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Properties of J− Regular modules

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Abstract
The present study introduces the concept of **J-pure** submodules as a generalization of pure submodules. We study some of its basic properties and by using this concept we define the class of J-regular modules, where an R-module M is called **J-regular** module if every submodule of M is J-pure submodule. Many results about this concept are proved.

Keywords: pure submodules**,** J-pure submodules, regular modules and J-regular modules.

المقاسات المنتظمة من النمط –J

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الخلاصة
في هذا البحث نقدم مفـهوم المقاسات الجزئية النقية من النمط –J كتعميم لمفهوم المقاسات الجزئية النقيةوباستخدام
المفـهوم هذا نعرف المقاسات المنتظمة من النمط –J إذ يقال ان المقاس M على الحلقة R بأنه منتظم من النمط –J اذا كان كل مقاس جزئي منة يكون نقياً من النمط –J أعطينا العديد من النتائج حول المفـهوم هذا

1- Introduction

Throughout this paper, all rings are a commutative with identity and every R-module is a unitary. The notion pure submodule is well known and there are several authors deal with this concept. For example [1] and [2]. Let M be an R-module. A submodule N of M is called pure if the sequence $0 \longrightarrow E\otimes N \longrightarrow E\otimes M$ is exact for every R-module E. Cohn [1]. Equivalently if for each $\sum_{i=1}^{n} r_{ji}m_i \in N$, $r_{ji} \in R$, $m_i \in M$, $j = 1,2,\dots,k$, there exist $x_i \in N$, $i = 1,2,\dots,n$ such that $\sum_{i=1}^{n} r_{ji}m_i = \sum_{i=1}^{n} r_{ji}x_i$ for each j. A submodule N of an R-module M is called pure in M if IN = N ∩ IM for every ideal I of R [3]. Recall that an R- module M is called regular module if every submodule of M is pure [2]. M is called a Von Neumann regular module if every cyclic submodule of M is a direct summand of M, [4].
This paper is structured in two sections. In section one we give a comprehensive study of J-pure submodules. Some results are analogous to the properties of pure submodules. In section two, we study the concept of J-regular modules.

Recall that an R- module M is called F- regular if for each submodule of M is pure. (Equivalently) an R-module M is said to be F- regular R- module if for each $m \in M, r \in R$, there exist $t \in R$ such that $r\,m = r\,t\,r\,m$. [5]. The intersection of all maximal submodules of M denoted by J (M) is called the Jacobson radical of M [4]. Recall that, a ring R is said to be a good ring if J(R).M = J (M); M be an R- module. (Equivalently), if R is a good ring, then J (M) ∩ N =

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J (N), [4.p.234].
Recall that the annihilator of an element x of an R-module M denoted by ann(x) is defined to be ann(x)={r∈R: rx= 0} and the annihilator of M denoted by ann(M) is defined to be ann(M) = {r ∈ R: rx = 0 for every x ∈ M}. Clearly ann(x) and ann(M) are ideals of R, [3]. An R -module M is simple if 0 and M are the only submodules of M, and an R-module M is said to be semisimple if M is a sum of simple modules (may be infinite). A ring R is semisimple if it is semisimple as an R-module [4]. It is known that over any ring R, a semisimple R-module is F-regular [6, 7].
It is clear that every F- regular module is J-regular, but the converse is not true (see Remarks and Examples (3.3) (1).

2-Properties of J-pure submodules

In this section we introduce the concept of J-pure submodule. Also we investigate the basic properties of this type of submodules some of these properties are analogous to the properties of pure submodules.

Definition (2.1):

A submodule N of an R-module M is called a **J-pure** if N is pure in J (M), i.e. for each ideal I of R, **I** J (M) ∩ N = IN, where J (M) is the Jacobson radical of M**.**

Remarks and Examples (2.2):

