Properties of $J$– Regular modules

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Abstract
The present study introduces the concept of $J$-pure submodules as a generalization of pure submodules. We study some of its basic properties and by using this concept we define the class of $J$-regular modules, where an $R$-module $M$ is called $J$-regular module if every submodule of $M$ is $J$-pure submodule. Many results about this concept are proved.

Keywords: pure submodules, $J$-pure submodules, regular modules and $J$-regular modules.

1- Introduction
Throughout this paper, all rings are a commutative with identity and every $R$-module is a unitary. The notion pure submodule is well known and there are several authors deal with this concept. For example [1] and [2]. Let $M$ be an $R$-module. A submodule $N$ of $M$ is called pure if the sequence $0 \rightarrow E \otimes N \rightarrow E \otimes M$ is exact for every $R$-module $E$. Cohn [1]. Equivalently if for each $\sum_{i=1}^{n} r_{ij}m_{i} \in N, r_{ij} \in R, m_{i} \in M, j = 1,2,\ldots,k$, there exist $x_{i} \in N, i = 1,2,\ldots,n$ such that $\sum_{i=1}^{n} r_{ij}m_{i} = \sum_{i=1}^{n} r_{ij}x_{i}$ for each $j$.
A submodule $N$ of an $R$-module $M$ is called pure in $M$ if $IM \cap N = N$ for every ideal $I$ of $R$ [3]. Recall that an $R$- module $M$ is called regular module if every submodule of $M$ is pure [2]. $M$ is called a Von Neumann regular module if every cyclic submodule of $M$ is a direct summand of $M$, [4].

This paper is structured in two sections. In section one we give a comprehensive study of $J$-pure submodules. Some results are analogous to the properties of pure submodules. In section two, we study the concept of $J$-regular modules.

Recall that an $R$- module $M$ is called F- regular if for each submodule of $M$ is pure. (Equivalently) an $R$-module $M$ is said to be F- regular $R$- module if for each $m \in M, r \in R$, there exist $t \in R$ such that $r m = t r m$ [5]. The intersection of all maximal submodules of $M$ denoted by $J(M)$ is called the Jacobson radical of $M$ [4]. Recall that, a ring $R$ is said to be a good ring if $J(R).M = J(M)$; $M$ be an $R$- module. (Equivalently), if $R$ is a good ring, then $J(M) \cap N =$
J (N), [4,p.234].

Recall that the annihilator of an element \( \chi \) of an R-module \( M \) denoted by \( \text{ann}(\chi) \) is defined to be \( \text{ann}(\chi) = \{ r \in R : r\chi = 0 \} \) and the annihilator of \( M \) denoted by \( \text{ann}(M) \) is defined to be \( \text{ann}(M) = \{ r \in R : r\chi = 0 \text{ for every } \chi \in M \} \). Clearly \( \text{ann}(\chi) \) and \( \text{ann}(M) \) are ideals of \( R \), [3]. An R-module \( M \) is simple if \( 0 \) and \( M \) are the only submodules of \( M \), and an R-module \( M \) is said to be semisimple if \( M \) is a sum of simple modules (may be infinite). A ring \( R \) is semisimple if it is semisimple as an R-module [4]. It is known that over any ring \( R \), a semisimple R-module is \( F \)-regular [6,7].

It is clear that every \( F \)-regular module is \( J \)-regular, but the converse is not true (see Remarks and Examples (3.3) (1)).

2-Properties of J-pure submodules

In this section we introduce the concept of J-pure submodule. Also we investigate the basic properties of this type of submodules some of these properties are analogous to the properties of pure submodules.

**Definition (2.1):**

A submodule \( N \) of an \( R \)-module \( M \) is called a **J-pure** if \( N \) is pure in \( J(M) \), i.e. for each ideal \( I \) of \( R \), \( IJ(M) \cap N = IN \), where \( J(M) \) is the Jacobson radical of \( M \).

**Remarks and Examples (2.2):**

(1) It is clear that every pure submodule is J-pure but the converse is not true in general. For example the submodule \( \{0, 2\} \) of the module \( Z_4 \) as \( Z \)-module is J-pure submodule since if \( I = 2Z \) is an ideal of \( Z \), then \( I \{0, 2\} \cap \{0, 2\} = 2\{0, 2\} \cap \{0, 2\} = \{0\} \).

