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Optimal Harvesting Strategy of a Discretization Fractional-Order Biological Model

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Abstract
Optimal control methods are used to get an optimal policy for harvesting renewable resources. In particular, we investigate a discretization fractional-order biological model, as well as its behavior through its fixed points, is analyzed. We also employ the maximal Pontryagin principle to obtain the optimal solutions. Finally, numerical results confirm our theoretical outcomes.

Keywords: Discrete fractional-order, ratio-dependent prey- predator, optimal strategy

استراتيجية الحصاد الامثل لنموذج بايولوجي كسري متقطع

صادق ال ناصر
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصه
في هذا البحث تم استعمال طرق السيطرة المثلى للحصول على السياسة المثلى لحصاد الموارد المتجددة, على وجه الخصوص تم دراسة وتحليل سلوك نموذج كسري متقطع بايولوجي من خلال نقاط اتزانه. كذلك تم استعمال مبدأPontryagin الأقصى للحصول على الحلول المثلى . أخيرًا ، تؤكد النتائج العددية نتائجنا النظرية.

1-Introduction

Harvesting is an important theme in renewable resources management, so that the dilemma of harvest biological systems have been analyzed and investigated by many researchers to get optimal exploitation polices.

The books of C.W. Clark [1], and Mark Kot [2] are extremely applicable and relevant to the optimal harvesting problems. Rassi and Jerry[3] related to the maximization of the total net gains derived by the harvesting of the resources. They also developed and studied the exploitation policy to the optimal control problems.

The original work of Lotka and Volterra [4, 5] is inception of the prey-predator theory, then after it became the most important subject in mathematical ecology. So that many

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authors have been adapted the work of Lotka-Volterra system employing difference equations, ordinary differential equations or partial differential equations, as well as fractional-order derivative [6-13], and references therein. However, there are many types of functional response , namely Holling type I , II, III, and type IV, Beddington-Deaugelis, Leslie-Gower, Corwley-Marin, and others [8,14,15,16].

Fractional-order derivative provides a precise description of the dynamics of biological or epidemiological models due to in consideration of information about a population memory compared to the other descriptions for that many researchers prefer to model their systems by fractional-order derivative. For more details about the fractional-order derivative we refer to these references [6, 11, 12, 17-19].

A general system of two dimensional prey-predator without harvesting is described by ordinary differential equations is as following:

$$\begin{aligned} \frac{dx(t)}{dt} &= xf(x) - g(x,y)\mathrm{y} \\ \frac{dy(t)}{dt} &= dg(x,y)\mathrm{y} - c(y)y(t) \end{aligned} \tag{1}$$

Here the variables $x(t)$, and $y(t)$ denote to the size of prey and predator population at time t, respectively. Parameter d is the conversion rate. Parameter c denotes the death natural rate of predator species. The function f represents growth rate of prey , while the function $g(x,\mathrm{y})$ is called the functional response of predator to prey density.
This work is organized as follows: The fractional-order derivative model is described in section 2, as well as its discretization is analyzed and investigated through its fixed points. Then we extend the discretization system to an optimal control problem, this is done in section 3. In section 4 numerical results are presented to clarify the theoretical analysis. A discussion follows in section 5.

2-The fractional-order derivative system, and its discretization

Definition 1 [20] The $\theta -$ order Caputo differential operator is defined as follows:

$$\mathrm{D}_t^{\theta}\mathrm{f}(\mathrm{x}) = \mathrm{I}^{l-\theta}\mathrm{f}(\mathrm{x}), \quad \theta > 0$$

Such that $l = [\theta]$, and $\mathrm{I}^{\beta}\mathrm{u}(\mathrm{t}) = \frac{1}{\Gamma(\beta)}\int_0^t (\mathrm{t}-\mu)^{\beta-1}\mathrm{u}(\mathrm{t})\mathrm{dt}, \quad \beta > 0.$