(1) It is clear that every pure submodule is J-pure but the converse is not true in general. For example the submodule $\{\bar{0}, \bar{2}\}$ of the module Z_4 as Z-module is J-pure submodule since if I = 2Z is an ideal of Z, then I$\{\bar{0}, \bar{2}\}$ ∩ $\{\bar{0}, \bar{2}\}$ = 2$\{\bar{0}, \bar{2}\}$ ∩ $\{\bar{0}, \bar{2}\}$ = $\{\bar{0}\}$.
On the other hand, I $\{\bar{0}, \bar{2}\}$ = 2$\{\bar{0}, \bar{2}\}$ = $\{\bar{0}\}$. By the similar simple calculation one can easily show that I J(Z_4) ∩ $\{\bar{0}, \bar{2}\}$ = I$\{\bar{0}, \bar{2}\}$ for every ideal I = nZ of Z where n is any positive integer. Thus $\{\bar{0}, \bar{2}\}$ is a **J-pure** submodule of Z_4 but is not pure since if I = 2Z, then IZ_4 ∩ $\{\bar{0}, \bar{2}\}$ = 2Z_4 ∩ $\{\bar{0}, \bar{2}\}$ = $\{\bar{0}, \bar{2}\}$ but I$\{\bar{0}, \bar{2}\}$ = 2$\{\bar{0}, \bar{2}\}$ = $\{\bar{0}\}$.
(2) Every direct summand of an R- module M is J-pure, since every direct summand of M is a pure submodule in M, [2] hence by remark (1) is J-pure submodule. But, the converse is not true, for example the submodule $\{\bar{0}, \bar{3}, \bar{6}\}$ of the module Z_9 as Z-module. It is easily to check that IJ(Z_9) ∩ $\{\bar{0}, \bar{3}, \bar{6}\}$ = I $\{\bar{0}, \bar{3}, \bar{6}\}$ for each ideal I of Z. So, $\{\bar{0}, \bar{3}, \bar{6}\}$ is J -pure in Z_9 but not direct summand.
(3) Every nonzero cyclic submodule of the module Q as Z-module is not J-pure submodule.

Proof:

Let N be a cyclic submodule of Q as Z-module, generated by an element $\frac{a}{b}$ where a and b are two nonzero elements in Z. If we take an ideal <n> of Z where n is greater than one, then <n> $\frac{a}{b}$= <$\frac{na}{b}$>. Also Q = <n> Q, because for any element $\frac{x}{y}$ ∈ Q we have $\frac{x}{y} = \frac{x}{ny} n$ ∈ <n> Q, thus Q = $< n >$Q. Therefore <n> J (Q) ∩ <$\frac{a}{b}$> = <$\frac{a}{b}$> implies that $< n >$ Q ∩ <$\frac{a}{b}$> $\neq$ $< n >$<$\frac{a}{b}$>.
(4) If N_1 and N_2 are J-**pure** submodule of an R-module M, then $N_1 \cap N_2$ is not necessarily **J-pure**. For example: Let M = $Z_8 \oplus Z_2$ as a Z-module, and let $N_1 = Z_8 \oplus 0$ and N_2 = Z (2, 1).It is easily seen that N_1 and N_2 are J-pure. But $N_1 \cap N_2$ = {(0, 0), (4, 0)} is not J-pure.
(5) The sum of two J-pure submodules may not be J-pure. To show this consider M = $Z_4 \oplus Z_2$ as a Z-module, and let A= Z(2,1) and B = Z(2,0). It is easily seen that A is **pure**, hence it is J-**pure** and B is J-pure. But A + B = {(0,0),(2,0),(2,1),(0,1)} is not J-pure.

Remark (2.3):

Let M be an R-module and N be a **J-pure** submodule of M. If B is a J-**pure** submodule of N, then B is a J-**pure** submodule of M.

Proof:

Let I be an ideal of R. Since N is a **J-pure** sub module in M and B is a **J-pure** submodule in N, then I J (M) ∩ N = IN and I J (N)∩ $B = IB$.
Now
I J (M) ∩ B ⊆ I J (M) ∩ $N = IN$, implies that I J (M) ∩ B ⊆ IN, then
I J (M) ∩ B = (IJ(M) ∩ B) ∩ IN

= (I J(M) $\cap IN$) ∩ B .
= I (J(M) ∩ N) ∩ B [since N is **J-pure** in M].
Hence, I J (M) ∩ B = IN∩ B = IB and IB ⊆ IN∩ B ⊆ I J (M) ∩ B.

Proposition (2.4):
Let R be a good ring. Suppose that M be an R-module and N is a J-pure submodule of M. If B is a J-pure submodule of M containing N, then N is a J-pure submodule of B.

Proof:
Let I be an ideal of R. Since N is **J-pure** submodule in M, hence I J (M) ∩ $N = IN$, now
I J (B) ∩ N ⊆ I J (M) ∩ N = IN, implies that I J (B) ∩ $N \subseteq IN$. Since N is **J-pure** submodule in M, then N ⊆ J(M) and N ⊆ B implies that N ⊆ J(M) ∩ B. Since R is a good ring, then J (M) ∩ B = J (B). [4]. Hence I J (B) ∩ N = IN.

Proposition (2.5):
Let Me be an R-module and N is a **J-pure** submodule of M. If K is small a submodule of N, then $\frac{N}{K}$ is a J-pure submodule in $\frac{M}{K}$.