On the other hand, \( I \{0, 2\} = 2\{0, 2\} = \{0\} \). By the similar simple calculation one can easily show that \( IJ(Z_4) \cap \{0, 2\} = \{0, 2\} \) for every ideal \( I = nZ \) where \( n \) is any positive integer.

Thus \( \{0, 2\} \) is a **J-pure** submodule of \( Z_4 \) but is not pure since if \( I = 2Z \), then \( IZ_4 \cap \{0, 2\} = 2Z \cap \{0, 2\} = \{0, 2\} \) but \( I \{0, 2\} = 2\{0, 2\} = \{0\} \).

(2) Every direct summand of an \( R \)-module \( M \) is J-pure, since every direct summand of \( M \) is a pure submodule in \( M \), [2] hence by remark (1) is J-pure submodule. But, the converse is not true, for example the submodule \( \{0, 3, 6\} \) of the module \( Z_9 \) as \( Z \)-module. It is easily to check that \( IJ(Z_9) \cap \{0, 3, 6\} = I\{0, 3, 6\} \) for each ideal \( I \) of \( Z \). So, \( \{0, 3, 6\} \) is J-pure in \( Z_9 \) but not direct summand.

(3) Every nonzero cyclic submodule of the module \( Q \) as \( Z \)-module is not J-pure submodule.

**Proof:**

Let \( N \) be a cyclic submodule of \( Q \) as \( Z \)-module, generated by an element \( \frac{a}{b} \) where \( a \) and \( b \) are two nonzero elements in \( Z \). If we take an ideal \( \langle n \rangle \) of \( Z \) where \( n \) is greater than one, then \( \langle n \rangle = \langle \frac{a}{b} \rangle \). Also \( Q = \langle n \rangle Q \), because for any element \( \frac{x}{y} \in Q \) we have \( \frac{x}{y} = \frac{x}{ny} n \in \langle n \rangle Q \), thus \( Q = \langle n > Q \). Therefore \( \langle n > Q \cap \langle \frac{a}{b} \rangle = \langle \frac{a}{b} \rangle \) implies that \( Q \cap \langle \frac{a}{b} \rangle = \langle \frac{a}{b} \rangle > Q \). Hence \( Q \cap \langle \frac{a}{b} \rangle = \langle \frac{a}{b} \rangle \).

(4) If \( N_1 \) and \( N_2 \) are J-pure submodule of an \( R \)-module \( M \), then \( N_1 \cap N_2 \) is not necessarily J-pure. For example: Let \( M = Z_8 \oplus Z_2 \) as \( Z \)-module, and let \( N_1 = Z_8 \oplus 0 \) and \( N_2 = Z \langle 2, 1 \rangle \). It is easily seen that \( N_1 \) and \( N_2 \) are J-pure. But \( N_1 \cap N_2 = \{(0, 0), (4, 0)\} \) is not J-pure.

(5) The sum of two J-pure submodules may not be J-pure. To show this consider \( M = Z_4 \oplus Z_2 \) as a \( Z \)-module, and let \( A = Z\langle 2,1 \rangle \) and \( B = Z\langle 2,0 \rangle \). It is easily seen that \( A \) is pure, hence it is J-pure and \( B \) is J-pure. But \( A + B = \{(0,0),(2,0),(2,1),(0,1)\} \) is not J-pure.

**Remark (2.3):**

Let \( M \) be an \( R \)-module and \( N \) be a J-pure submodule of \( M \). If \( B \) is a J-pure submodule of \( N \), then \( B \) is a J-pure submodule of \( M \).

**Proof:**

Let \( I \) be an ideal of \( R \). Since \( N \) is a J-pure sub module in \( M \) and \( B \) is a J-pure submodule in \( N \), then \( IJ(M) \cap N = IN \) and \( IJ(N) \cap B = IB \).

