DOI: 10.24996/ijs.2022.63.12.34

ISSN: 0067-2904

Purely Small Submodules and Purely Hollow Modules

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Received: 27/2/2022 Accepted: 1/6/2022 Published: 30/12/2022

Abstract
The main goal of this paper is to give a new generalizations for two important classes in the category of modules, namely the class of small submodules and the class of hollow modules. They are purely small submodules and purely hollow modules respectively. Various properties of these classes of modules are investigated. The relationship between purely small submodules and P-small submodules which is introduced by Hadi and Ibrahim, is studied. Moreover, another characterization of purely hollow modules is considered.

Keywords: Pure submodules, Purely small submodules, Purely hollow modules.

المقاسات الجزئيّة الصغيرة النقيّة والمقاسات المجوّفة النقيّة

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الخلاصة
في هذا البحث تم تقديم إعماماً جديداً لصَنفين مُهمّين من فئة المقاسات، هما المقاسات الجزئية الصغيرة والمقاسات المجوفة, سنطلق عليهما اسم المقاسات الجزئية الصغيرة النقيّة والمقاسات المجوفة النقيّة على التوالي. أعطينا عدد من الخصائص الرئيسة والمهمة لهذين النوعين من المقاسات. كما تم دراسة علاقة المقاسات الجزئية الصغيرة النقيّة بالمقاسات الجزئية الصغيرة من النمط – P والتي قُدّمت من قبل الباحثتان انعام محمد علي هادي وتماضرعارف ابراهيم، إضافة الى ذلك فقد تم إعطاء تشخيصاً اخراً للمقاسات المجوفة النقيّة.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary left modules. A ring and module are denoted by R and M respectively. It is well known that a submodule N of M is called pure if $IM \cap N = IN$ for every finitely generated I of R [1, P.31]. A submodule N of M is small if $N+V \neq M$ for every proper submodule V of M. It is shortly denote by $N \ll M$ [2, Exercise (20), P.20]. An R-module M is called hollow if every proper submodule of M is small submodule [3].

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The main purpose of this paper is to extend the notions of small submodules and hollow modules by new generalizations. They are called purely small submodule and purely hollow module, respectively.

Section 2 is devoted to study purely small submodules. Various properties of this kind of submodules are given. In section 3, we study the relationships of purely small submodules with P-small submodules, where a proper submodule N of an R-module M is called P-small if N+P≠M for any prime submodule p of M. In section 4, an extension of hollow modules is given, it is called purely hollow modules.

2. Purely Small Submodules

In this section, a new class of submodules is introduced and called purely small submodules. We choose the symbols $N \ll_{pu} M$ to mean that N is a purely small submodule of M, We investigate the main properties of this class of submodules.

Definition (2.1): A submodule N of an R-module M is called purely small, if N+V≠M for every proper pure submodule V of M. Equivalently; $N \ll_{pu} M$, if whenever V is a pure submodule of M with N+V=M, then V=M. An ideal I of R is called purely small if I+J≠R for every proper pure ideal J of R.

Examples and Remarks (2.2):

1. It is clear that (0) is purely small submodules.

2. It can be easily show that every small submodule is purely small.

3. The converse of (2) is not true in general, for example: 2Z is a purely small submodule of the Z-module Z. In fact, the only pure submodule of Z is (0) and 2Z+(0)≠Z. But 2Z is not small submodule, since 2Z+3Z=Z, while 3Z≠Z.

4. Consider the Z-module of all rational numbers Q. The only proper pure submodules in Q is (0) [1], and for each proper submodule N of M ; N+(0)≠Q. Thus every proper submodule of Q is purely small. On the other hand, not every proper submodule of Q is small. In fact, there exists proper submodules say X and Y of Q such that X+Y=Q [4, Exercise (6)b, P.186], so that both of X and Y are not small submodules of Q.

Recall that a module M is called pure simple if (0) and M are the only pure submodule of M [5].

