point E_0 of system (28) when $\varepsilon = 0$, into its real Jordan normal form

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

This change of variables is verified as, where $\omega^2 = -(a+1)^2 - b$,

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -(a+1) & -\omega \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \tag{29}$$

In the new introduced variables (U,V) , the system (28) is written as follows:

$$\begin{aligned}\dot{U} &= -\omega V - \varepsilon d_1 U - \varepsilon^2 d_2 U,\\ \dot{V} &= \omega U + \frac{\varepsilon U}{\omega}\big(2(\omega V + U + aU) + d_1(a+1)\big) - \frac{\varepsilon^2 U}{\omega}\big((a+1)U^2 + \omega UV - (a+1)\,d_2\big).\end{aligned} \tag{30}$$

Therefore, using angle θ and we write the system (30) in the polar coordinates as follows: $U = r\cos\theta$ and $V = r\sin\theta$, and we can apply the averaging theory, we obtain

$$\dot{r} = -\varepsilon\frac{r\cos\theta}{\omega}(2r\omega\cos^2\theta + \cos\theta\,(-2r(a+1)\sin\theta + \omega d_1) - (a+1)\cos\theta\, d_1 - 2r\omega) - \frac{\varepsilon^2}{\omega} r\cos\theta\,(r^2(a+1)\sin\theta\cos^2\theta - \omega r^2\cos^3\theta + \omega(r^2 + d_2)\cos\theta - d_2(a+1)\sin\theta),$$

(31)

$$\dot{\theta} = \omega + \frac{\varepsilon}{\omega}\big(\cos\theta\,(2\,r\cos\theta + d_1)(\cos\theta\,(a+1) + \omega\sin\theta)\big) - \frac{\varepsilon^2}{\omega}\big(\cos\theta\,(r^2\cos^2\theta - d_2)(\cos\theta\,(a+1) + \omega\sin\theta)\big).$$

We apply the averaging theory to the angular variable θ as the new independent variable. We compute $\frac{dr}{d\theta}$ and develop the new equation for the system (31) in the variable ε up to the second order in the form

$$\frac{dr}{d\theta} = \varepsilon\, f_{0.1} + \varepsilon^2\, f_{0.2} + O(\varepsilon^3), \tag{32}$$

where

$$f_{0.1} = -\frac{r\cos\theta}{\omega^2}(2r\omega\cos^2\theta) - 2r\sin\theta\cos\theta\,(a+1) + \omega d_1\cos\theta - d_1\sin\theta\,(a+1) - 2r\omega),$$

$$f_{0.2} = \frac{r\cos\theta}{\omega^4}(4r^2\cos^4\theta\sin\theta\,(\omega^2 - a^2 - 4a) + 8\omega r d_1\cos^2\theta + 2a\omega d_1^2\cos\theta - a^2 d_1^2\sin\theta\cos^2\theta + \omega^2 d_1^2\sin\theta\cos^2\theta - 4rd_1\sin\theta\cos^3\theta - 2ad_1^2\sin\theta\cos^2\theta + 8\omega r^2\cos^5\theta + \omega^3 r^2\cos^3\theta - 8\omega r^2\cos^3\theta + 8a\omega r^2\cos^5\theta - 8a\omega r^2\cos^3\theta - 5\omega^2 r^2\sin\theta\cos^2\theta - 6\omega r d_1\cos^2\theta - ar^2\omega^2\sin\theta\cos^2\theta - 6a\omega r d_1\cos^2\theta - 2\omega^2 r d_1\sin\theta\sin\theta + a\omega^2 d_2\sin\theta - a\omega d_1^2\cos\theta - \omega d_1^2\cos\theta - d_2\omega^3\cos\theta + \omega^2 d_2\sin\theta - \omega^3 r^2\cos\theta - 4r^2\sin\theta\cos^4\theta + 2\omega d_1^2\cos^3\theta - d_1^2\sin\theta\cos^2\theta - 4a^2 r d_1\sin\theta\cos^3\theta + 4\omega^2 r d_1\sin\theta\cos^3\theta - 8ard_1\sin\theta\cos^3\theta + 8a\omega r d_1\cos^4\theta).$$