Now \( IJ(M) \cap B \subseteq IJ(M) \cap N = IN \), implies that \( IJ(M) \cap B \subseteq IN \), then \( IJ(M) \cap B = (IJ(M) \cap B) \cap IN \).
= (I (J(M) \cap IN) \cap B).
= I ((J(M) \cap N) \cap B) \ [\text{since } N \text{ is J-pure in } M].
Hence, I (J(M) \cap B) = IN \cap B = IB and IB \subseteq IN \cap B \subseteq I (J(M) \cap B).

**Proposition (2.4):**
Let R be a good ring. Suppose that M be an R-module and N is a J-pure submodule of M. If B is a J-pure submodule of M containing N, then N is a J-pure submodule of B.

**Proof:**
Let I be an ideal of R. Since N is J-pure submodule in M, hence I (J(M) \cap N) = IN, now I (J(B) \cap N) \subseteq I (J(M) \cap N) = IN, implies that I (J(B) \cap N) \subseteq IN. Since N is J-pure submodule in M, then N \subseteq J(M) and N \subseteq B implies that N \subseteq J(M) \cap B. Since R is a good ring, then J(M) \cap B = J(B).

**Proposition (2.5):**
Let Me be an R-module and N is a J-pure submodule of M. If K is small a submodule of N, then \( \frac{N}{K} \) is a J-pure submodule in \( M_K \).

**Proof:**
Let I be an ideal of R. Since N is a J-pure submodule of M, then I (J(M) \cap N) = IN

\[
I J M K = \frac{I J (M) + K \cap N}{K} = \frac{I J (M) \cap N}{K} = \frac{(I J (M) + N) \cap K}{K} \ [\text{by Modular law}]
\]

**Proposition (2.6):**
If \( N_1 \) is a J-pure submodule of \( M_1 \) and \( N_2 \) is a J-pure submodule of \( M_2 \), then \( N_1 \oplus N_2 \) is J-pure submodule in \( M_1 \oplus M_2 \).

**Proof:**
Let \( M = M_1 \oplus M_2 \) be an R-module, let I be an ideal of R. We have to show I (J(M) \cap (N_1 \oplus N_2)) = I (N_1 \oplus N_2). Let \( x \in I (J(M) \cap (N_1 \oplus N_2)) \), then \( x = \sum_{i=1}^{n} r_i (a_i, b_i) = (x_1, x_2) \), where \( a_i \in J(M_1), b_i \in J(M_2) \) and \( x_1 \in N_1, x_2 \in N_2 \), so \( \sum_{i=1}^{n} r_i a_i, \sum_{i=1}^{n} r_i b_i \) = \( (x_1, x_2) \) then \( \sum_{i=1}^{n} r_i a_i = x_1 \in IN_1 \), since \( N_1 \) is J-pure submodule of \( M_1 \) then \( \sum_{i=1}^{n} r_i b_i = x_2 \in IN_2 \), since \( N_1 \) is J-pure submodule of \( M_2 \) then \( \sum_{i=1}^{n} r_i a_i, \sum_{i=1}^{n} r_i b_i \) \( \in N_1 \oplus N_2 = I (N_1 \oplus N_2) \). Hence \( \sum_{i=1}^{n} r_i (a_i, b_i) \in I (N_1 \oplus N_2) \). Thus \( (N_1 \oplus N_2) \) is J-pure submodule of \( M_1 \oplus M_2 \).

**3- Basic Results for J-regular modules**

In this section, we introduce and study the class of J-regular modules.

**Definition (3.1):**
An R-module M is said to be J-regular module if for each \( m \in J(M), \ r \in R \), there exists \( t \in R \) such that \( rm = trm \).

**Proposition (3.2):**
An R-module M is J-regular if and only if every submodule of J(M) is pure.

**Proof:**
Suppose that M is a J-regular R-module and let N be any submodule of J(M). For each \( r \in R \), let \( x \in I J(M) \cap N \), then there exists \( y \in J(M) \) such that \( x = ry \). Since M is J-regular, then there exists \( t \in R \) such \( ry = rtry \). Put \( e = tr \), then \( ry = ery \) which implies that \( x = ex \), but \( x \in N \), so \( x = ex \in IN \) and hence \( I J(M) \cap N \subseteq IN \). On the other hand, it is clear that \( IN \subseteq I J(M) \cap N \), thus \( I J(M) \cap N = IN \). Thus, N is a pure submodule of J(M).