I^{β} represents the β order Riemann-Liuouville integral operator, $\Gamma(.)$ denotes the Gamma function.
In this work the ratio-dependent predator–prey or Michaelis–Menten type prey–predator model [21] is modified to the following fractional-order model.

$$\begin{aligned} D^{\theta}x(t) &= x(t)\big(1-x(t)\big) - \frac{x(t)y(t)}{ax(t)+by(t)} - \mathrm{h}_1 x(t) \\ D^{\theta}y(t) &= \frac{dx(t)y(t)}{\mathrm{a}x(t)+by(t)} - cy(t) - \mathrm{h}_2 y(t) \end{aligned} \tag{2}$$

Where $x(t)$, and $y(t)$ denote the densities of prey, and predator species at time t, respectively. In this system the prey growths logistically. The parameter d represents the conversion rate part from the prey species to the predator species. The parameter c denotes the death rate of predator species. The functional response is the ratio-dependent predator-prey. $\mathrm{a},$ *and* b are the half saturation constants. h_1, *and* h_2 are the rate harvesting or the removal rate of prey and predator, respectively. Throughout this article we assume that $\mathrm{h}_2 = 0$,and $\mathrm{h}_1 = h$.

Applying discretization method to the fractional-order system (2). For more details we refer to [6,10]. The system (2) is reduced to

$$\begin{aligned} x_{n+1} &= x_n + \frac{s^\theta}{\Gamma(\theta+1)}\left[x_n(1-x_n) - \frac{x_n y_n}{(ax_n+by_n)} - hx_n\right] \\ y_{n+1} &= y_n + \frac{s^\theta}{\Gamma(\theta+1)}\left[\frac{dx_n y_n}{(ax_n+by_n)} - cy_n\right] \end{aligned} \tag{3}$$

Definition 2 [22]: Let $\overrightarrow{x_{t+1}} = f(\overrightarrow{x_t}) \quad t = ,2,3,\dots$ (4)

be a discrete time system the point e^* is called a fixed point of equation (4) if $e^* = f(e^*)$. If $|\lambda_i| < 1$ for $i = 1,2,\dots\dots,n$ λ_i are the eigenvalues of the Jacobian matrix J at e^* then it is called local stable point. Otherwise e^* is called unstable point. While if $|\lambda_i| = 1$ for some $1 \le i \le n$ then e^* is called a non-hyperbolic point.

The system (3) has the following fixed points:

1- The $e_0 = (0,0)$ is the trivial fixed point which always exists, while the fixed point $e_1 = (1-h, 0)$ exists only when $1 > h$. .

2- The unique positive fixed point $e_2 = (x_p, y_p)$ exits if $bd(1-h) > (d-ac)$ and $d > ac$ where $x_p = \frac{bd(1-h)-(d-ac)}{bd}$, and $y_p = \frac{(d-ac)}{bc} x_p$.

To discuss the dynamic behavior of the system (3) we have to compute the Jacobian matrix of (3). The Jacobian matrix at (x, y) is as follows :

$$J(x,y) = \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix}$$

Where $j_{11} = 1 + m - 2mx - \frac{mby^2}{k} - mh$, $j_{12} = -\frac{max^2}{k}$, $j_{21} = \frac{mbdy^2}{k}$, $j_{22} = 1 - mc + \frac{madx^2}{k}$.

For the local stability of the fixed points e_0, and e_1 of system (3) we have the following theorem.

Theorem 1

1- The e_0 is never to be locally stable point.

2- The e_1 is locally stable if $h \in \left(\frac{m-2}{m}, 1\right)$, and $c \in \left(\frac{d}{a}, \frac{2a+md}{am}\right)$

Proof: The Jacobian matrices at e_0 and e_1 are

$$J_{e_0} = \begin{bmatrix} 1+m-mh & 0 \\ 0 & 1-mc \end{bmatrix} \text{ and}$$

$$J_{e_1} = \begin{bmatrix} 1-m+mh & \frac{-m}{a} \\ 0 & 1-mc+\frac{md}{a} \end{bmatrix}, \text{ respectively.}$$

Now the eigenvalues of J_{e_0} are $\lambda_1 = 1 + m(1-h), \lambda_2 = 1 - mc$. Since h is always less than 1, therefore $\lambda_1 > 1$,and the point e_0 is never to be stable point.