5. If M is a pure simple module, then every proper submodule of M is purely small. For example Z_{p^∞} is pure simple Z-module, since the only pure submodule in Z_{p^∞} is (0), so that every proper submodule of Z_{p^∞} is purely small.

6. A non-zero direct summand of any R-module M is not purely small. For example: the submodule $(\bar{2})$ is a direct summand of the Z- module Z_{10}, but not purely small since $(\bar{2})+(\bar{5})=Z_{10}$, and $(\bar{5})$ is pure, but not equal to Z_{10}.

In order to prove the next proposition we need the following lemma which is appeared in [6, Remark (1.2.8)(5), P.19].

Lemma (2.3): Let N and H be submodules of an R-module M with H≤N. If H is pure submodule of M and N/H is pure in M/H, then N is a pure submodule of M.

Proposition (2.4): Let $N \ll_{pu} M$, and H≤N with H is pure in M, then $N/H \ll_{pu} M/H$.

Proof: Suppose that N/H+ L/H= M/H, where L/H is pure submodule of M/H. This implies that N+L=M. Since H is pure in M and L/H is pure submodule of M/H, so from Lemma

(2.3), we have L is pure in M. But $N \ll_{pu} M$, then L=M, and L/H= M/H. Therefore $N/H \ll_{pu} M/H$.

An R-module M is called multiplication if for each submodule N of M, there exists an ideal I of R such that N=IM [7].

Proposition (2.5): Let M be an R-module and let H, K and N be submodules of M with H≤K≤N≤M. Assume that N is multiplication with pure annihilator. If $K \ll_{pu} N$ then $H \ll_{pu} M$.

Proof: Assume that there exists a pure submodule L of M with H+L=M. Now, N∩(K+L)=N∩M. Since K≤N, then K+(N∩L)=N. Note that L is pure in N [1, Remark (1.4)(3), P.31], and N is pure in itself. Since N is multiplication with pure annihilator, then N∩L is pure in N [8, Corollary (1.3)(2)]. But $K \ll_{pu} N$ then N∩L=N, hence N⊆L. By assumption, we have H⊆N, thus L=M. That is $H \ll_{pu} M$.

Al-Bahraany in her Ph.D thesis introduced the property PIP, where an R-module M has PIP if the intersection of any two pure submodules is again pure [1, Definition (2.1), P.33].

Proposition (2.6): Let M be an R-module and let H, K and N be submodules of M with H≤K≤N≤M. Assume that N has PIP property. If $H+K \ll_{pu} N$ then $H \ll_{pu} M$.

Proof: Suppose there exists a pure submodule L of M with H+L=M. From the proof of Proposition (2.5), we obtain K+(N∩L)=N. Now, N is pure in itself, and since L is pure in M, then L is pure in N [1, Remark (1.4)(3), P.31]. On the other hand M has PIP property, therefore N∩L is pure in N [1, Definition (2.1), P.33]. Since K is a purely small submodule of N, then N∩L=N, i.e N⊆L, hence H⊆N⊆L, so that L=M. Therefore H is a purely small submodule of M.

Corollary (2.7): Let M be a multiplication R-module, and consider H, K and N are submodules of M such that H≤K≤N≤M and N is pure in M. If $H+K \ll_{pu} N$ then $H \ll_{pu} M$.

Proof: Since M is multiplication, then M has PIP [1,P.33]. But N is pure submodule of M, therefore N has PIP [1, Proposition (2.4)(1), P.33]. By Proposition (2.6) the result is followed.

Recall that a module M is called F-regular, if every submodule of M is pure [9]. Next proposition shows that, in the class of F-regular modules the classes small and purely small submodules are coincide.

Proposition (2.8): Let M be an F-regular module. A submodule N of M is small if and only if $N \ll_{pu} M$.

Proof: The necessity is clear. For the converse, we suppose that L is a submodule of M with N+L=M. Since M is F-regular, then L is a pure submodule of M. But $N \ll_{pu} M$, so that L=M. Therefore the result is obtained.

The next proposition deals with the transitive property of purely small submodules.