Conversely, assume that every submodule of J(M) is pure. Let \( x \in J(M) \) and \( r \in R \) such that \( Rrx = N \) which is a J-pure submodule of M, then \( I J(M) \cap N = IN \). For each \( t \in R \). In particular, if \( I = \langle r \rangle \) we get \( rtx \in I J(M) \cap N \subseteq IN = rRrx \). Therefor there exists \( t \in R \) such that \( rt = x \), so M is J-regular R-module.

**Remarks and Examples (3.3):**
(1) It is clear that every F-regular module is \textbf{J-regular}, but the converse may not be true in general for example, the module \( Z_4 \) as \( Z \)-module is J-regular since every submodule of \( Z_4 \) is \textbf{J-pure} submodule in \( Z_4 \), but \( Z_4 \) is not F-regular since the submodule \( \{0, 2\} \) of \( Z_4 \) is not pure, see remark and example (2.2) (1).

(2) The module \( Q \) as \( Z \)-modules are not J-regular modules, see remarks and examples (2.2) (3).

(3) The module \( Z_9 \) as \( Z \)-module is J-regular since every submodule of \( Z_9 \) is J-pure, but \( Z_9 \) is not regular since the submodule \( \{0, 3, 6\} \) is not pure, see remarks and examples (2.2) (2).

(4) It is clear that if \( N_1 \) and \( N_2 \) are two J-regular submodules of an \( R \)-module \( M \), then \( N_1 \cap N_2 \) is J-regular submodules in \( M \).

(5) It is not necessarily that if every submodule of an \( R \)-module \( M \) is J-regular implies \( M \) is J-regular. For example: the module \( Z_9 \) as \( Z \)-module is not J-regular. We know that \( \langle 4 \rangle \) is not J-pure submodule of \( Z_9 \) because \( 2.J \langle Z_9 \rangle \cap \langle 4 \rangle = \langle 4 \rangle \) while \( 2.J \langle Z_9 \rangle \cap \langle 4 \rangle \neq 2.J \langle 4 \rangle \). While every proper submodule of \( Z_9 \) is J-regular, since \( \langle 2 \rangle \cong Z_4 \) and \( \langle 4 \rangle \cong Z_2 \) are J-regular modules.

(6) It is clear that, if every submodule \( N \) of an \( R \)-module \( M \) is \textbf{J-regular} with \( J(M) = J(N) \), then \( M \) is \textbf{J-regular}.

(7) If \( J(M) = 0 \), then \( M \) is J-regular \( R \)-module. For example: In \( Z \) as \( Z \)-module, \( J(M) = 0 \), hence \( Z \) is J-regular \( R \)-module, but not regular.

(8) Every submodule \( N \) of J-regular \( R \)-module \( M \) such that \( J(N) \) is J-pure \( M \) is J-regular.

**Proof:**

Let \( K \) be a submodule in \( N \) and \( I \) be an ideal of \( R \). To show that \( K \) is J-pure in \( N \), we have:

\[ IJ(N) \cap K = (IJ(M) \cap N) \cap K \quad \text{[since \( J(N) \) is \textbf{J-pure} in \( M \).]}
\]

\[ = IJ(M) \cap (N \cap K) = IJ(M) \cap K = IK \quad \text{[since \( K \) is \textbf{J-pure} in \( M \).]}
\]

Therefore, \( K \) is \textbf{J-pure} in \( N \) implies \( N \) is \textbf{J-regular}.

The following theorem shows that the cyclic J-pure submodules is enough to make the module be J-regular.

**Theorem (3.4):**

Let \( M \) be an \( R \)-module. The following statements are equivalent:

(1) \( M \) is J-regular module.

(2) Every cyclic submodule of \( M \) is J-pure submodule of \( M \).

(3) Every finitely generated sub module of \( M \) is J-pure submodule.

(4) Every submodule of \( M \) is a J-pure submodule of \( M \).

**Proof:**

(1) \( \Rightarrow \) (2) it follows by definition (2.1).

(2) \( \Rightarrow \) (1) Assume that every cyclic submodule of \( M \) is J-pure.