The eigenvalues of J_{e_1} are $\lambda_1 = 1 - m + mh), \lambda_2 = 1 - mc + \frac{md}{a}$ hence if $h \in \left(\frac{m-2}{m}, 1\right)$, then $-2 + m < mh < m$, and $|\lambda_1| < 1$. Now we assume that $c \in \left(\frac{d}{a}, \frac{2a+md}{am}\right)$ this gives $\frac{d}{a} < c < \frac{2a+md}{am}$, and $|\lambda_2| < 1$. Therefore the e_1 is locally stable.

Lemma 1 [22] Let $P(x) = x^2 + p_1 x + q_1$, if the following conditions hold:

1. $P(1) > 0$

2. $P(-1) > 0$

3. $q_1 < 1$.

Then the roots of $P(x)$ are inside the unit disk.

Theorem 2

The point e_2 is locally stable if $h \in \left(Max\{\frac{z_5}{z_6}, \frac{z_1}{z_2}\}, \frac{z_3}{z_4}\right)$, and $c \in (\frac{adx^{*2}}{k}, \mathrm{Max}\{\frac{1}{m} + \frac{adx^{*2}}{k}, \frac{2}{m} + \frac{adx^{*2}}{k}\}$ where $z_1 = -c + 2x^*c + \frac{bcy^{*2}}{k} + \frac{adm^2x^{*2}}{k} \frac{-2adx^3}{k}, z_2 = \frac{adx^{*2}}{k} - c, z_3 = 4 - 2mc + \frac{2madx^{*2}}{k} + 2m - 2mx^* - \frac{2bmy^{*2}}{k} - 2m^2c + 2m^2x^*c + \frac{2bcm^2\, y^{*2}}{k} + \frac{adm^2x^{*2}}{k} - \frac{2adm^2\, x^{*3}}{k}, z_4 = 2m - m^2c + \frac{adm^2\, x^{*3}}{k}, z_5 = 1 - 2x^* - \frac{by^{*2}}{k} - c - mc - 2mcx^* + \frac{bmy^{*3}}{k} + \frac{ad\, x^{*3}}{k} + \frac{admx^{*3}}{k} - \frac{2mad\, x^{*3}}{k}, z_6 = 1 - mc + \frac{amd\, x^{*3}}{k}$

Proof: Jacobian matrix at e_2 is

$$\text{The } J_{e_2} = \begin{bmatrix} 1 + m - 2mx^* - \frac{mby^{*2}}{k} - mh & -\frac{max^{*2}}{k} \\ \frac{mbdy^{*2}}{k} & 1 - mc + \frac{madx^{*2}}{k} \end{bmatrix}$$

Then the characteristic polynomial of J_{e_2} is given as follows:

$P(\lambda) = \lambda^2 + p\lambda + q$, where $p = -2 + mc - \frac{madx^{*2}}{k} - m + 2mx^* + \frac{bmy^{*2}}{k} + mh$. And $q = 1 + m - 2mx^* - \frac{bmy^{*2}}{k} - mh - mc - m^2c + 2m^2x^*c + \frac{bcm^2\, y^{*2}}{k} + m^2ch + \frac{madx^{*2}}{k} + \frac{adm^2\, x^{*2}}{k} - \frac{2adm^2\, x^{*3}}{k} - \frac{adhm^2\, x^{*2}}{k}$. So that

If $h < \mathrm{Min}\left\{\frac{z_1}{z_2}, \frac{z_3}{z_4}\right\}$, with $\frac{adx^{*2}}{k} > c$ then the condition 1 and 2 in lemma 1 hold , while if $h > \frac{z_5}{z_6}$ then the condition3 in lemma 1 holds . Therefore the point e_2 is local stable point.