Proof:
Let I be an ideal of R. Since N is a J-pure submodule of M, then I J (M) ∩ $N = IN$
I J $(\frac{M}{K}) \cap \frac{N}{K} = \frac{I\,J(M)+K}{K} \cap \frac{N}{K}$ [since J $(\frac{M}{N}) = \frac{J(M)}{N}$]

$$= \frac{(IJ(M)+K)\cap N}{K}$$

I J $(\frac{M}{K}) \cap \frac{N}{K} = \frac{(I\,J(M)\cap N)+(K\cap N)}{K}$ [by Modular law]
$= \frac{IN+K}{K} = I(\frac{N}{K})$.

Proposition (2.6):
If N_1 is a **J-pure** submodule of M_1 and N_2 is a **J-pure** submodule of M_2, then $N_1 \oplus N_2$ is J-pure submodule in $M_1 \oplus M_2$.

Proof:
Let M = $M_1 \oplus M_2$ be an R-module, let I be an ideal of R. We have to show I J (M) ∩ $(N_1 \oplus N_2)$ = I $(N_1 \oplus N_2)$. Let $x \in$ I J(M) ∩ $(N_1 \oplus N_2)$, then $x = \sum_{i=1}^{n} r_i(a_i, b_i) = (x_1, x_2)$, where $a_i \in J(M_1)$, $b_i \in J(M_2)$ and $x_1 \in N_1$, $x_2 \in N_2$, so $(\sum_{i=1}^{n} r_i a_i, \sum_{i=1}^{n} r_i b_i) = (x_1, x_2)$ then $\sum_{i=1}^{n} r_i a_i = x_1 \in IN_1$, since N_1 is J-pure submodule of M_1 then $\sum_{i=1}^{n} r_i b_i = x_2 \in IN_2$, since N_1 is J-pure submodule of M_2 then $(\sum_{i=1}^{n} r_i a_i, \sum_{i=1}^{n} r_i b_i) \in IN_1 \oplus IN_2 = I\ (N_1 \oplus N_2)$. Hence $\sum_{i=1}^{n} r_i(a_i, b_i) \in I\ (N_1 \oplus N_2)$. Thus $(N_1 \oplus N_2)$ is J- pure submodule of $M_1 \oplus M_2$.

3- Basic Results for J-regular modules

In this section, we introduce and study the class of **J-regular** modules.

Definition (3.1):
An R-module M is said to be **J-regular** module if for each $m \in J(M)$, $r \in R$, there exists $t \in R$ such that $r\,m = r\,t\,r\,m$.

Proposition (3.2):
An R- module M is J-regular if and only if every submodule of J (M) is pure.

Proof:
Suppose that M is a **J-regular** R-module and let N be any submodule of J (M). For each $r \in R$, let $x \in$ I J (M) ∩ N, then there exists y ∈J (M) such that $x = ry$. Since M is **J-regular**, then there exists t ∈ R such $ry = rtry$. Put e = tr, then $ry = ery$ which implies that $x = ex$, but $x \in$ N, so $x = ex \in$ IN and hence I J (M) ∩ N ⊆ IN. On the other hand, it is clear that IN ⊆ I J (M) ∩ N, thus I J (M) ∩ N = IN. Thus, N is a pure submodule of J (M).
Conversely, assume that every submodule of J (M) is pure. Let $x \in$J (M) and r ∈ R such that Rrx = N which is a **J-pure** submodule of M, then I J (M) ∩ N=IN. For each I ∈ R. In particular, if I = <r> we get $rx \in$ I J (M) ∩ N ⊆ IN = $rRrx$. Therefor there exists t ∈ R such that $r\ t = x$, so M is J-regular R- module.

Remarks and Examples (3.3):