Let \( N \) be a submodule of \( J(M) \) and \( I \) be an ideal of \( R \). Let \( x \in IJ(M) \cap N \), implies that \( x \in IJ(M) \) and \( x \in N \). Therefore \( IJ(M) \cap \langle x \rangle = I \langle x \rangle \subseteq IN \).

(1) \( \Rightarrow \) (3) It follows by definition (2.1), and the proof of (2) \( \Rightarrow \) (1).

(3) \( \Rightarrow \) (2) It is clear.

(1) \( \Leftrightarrow \) (4) It follows by proposition (2.2).

**Proposition (3.5):**

Let \( M \) and \( M' \) be \( R \)-modules; and \( f: M \to M' \) be an \( R \) - epimorphism, \( \ker f \) is small of \( M \). If \( M \) is J-regular module, then \( M' \) is J-regular.

**Proof:**

Let \( f: M \to M' \) be two \( R \) - epimorphism and \( \ker f \) is small of \( M \). To show that \( M' \) is J-regular. Let \( y \in J(M') \), since \( f: M \to M' \) be an \( R \) - epimorphism and \( \ker f \) is small of \( M \), then \( f(J(M)) = J(M') \) [4] implies that \( y \in f(J(M)) \), then there exist \( x \in J(M) \) such that \( f(x) = y \). Since \( M \) is J-regular and \( x \in J(M) \), then there exist \( t \in R \) such that \( rx = rtx \) so

\[ f(rx) = f(rtx) \]
\[ rf(x) = rtf(x) \]
\[ ry = rty \]

Hence, \( M' \) is J-regular.

If \( M \) is F-regular \( R \)-module then \( J(R) = 0 \), but if \( M \) is \textbf{J-regular} we have the following:

**Proposition (3.6):**
If $M$ is $J$-regular $R$-module, then $J(R)M = 0$.

**Proof:**
For each $0 \neq x \in J(M)$ and for each $0 \neq r \in J(R)$, there exist $t \in R$ such that $r \cdot t \cdot r \cdot x = r \cdot x$, then $r \cdot x \cdot (r \cdot t - 1) = 0$. Since $r \in J(R)$, then $(r \cdot t - 1)$ is invertible, hence, $r \cdot x = 0$ which implies that $J(R)M = 0$.

**Lemma (3.7):**
Let $0 \neq x \in J(M)$. Then $\frac{R}{\text{ann}(x)}$ is regular ring if and only if for each $r \in R$, there exist $t \in R$ such that $r \cdot x = r \cdot t \cdot r \cdot x$.

**Proof:**
$\Rightarrow$) Let $0 \neq x \in J(M)$, let $r \in R$, $\bar{r} \in \frac{R}{\text{ann}(x)}$ since $\frac{R}{\text{ann}(x)}$ is regular ring. Then there exist $\bar{t} \in \frac{R}{\text{ann}(x)}$ such that $\bar{r} \cdot x = \bar{r} \cdot \bar{t} \cdot \bar{r}$ implies that $r \cdot t \cdot r \in \text{ann}(x)$, then $r \cdot x = r \cdot t \cdot r \cdot x$.

$\Leftarrow$) Let $0 \neq x \in J(M)$, let $r \in R$, since $r \cdot x = r \cdot t \cdot r \cdot x$ for some $t \in R$, implies that $r \cdot t \cdot r \in \text{ann}(x)$. Thus $\bar{r} \cdot x = \bar{r} \cdot \bar{t} \cdot \bar{r}$, then $\frac{R}{\text{ann}(x)}$ is regular ring.

**Proposition (3.8):**
Let $M$ be a $J$-regular $R$-module. Then $\frac{R}{\text{ann}(x)}$ is regular ring for each $x \in J(M)$.

**Proof:**
Let $x \in J(M)$, $r \in R$. Since $M$ is $J$-regular $R$-module, then there exist $t \in R$ such that $r \cdot t \cdot r \cdot x = r \cdot x$, then $r \cdot t \cdot r \in \text{ann}(x)$. Thus $\bar{r} \cdot x = \bar{r} \cdot \bar{t} \cdot \bar{r}$, which implies that $\frac{R}{\text{ann}(x)}$ is regular ring.