3-Optimal harvesting approach.

This part of the article deals with the optimal harvesting amounts so that the system (3) becomes as follows

$$\begin{aligned} x_{n+1} &= x_n + \frac{S^\theta}{\Gamma(\theta+1)}[x_n(1 - x_n) - \frac{x_n y_n}{(ax_n + by_n)} - h_n x_n] \\ y_{n+1} &= y_n + \frac{S^\theta}{\Gamma(\theta+1)}[\frac{dx_n y_n}{(ax_n + by_n)} - cy_n] \end{aligned} \tag{5}$$

The all parameter are the same previous interpolation, while the parameter h_n represents the control variable. We form the objective functional as follows:

$$J(h_n) = \sum_{n=0}^{T-1} c_1 h_n x_n - c_2 h_n \tag{6}$$

Subject to the considered system (5) the parameters c_1 and c_2 are positive constants. Now we have to find out the optimal solution h_n^* that satisfies $J(h_n^*) = \mathrm{Max}\, J(h_n)$ for all $0 \le h_n \le h_{Max}$, h_{Max} represents the maximum harvesting. We apply the Pontryagin's Maximum Principle [1,3, 23-25] to get the necessary conditions for the optimal variable control and corresponding states.

Theorem 3

If h_n^* represents an optimal solution with the optimal corresponding states x_n^* and y_n^*, then for $n = 1,2,\ldots.T-1$ the adjoint functions λ_n, and μ_n exist that satisfy:

$$\lambda_n = c_1h_n + \lambda_{n+1}[1 + m - 2mx - \frac{mby^2}{k} - mh_n] + \mu_{n+1}(\frac{mbdy^2}{k})$$

$$\mu_n = \lambda_{n+1} - \frac{max^2}{k} + \mu_{n+1}[1 - mc + \frac{madx^2}{k}] \qquad (7)$$

$\lambda_T = 0$, $\mu_T = 0$, and $m = \frac{s^\theta}{\Gamma(\theta+1)}$. The optimal control is given by $h_n^* = \frac{c_1x_{n-}\lambda_{n+1}x_n}{2c_2}$ for $0 < \frac{c_1x_{n-}\lambda_{n+1}x_n}{2c_2} < h_{max}$,while $h_n^* = h_{max}$ if $\frac{c_1x_{n-}\lambda_{n+1}x_n}{2c_2} > h_{max}$

Proof:

The Hamiltonian function is

$H_n = c_1h_nx_n - c_2h_n^2 + \lambda_{n+1}\left[x_n + \frac{s^\theta}{\Gamma(\theta+1)}[x_n(1-x_n) - \frac{x_ny_n}{(ax_n+by_n)} - h_nx_n\right] + \mu_{n+1}[y_n + \frac{s^\theta}{\Gamma(\theta+1)}[\frac{dx_ny_n}{(ax_n+by_n)} - cy_n]$.

By the necessary conditions of Pontryagin maximum principle , we have for $n = 1,2,\ldots.T - 1$.

$$\lambda_n = c_1h_n + \lambda_{n+1}[1 + m - 2mx - \frac{mby^2}{k} - mh] + \mu_{n+1}(\frac{mbdy^2}{k})$$

And $\mu_n = \lambda_{n+1} - \frac{max^2}{k} + \mu_{n+1}[1 - mc + \frac{madx^2}{k}]$. Now the optimal variable will be $h_n^* = \frac{c_1x_{n-}\lambda_{n+1}x_n}{2c_2}$ for $0 < \frac{c_1x_{n-}\lambda_{n+1}x_n}{2c_2} < h_{max}$, and $h_n^* = h_{max}$ if $\frac{c_1x_{n-}\lambda_{n+1}x_n}{2c_2} > h_{max}$.