We shall apply the averaging differential system of the first order for system (32). Via the notation of Theorem 1, we have $t = \theta, T = 2\pi$ and $x = r$. Also, we have interval $I = \{r: 0 < r < \bar{r}\}$ for some $\bar{r} > 0$, given the following result by Theorem 1 formula (b)

$$\mathcal{F}_{01} = \frac{1}{2\pi}\int_0^{2\pi} f_{0.1}(r)\, d\theta = -\frac{rd_1}{2\omega}.$$

Hence, $\mathcal{F}_{01}(r)$ has no solution in the interval I. We move to the second-order averaging theory $\mathcal{F}_{01} \equiv 0$. This makes $d_1 = 0$ and by Theorem 1 formula (b) after the same calculation for the $f_{0.2}$ to

$$\mathcal{F}_{02} = \frac{1}{2\pi}\int_0^{2\pi}\left[f_{0.2}(r) + \left(\frac{\partial}{\partial r}f_{0.1}\right)\int_0^{\theta} f_{0.1}(r)\, d\theta\right] d\theta = -\frac{r(r^2(\omega^2 + 4a + 4) + 4\omega^2 d_2)}{8\omega^3}.$$

Therefore, since $\mathcal{F}_{02} = 0$ has one positive real root $r^* = 2\omega\sqrt{-\frac{d_2}{\omega^2+4a+4}}$ in the interval I. If $\frac{d_2}{\omega^2+4a+4} < 0$, then the derivative of $\mathcal{F}_{02}$ at r^* is $\frac{d\mathcal{F}_{02}}{dr(r^*)} = \frac{d_2}{\omega} \neq 0$. Moreover, we obtain that the small limit cycle is stable if $d_2 < 0$, and unstable if $d_2 > 0$. For $\varepsilon > 0$ is sufficiently small, Theorem 1, guarantees the existence of a 2π −periodic solution r^* such that $r^*(\theta, \varepsilon) \to 2\omega\sqrt{-\frac{d_2}{\omega^2+4a+4}}$, when $\varepsilon \to 0$. Now we shall look at the system (28), similarly, the current system has the periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ bifurcating from the origin with a period tends to 2π when $\varepsilon \to 0$. □

We translate the stationary point E_1 to the origin, can you see the above result system (10).

Theorem 8: Consider the forest pest system (10), if $b = -d^2 + \varepsilon^2\beta$, $\omega^2 = 2\left(d(a+d) + \sqrt{-d^2(a+d)}\right)$, with $\omega > 0$, $\left(2\left(d(a+d) - \sqrt{-d^2(\,a+d)}\right)\right) > 0, and\ d^2(a+d) < 0$ with $\beta > 0$ with $\varepsilon > 0$ is a sufficiently small parameter. Only one limit cycle bifurcates from the Hopf stationary point at E_1 using the averaging theory of the second order. The limit cycle is stable if $(a+d)(-2\sqrt{-d^2(\,a+d)} + d(a+d-1) < 0$, and it is unstable if $(a+d)(-2\sqrt{-d^2(a+d)} + d(a+d-1) > 0$.

Proof: Let the parameters for perturbations $b = -d^2 + \varepsilon^2\beta$, then the forest pest system (10) becomes

$$\dot{x} = -\sqrt{-d(ad + d^2 - \varepsilon^2\beta)}\left(2y + \frac{2xy}{d} + y^2\right) + \left(d - \frac{\varepsilon^2\beta}{d}\right)x + (2ad + d^2 - \varepsilon^2\beta)y - dy^2 - xy^2, \tag{33}$$
$$\dot{y} = x - dy.$$