(1) It is clear that every F-regular module is **J-regular**, but the converse may not be true in general for example, the module Z_4 as Z-module is J-regular since every submodule of Z_4 is **J-pure** submodule in Z_4, but Z_4 is not F-regular since the submodule $\{\bar{0}, \bar{2}\}$ of Z_4 is not pure, see remark and example (2.2)(1).
(2) The module Q as Z-modules are not J-regular modules, see remarks and examples (2.2) (3).
(3) The module Z_9 as Z-module is J-regular since every submodule of Z_9 is J-pure, but Z_9 is not regular since the submodule $\{\bar{0}, \bar{3}, \bar{6}\}$ is not pure, see remarks and examples (2.2) (2).
(4) It is clear that if N_1 and N_2 are two J-regular submodules of an R-module M, then $N_1 \cap N_2$ is J-regular submodules in M.
(5) It is not necessarily that if every submodule of an R-module M is J- regular implies M is J- regular. For example: the module Z_8 as Z-module is not J-regular. We know that $\langle\bar{4}\,\rangle$ 4 is not J-pure submodule of Z_8 because 2.J $(Z_8) \cap \langle\bar{4}\,\rangle =$ $\langle\bar{4}\,\rangle$ while 2. $\langle\bar{4}\,\rangle$ =$\langle\bar{0}\,\rangle$, implies 2.J $(Z_8) \cap \langle\bar{4}\,\rangle \neq$ 2. $\langle\bar{4}\,\rangle$. While every proper submodule of Z_8 is J-regular, since $\langle\bar{2}\,\rangle \cong Z_4$ and $\langle\bar{4}\,\rangle \cong Z_2$ are J-regular modules.
(6) It is clear that**,** if every submodule N of an R-module M is **J- regular** with J(M) = J(N), then M is **J- regular.**
(7) If J (M) $= 0$, then M is J- regular R-module. For example: In Z as Z- module, J (M) $= 0$, hence Z is J- regular R-module, but not regular.
(8) Every submodule N of J- regular R-module M such; that J (N) is J- pure in M is J- regular.
Proof:
Let K be a submodule in N and I be an ideal of R. To show that K is J- pure in N, we have:
IJ(N) ∩ K = (IJ(M) ∩ N) ∩ K [since J(N) is **J- pure** in M].
= IJ (M) ∩ (N ∩ K) = IJ (M) ∩ K = IK [since K is **J- pure** in M].
Therefore, K is **J- pure** in N implies N is **J- regular**.
The following theorem shows that the cyclic J-pure submodules is enough to make the module be **J-regular**.
Theorem (3.4):
Let Me be an R-module. The following statements are equivalent:
(1) M is **J-regular** module.
(2) Every cyclic submodule of M is J-pure submodule of M.
(3) Every finitely generated sub module of M is **J-pure** submodule.
(4) Every submodule of M is **a J-pure** submodule of M.
Proof:
(1) ⇒ (2) it follows by definition (2.1).
(2) ⇒ (1) Assume that every cyclic submodule of M is **J-pure**.
Let N be a submodule of J(M) and I be an ideal of R. Let $x \in$ IJ (M) ∩ N, implies that $x \in$ IJ (M) and $x \in$ N. Therefore ∈ IJ (M) ∩ <x> = I <x> ⊆ IN.
(1) ⇒ (3) It follows by definition (2.1), and the proof of (2) ⇒ (1).
(3) ⇒ (2) It is clear.
(1) ⇔ (4) It follows by proposition (2.2).
Proposition (3.5):
Let M and M′ be R- modules; and f: M → M′ be an R − epimorphism, ker f is small of M. If M is **J-regular** module, then M′ is J**-regular**.
Proof:
Let f: M → M′ be two R − epimorphism and Ker f is small of M. To show that M′ is **J-regular**. Let y∈ J(M′), since f: M ⟶ M′ be an R − epimorphism and Ker f is small of M, then f(J(M))= J(M′) [4] implies that y ∈ f(J(M), then there exist $x \in$ J (M) such that $f(x) = y$. Since M is J - regular and $x \in$ J (M), then there exist $t \in R$ suth that $rx = rtrx$ so

$$f(rx) = f(rtrx)$$
$$rf(x) = rtrf(x)$$
$$ry = rtry$$

Hence, M′ is J-regular.
If M is F-regular R-module then J (R).M = 0, but if M is **J- regular** we have the following:
Proposition (3.6):

If M is **J-regular** R-module, then J (R).J (M) = 0.

Proof:
For each $0 \neq x \in$ J (M) and for each $0 \neq r \in$ J (R), there exist t∈ R such that $r\,t\,r\,x = r\,x$, then $r\,x\,(r\,t - 1) = 0$. Since r ∈ J (R), then $(r\,t - 1)$ is invertible, hence, $r\,x = 0$ which implies that J (R). J (M) = 0.

Lemma (3.7):
Let $0 \neq x \in$ J (M). Then $\frac{R}{\mathrm{ann}(x)}$ **regular ring** if and only if for each r ∈ R, there exist t ∈ R such that r x = r t r x.

Proof:
$\Longrightarrow$) Let $0 \neq x \in$ J (M), let r ∈ R, $\bar{r} \in \frac{R}{\mathrm{ann}(x)}$ since $\frac{R}{\mathrm{ann}(x)}$ is **regular ring**. Then there exist $\bar{t} \in \frac{R}{\mathrm{ann}(x)}$, such that $\bar{r}x = \bar{r}\bar{t}\bar{r}$ implies that r – r t r ∈ ann (x), then r x = r t r x.
($\Longleftarrow$ Let $0 \neq x \in$ J (M), let r ∈ R, since r x = r t r x for some t ∈ R, implies that r – r t r ∈ ann(x). Thus $\bar{r}x = \bar{r}\bar{t}\bar{r}$, then $\frac{R}{\mathrm{ann}(x)}$ is **regular ring**.