4-Numerical results

This section verifies the effectiveness of our theoretical results, so that some numerical simulations are given. To confirm the behavior of the system (3) through the local stability of its fixed points. Some numerical simulations have been given. To confirm of the point $e_1 = (1 - h, 0)$ is local stable point we use the following values of parameters : $a = 0.6$; $b = 0.8$; $d = 0.3$; $c = 0.6$; $h = 0.1$; $\alpha = 0.98$, and the initial point is (1.8, 1.9). Hence the condition 2 in Theorem (1) is established. Figure 1 displays the local stability of e_1.

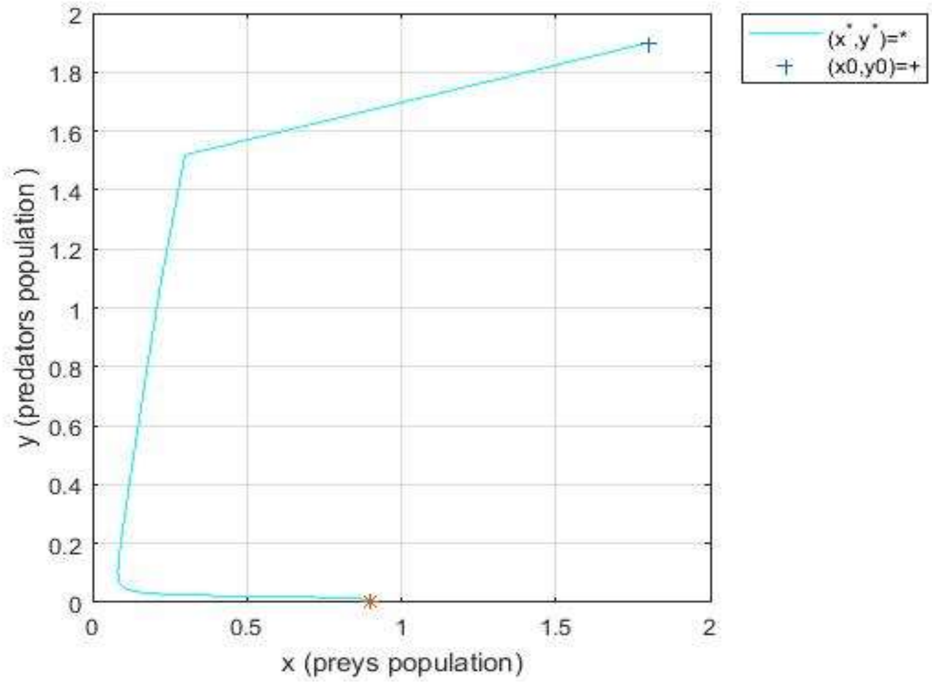

Figure 1: Local stability of e_1 is illustrated in this figure.

For the point e_2 the values of parameters are set as follows: $a = 0.4$; $b = 0.8$; $d = 0.33$; $c = 0.4$; $h = 0.15$; $\alpha = 0.98$, and the initial point is (0.4,0.5). Hence the Theorem 2 is verified, and the point is stable. This is displayed in Figure 2. Trajectories of the prey species and the predator species as a function of time which
Indicates that the point e_2 is local stability. This is done in Figure 3.