Doing the rescaling of variable $(x,y)=(\varepsilon X,\varepsilon Y)$, then the system (33) in the new variable (X,Y) is

$$\dot{X}=-\sqrt{-d(ad+d^2-\varepsilon^2\beta)}\left(2Y+\frac{2\varepsilon xy}{d}+\varepsilon Y^2\right)+\left(d-\frac{\varepsilon^2\beta}{d}\right)X+(2ad+d^2-\varepsilon^2\beta)Y-\varepsilon dY^2-\varepsilon^2 XY, \tag{34}$$
$$\dot{Y}=X-d\,Y.$$

Now, the linear can be written at the stationary point E_1 of the system (34) when $\varepsilon=0,$ into its real Jordan normal form

$$\begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}.$$

This change of variables is verified as,

$$\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}d & -\omega\\ 1 & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}. \tag{35}$$

In the new variables (u,v), the system (34) is written as follows:

$$\dot{u}=-\omega\,v,$$
$$\dot{v}=\omega u+\varepsilon\left(\frac{(d-3\sqrt{-d(d(a+d)-\varepsilon^2\beta)}}{\omega}u^2+\frac{2\sqrt{-d(d(a+d)-\varepsilon^2\beta)}}{d}uv\right)+\varepsilon^2\left(\frac{2\beta u+du^3}{\omega}-u^2-\frac{\beta}{d}v\right). \tag{36}$$

Therefore, using angle θ and we write the differential system (36) in polar coordinates as follows: $u=r\cos\theta$ and $v=r\sin\theta,$ we can apply the averaging theory, and we obtain

$$\dot{r}=\varepsilon\,r^2\sin\theta\cos\theta\left(\frac{(d-3\sqrt{-d(d(a+d)-\varepsilon^2\beta)}}{\omega}+\frac{2\sqrt{-d(d(a+d)-\varepsilon^2\beta)}}{d}\sin\theta\right)-\varepsilon^2 r\sin\theta\left(\frac{2\beta\cos\theta+d\,r^2\cos^3\theta}{\omega}-\frac{\beta\sin\theta}{d}-r^2\sin\theta\cos^2\theta\right)+O(\varepsilon^3), \tag{37}$$
$$\dot{\theta}=\omega+\varepsilon\,r\cos^2\theta\left(2\sqrt{-d(d(a+d)-\varepsilon^2\beta)}\left(\frac{\sin\theta}{d}-\frac{3\cos\theta}{\omega}\right)+\frac{d\cos\theta}{\omega}\right)-\frac{\varepsilon^2\cos\theta}{d\,\omega}(-\omega\sin\theta\,(dr^2\cos^2\theta+\beta)+d^2r^2\cos^3\theta+2\beta d\cos\theta)+O(\varepsilon^3).$$

We applying the averaging theory to the angular variable θ as the new independent variable. We compute $\frac{dr}{d\theta}$ and develop the new equation for the system (37) in the variable ε up to the second order in the form

$$\frac{dr}{d\theta}=\varepsilon\,f_{0.1}+\varepsilon^2\,f_{0.2}+O(\varepsilon^3), \tag{38}$$

where

$$f_{0.1}=-\frac{r^2\sin\theta\cos\theta}{d\omega}\left(2\sqrt{-d^2(a+d)}\sin\theta+\omega+d\cos\theta\left(d-3\sqrt{-d^2(a+d)}\right)\right),$$
$$f_{0.2}=-\frac{r\sin\theta}{d^2\left(d(a^2+2ad+d^2-a-d+2\sqrt{-d^2(a+d)}(\,a+d)\right)}\Big(\omega\,\Big(12ar^2\sin\theta\cos^4\theta+12r^2d^3\sin\theta\cos^4\theta+4r^2d\sqrt{-d^2(a+d)}\sin\theta\cos^4\theta-2ar^2d^2\sin\theta\cos^2\theta-2r^2d^3\sin\theta\cos^2\theta-2r^2d\sqrt{-d^2(a+d)}\sin\theta\cos^2\theta-2\beta ad\sin\theta-2\beta d^2\sin\theta-2\beta\sqrt{-d^2(a+d)}\sin\theta\Big)+\sqrt{-d^2(a+d)}(4\beta d\cos\theta+8ar^2d\cos^3\theta-14r^2d^2\cos^5\theta-8ar^2d\cos^5\theta)+r^2d^3\cos^5\theta-8a^2r^2d^2\cos^5\theta-25ar^2d^3\cos^5\theta-17r^2d^4\cos^5\theta+8a^2r^2d^2\cos^3\theta+18ar^2d^3\cos^3\theta+10r^2d^4\cos^3\theta+4\beta ad^2\cos\theta+4\beta d^3\cos\theta\Big).$$