Proposition (3.8):
Let M be a **J- regular** R- module. Then $\frac{R}{\mathrm{ann}(x)}$ is **regular ring** for each $x \in$ J (M).

Proof:
Let $x \in$ J (M), r ∈ R. Since M is **J- regular** R- module, then there exist t∈ R such that $r\ t\ r\ x = rx$, then r- $r\ t\ r \in$ ann(x). Thus $\bar{r}x = \bar{r}\bar{t}\bar{r}$, which implies that $\frac{R}{\mathrm{ann}(x)}$ is **regular ring**.

Theorem (3.9):
For any R-module M, The following statements are equivalent:
(1) M is a **J- regular** R-module.
(2) Every cyclic submodule of M is **J- regular**.
(3) For every $0 \neq x \in$ J (M), $\frac{R}{\mathrm{ann}(x)}$ **regular ring**.

Proof:
(1) $\Longrightarrow$ **(2)** It follows by Theorem (3.4).
(2) $\Longrightarrow$ **(3)** Let $0 \neq x \in$ J (M), and r ∈ R, then P = <r x > the submodule generated by r x. By (2), P is J- pure, then there exist y = r x has a solution in P, i.e. There exist z ∈ P such that y = r x = r z, z ∈ P implies that z = t r x; t ∈ R , hence r x = r t r x. By lemma (3.7), $\frac{R}{\mathrm{ann}(x)}$ regular ring.
(3) $\Longrightarrow$ **(1)** Let N be any submodule of M, and I an ideal in R. Let $x \in$ N ∩ IJ (M), $x = \sum_{i=1}^{n} r_i x_i$, $r_i \in$ I, $x_i \in$ J (M). By lemma (3.7), for each I, there exist t ∈ R such that $r_i x_i = r_i t_i r_i x_i$. If we put $e_i = t_i r_i$, and $e = 1 - \prod_{i=1}^{n}(1 - e_i)$, then it can easily be seen that e∈ I, ${e^2}_i x_i = e_i x_i$, $r_i x_i = e_i r_i x_i$ and $e\,e_i x_i = e_i x_i$. Thus $e\ x = \sum_{i=1}^{n} e_i r_i x_i = \sum_{i=1}^{n} e e_i r_i x_i = \sum_{i=1}^{n} r_i x_i = x$. Hence $x \in$ IN and N ∩ IJ(M) ⊆ IN. Thus, N is J- pure.

Proposition (3.10):
Let M be an R-module. If $\frac{R}{\mathrm{ann}(J(M))}$ is a regular ring, then M is **J- regular**.

Proof:
Let $0 \neq x \in$ J (M), since ann(J(M)) ⊆ ann(x), for each $x \in$ J(M),so there exist an R − epimorphism φ: $\frac{R}{\mathrm{ann}(J(M))} \longrightarrow \frac{R}{\mathrm{ann}(x)}$ defined by φ (r + ann(J(M)) = r + $ann(x)$. Since $\frac{R}{\mathrm{ann}(J(M))}$ is regular ring, then $\frac{R}{\mathrm{ann}(x)}$ is a regular ring, then by Theorem (3.9). Therefor M is **J- regular**.
Consequently, it is J-regular. Furthermore, it is known that over a local ring, every F-regular module is semisimple [8]. We can generalize the latest statement as the following.

Proposition (3.11):
If M is a **J- regular** module over a local ring, then J (M) is a semisimple R- module.

Proof:

Let I be the only maximal ideal of R. Since M is J-regular, then for each $0 \neq x \in$ J (M) we have that $\mathrm{R}/\mathrm{ann}(x)$ is J-regular local ring, which implies that $\mathrm{R}/\mathrm{ann}(x)$ is a field [9] hence, $\mathrm{ann}(x)$ is a maximal ideal, so $\mathrm{I} = \mathrm{ann}(x)$ for each $0 \neq x \in$ J (M). Therefore, $\mathrm{I} = \mathrm{ann}(x)$ = ann (J (M)).
On the other hand, R/ I $\simeq \mathrm{R}/\mathrm{ann}$ (J (M)) is a field, which implies that J (M) is a vector space over the field R/ann (J (M)) is simple ring. Then J (M) is a semisimple R-module over the ring R/ann (J (M)). Thus, J (M) is a semisimple R-module. [4].

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