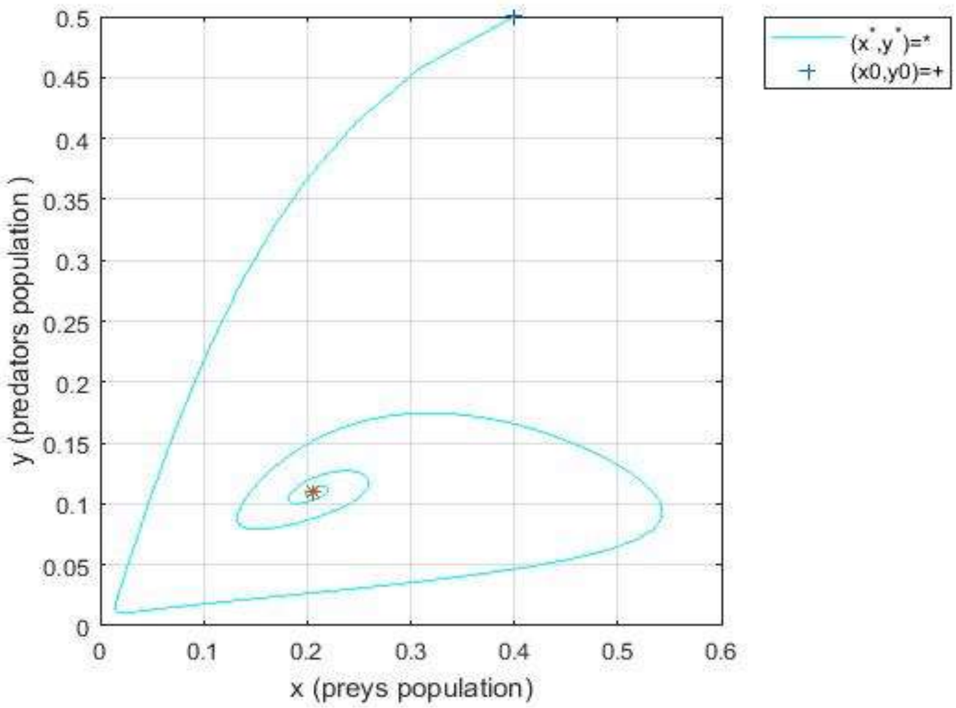

Figure 2: This figure shows the point e_2 is locally stable point.

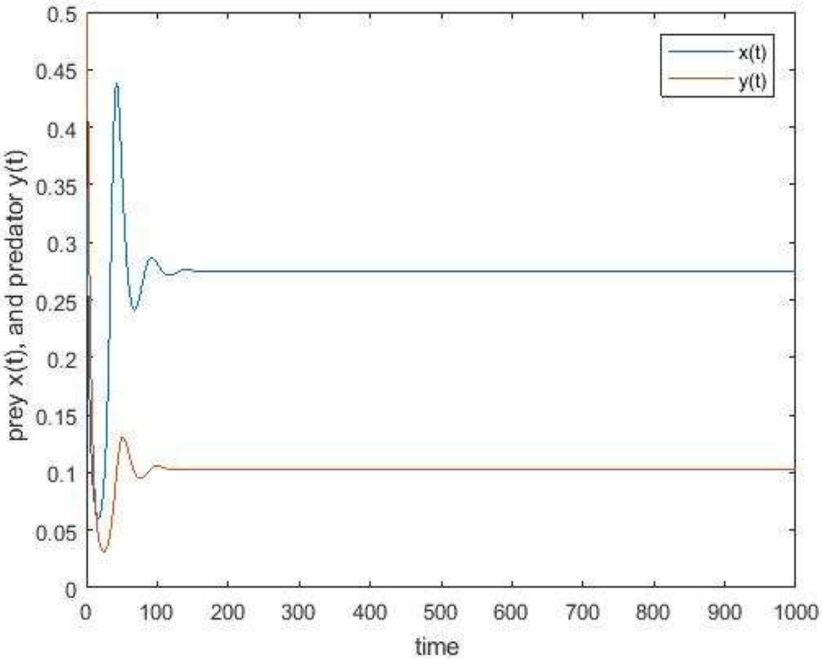

Figure 3: This figure shows the time series of prey density and predator density that indicates the local stability of e_2.