We shall apply the averaging differential system as represented in Theorem 1 to the differential system (38). This is done by using Theorem 1, we have $t=\theta, T=2\pi$ and $x=r$. Also, we have interval $I=\{r\colon 0<r<\bar{r}\}$ for same $\bar{r}>0$,given the following results

$$\mathcal{F}_{01} = \frac{1}{2\pi}\int_0^{2\pi} f_{0.1}(r)\, d\theta = 0.$$

$\mathcal{F}_{02} =$
$\frac{1}{2\pi}\int_0^{2\pi}\left[f_{0.2}(r) + \left(\frac{\partial}{\partial r} f_{0.1}\right)\int_0^{\theta} f_{0.1}(r)\, d\theta\right] d\theta = -$
$\frac{r\omega}{8\left(-2\sqrt{-d^2(a+d)}+d(a+d-1)\right)\left(d^2(a+d)\right)}\left(-2\beta\sqrt{-d^2(a+d)} + d(dr^2 - 2\beta)(a+d)\right).$

Therefore, since $\mathcal{F}_{02} = 0$ has one positive real root $r^* = \sqrt{\frac{2\beta\left(ad+d^2-\sqrt{-d^2(a+d)}\right)}{d^2(a+d)}}$, when $d^2(a+d) < 0$ and $\beta > 0$, then derivative of $\mathcal{F}_{02}$ at r^* is
$\frac{d\mathcal{F}_{02}}{dr(r^*)} = \frac{-\beta\omega\left(ad+d^2+2\sqrt{-d^2(a+d)}\right)}{2d^2(a+d)(-2\sqrt{-d^2(a+d)}+d(a+d-1)} \neq 0$, and must be $a > 0, d < 0$ or $a < 0, d > 0$. Moreover, we obtain that the small limit cycle is stable if $(a+d)(2\sqrt{-d^2(a+d)} + d(a+d-1) < 0$, and it is unstable if $(a + d)(2\sqrt{-d^2(a+d)} + d(a+d-1) > 0$. For $\varepsilon > 0$ is sufficiently small, Theorem 1 guarantees the existence of a 2π −periodic solution r^* such that

$r^*(\theta,\varepsilon) \to \sqrt{\frac{2\beta\left(ad+d^2-\sqrt{-d^2(a+d)}\right)}{d^2(a+d)}}$, when $\varepsilon \to 0$. Now we have to look back to system (28), it also has periodic solution $(x(t,\varepsilon), y(t,\varepsilon))$ bifurcating from the origin with a period tends to 2π when $\varepsilon \to 0$. □

5. Conclusions

This paper sheds light on the analysis of stability and bifurcation of the forest pest system. In this system, we have shown that the Hopf bifurcation occurs at stationary points E_0 and $E_{1,2}$. We studied the limit cycles bifurcating from these stationary points via the first Lyapunov coefficient and averaging theory of the first order and the second order. Moreover, it is shown that six limit cycles can bifurcate from stationary points and provide an estimation of the bifurcating limit cycles with the direction of Hopf bifurcation and bifurcating periodic solutions stability are completely studied. Moreover, the local stability of stationary points is discussed, as well as we prove that this system has no limit cycles if either $a + d = 0$, or $a, d > 0$ or $b > 0$ in the region $D = \{(x, y): xy \neq 0\}$,via the Bendixson-Dulac criteria. Also, some numerical results are presented.

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