**Theorem (3.9):**
For any $R$-module $M$, the following statements are equivalent:
1. $M$ is a $J$-regular $R$-module.
2. Every cyclic submodule of $M$ is $J$-regular.
3. For every $0 \neq x \in J(M)$, $\frac{R}{\text{ann}(x)}$ is regular ring.

**Proof:**
1. $\Rightarrow$ 2. It follows by Theorem (3.4).
2. $\Rightarrow$ 3. Let $0 \neq x \in J(M)$, and $r \in R$, then $P = \langle r \cdot x \rangle$ the submodule generated by $r \cdot x$. By (2), $P$ is $J$-pure, then there exist $y = r \cdot x$ has a solution in $P$, i.e. There exist $z \in P$ such that $y = r \cdot x = r \cdot z$, $z \in P$ implies that $z = t \cdot r \cdot x$; $t \in R$, hence $r \cdot x = r \cdot t \cdot r \cdot x$. By lemma (3.7), $\frac{R}{\text{ann}(x)}$ is regular ring.
3. $\Rightarrow$ 1. Let $N$ be any submodule of $M$, and $I$ an ideal in $R$. Let $x \in N \cap J(M)$, $x = \sum_{i=1}^{n} r_i \cdot x_i$, $r_i \in I$, $x_i \in J(M)$. By lemma (3.7), for each $I$, there exist $t \in R$ such that $r_i \cdot x_i = r_i \cdot t \cdot r_i \cdot x_i$. If we put $e_i = t_i \cdot r_i$, and $e_i = 1 - \prod_{i=1}^{n} (1 - e_i)$, then it can easily be seen that $e_i \cdot x_i = e_i \cdot x_i$, $r_i \cdot x_i = e_i \cdot r_i \cdot x_i$ and $e_i \cdot x_i = e_i \cdot x_i$. Thus $x = \sum_{i=1}^{n} e_i \cdot r_i \cdot x_i = \sum_{i=1}^{n} e_i \cdot r_i \cdot x_i = \sum_{i=1}^{n} r_i \cdot x_i = x$. Hence $x \in N$ and $N \cap J(M) \subseteq N$. Thus, $N$ is $J$-pure.

**Proposition (3.10):**
Let $M$ be an $R$-module. If $\frac{R}{\text{ann}(I(M))}$ is a regular ring, then $M$ is $J$-regular.

**Proof:**
Let $0 \neq x \in J(M)$, since $\text{ann}(I(M)) \subseteq \text{ann}(x)$, for each $x \in J(M)$, so there exist an $R$-epimorphism $\varphi$: $\frac{R}{\text{ann}(I(M))} \to \frac{R}{\text{ann}(x)}$ defined by $\varphi(r + \text{ann}(I(M)) = r + \text{ann}(x))$. Since $\frac{R}{\text{ann}(I(M))}$ is regular ring, then $\frac{R}{\text{ann}(x)}$ is a regular ring, then by Theorem (3.9). Therefore $M$ is $J$-regular.

Consequently, it is $J$-regular. Furthermore, it is known that over a local ring, every $F$-regular module is semisimple [8]. We can generalize the latest statement as the following.

**Proposition (3.11):**
If $M$ is a $J$-regular module over a local ring, then $J(M)$ is a semisimple $R$-module.

**Proof:**
Let I be the only maximal ideal of R. Since M is J-regular, then for each $0 \neq x \in J(M)$ we have that $R/\text{ann}(x)$ is J-regular local ring, which implies that $R/\text{ann}(x)$ is a field [9] hence, $\text{ann}(x)$ is a maximal ideal, so $1 = \text{ann}(x)$ for each $0 \neq x \in J(M)$. Therefore, $1 = \text{ann}(x) = \text{ann}(J(M))$.

On the other hand, $R/I \cong R/\text{ann}(J(M))$ is a field, which implies that $J(M)$ is a vector space over the field $R/\text{ann}(J(M))$ is simple ring. Then $J(M)$ is a semisimple R-module over the ring $R/\text{ann}(J(M))$. Thus, $J(M)$ is a semisimple R-module. [4].

References