We use and employ iterative method to find the optimal control solution. We use an iterative algorithm. For more details we refer to [5, 14, 25]. The values of parameters as follows: $= 0.3$; $b = 0.8$; $d = 0.45$; $c = 0.5$; $\alpha = 0.98$, $c_1 = 0.2$, . and $c_2 = 0.2$ with initial guess $x_0 = 0.4, and\ y_0$ =0.5 for prey , and predator , respectively. We obtain the total net optimal harvesting is $J(h_n^*) = 0.1090$. Figure 4 shows the optimal solution variable as function of time, while Figures 5-6 indicate the effect of optimal solution and the fixed harvesting amount on the prey, predator, respectively.

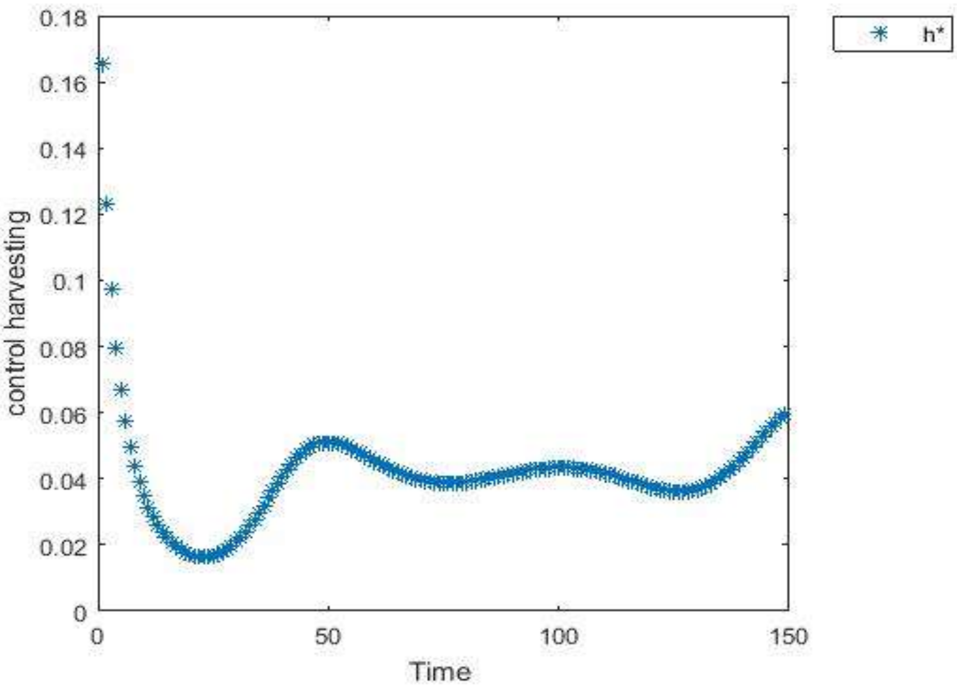

Figure 4: The optimal solution of the system 5 is plotted as function of time.

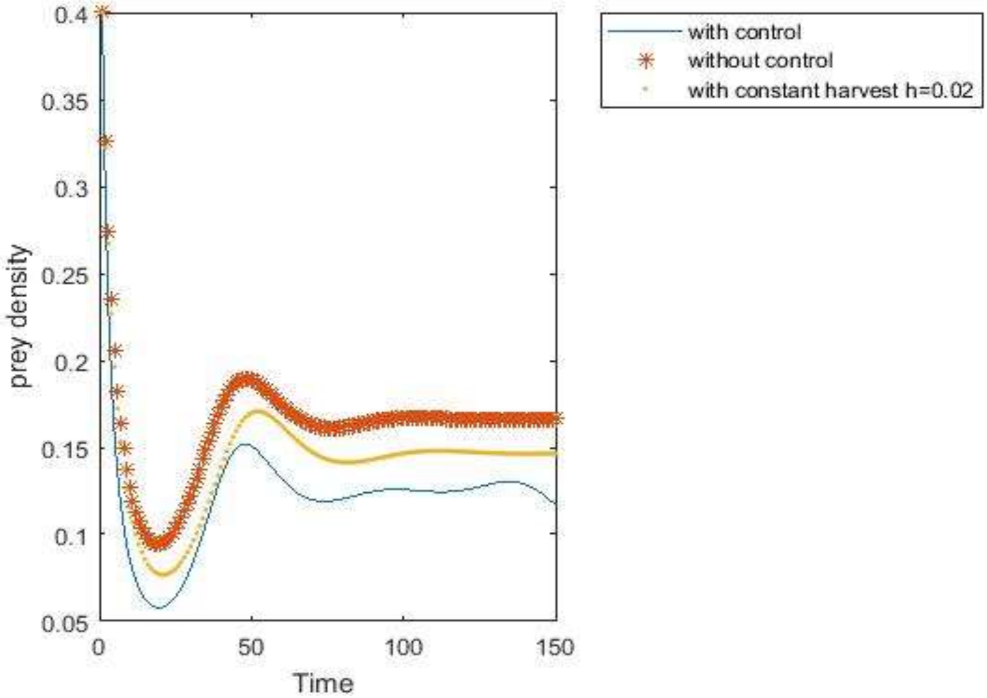

Figure 5: This figure shows prey species in system (6) with control, without control, and with fixed harvest amount.

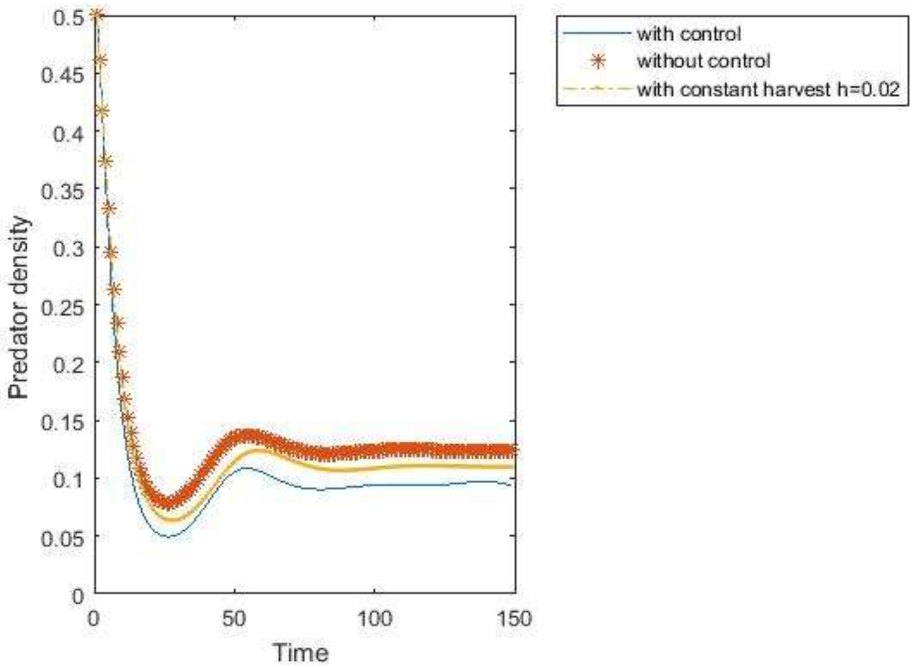

Figure 6: The predator species in system (6) is plotted with control, without control, and with fixed harvest amount.

5- Discussions and Conclusions

In this paper, a discretization of fractional-order prey-predator system with ratio-dependent predator–prey functional response has been presented and analyzed. The local stability of its fixed point is studied. Our analysis shows the considered system has three fixed points as well as the trivial fixed point is never to be stable point, while the other points are locally stable under certain conditions. We also conclude that the equilibrium harvesting amount as well as any constant harvesting amount cannot be the optimal solution. We can see in Figures 4 and 5 that the level of prey species density, predator density with optimal control are lower than their equilibrium level. It is also seen that the heavily harvesting will lead to increase the possibility of extinction.

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