Proposition (2.9): Suppose that a module M has PIP, and let H and N be submodules of M with H≤N≤M. If $H \ll_{pu} N$, and N is pure in M, then $H \ll_{pu} M$.

Proof: Assume that L is a pure submodule of M with H+L=M. We have to show that L=M. Now, N∩(H+L)=N∩M. Since H≤N, so by Modular Law we obtain that H+ (N∩L)=N. Since N and L are pure in M and M is PIP, then N∩L is pure in M. This implies that N∩L is pure in

N [1, Remark (1.4)(3), P.31]. But H$\ll_{pu}$N, therefore N∩L=N. That is N⊆L. On the other hand, we have H+L=M and H⊆N, therefore H⊆L, and L=M, that means H$\ll_{pu}$M.

Since every F-regular module has PIP property [1, Remark (2.2)(3), P.33], and also every multiplication has PIP property [1,Proposition(2.3),P.33], then as a consequence of Preposition (2.9), we have the following corollary.

Corollary (2.10): Let M be an F-regular (or multiplication) module, and let H and N be submodules of M with H≤N≤M. If H$\ll_{pu}$N, and N is pure in M then H$\ll_{pu}$M.

An R-module M is called a cancellation module if whenever AM = BM with A and B are ideals of R then A = B [10, P.6], and from [13], M is called purely cancellation if whenever AM = BM, then A = B, where A is a pure ideal of R and B is any ideal of R.

Proposition (2.11): Let M be a purely cancellation module, and I be an ideal of R. If IM is a purely small submodule of M then I is purely small ideal of R.

Proof: Suppose that I+J=R, where J is a pure ideal of R then (I+J)M=RM. This implies that IM+JM=M. Since J is a pure ideal of R then JM is a pure submodule of M [11, Proposition (1.4)]. But IM is a purely submodule of M, therefore JM=M, that is JM=RM. Because of M is purely cancellation, then we have J=R [13]. Thus I is a purely small ideal.

Corollary (2.12): For any purely cancellation module M, if N $\ll_{\mathrm{pu}}$M then (N:M) $\ll_{\mathrm{pu}}$R.

Proof: It is known that N=(N:M)M [12]. That is N=IM, where I=(N:M), and by Proposition (2.11), we conclude that (N:M) $\ll_{\mathrm{pu}}$R.

Corollary (2.13): Let M be a cancellation module, and I be an ideal of R. If IM$\ll_{\mathrm{pu}}$M, then I$\ll_{\mathrm{pu}}$R.

Proof: Assume that IM$\ll_{\mathrm{pu}}$M. Since every cancellation module is purely cancellation [13], then the result is directly followed by Proposition (2.11).

The following theorem gives conditions under which the hereditary property of "purely small" is satisfying between R and any R-module .

Theorem (2.14): Let M be a finitely generated faithful multiplication R-module. An ideal I of R is purely small if and only if IM is a purely small submodule of M.

Proof: Suppose that I is a purely small ideal of R, and consider IM+N=M, where N is pure submodule of M. Since M is multiplication, then N=JM for some ideal J of R [7]. So that we have IM+JM=M. This implies (I+J)M=RM. Since M is finitely generated faithful multiplication, then I+J=R [12, Theorem (3.1), P.768]. On the other hand, because of JM is a pure submodule of M, then we have J is a pure ideal of R [8, Theorem (1.4)]. Now, since I is purely small ideal of R, then J=R. But M is finitely generated, thus JM=RM=M [12, Theorem (3.1), P.768]. So that N=M, that is IM is purely small submodule of M. Conversely, assume that I+J=R, then (I+J)M=RM. This implies that IM+JM=M. Since J is a pure ideal of R, then JM is a pure submodule of M [11, Proposition (1.4)]. But IM is purely submodule of M, therefore JM=M, that is JM=RM. On the other hand, M is finitely generated faithful and multiplication, so that J=R [12, Theorem (3.1), P.768].

3. Purely Small Submodules and P-small Submodules

A proper submodule N of an R-module M is called P-small if N+P≠M for any prime submodule P of M [14]. In this section we study the relationship between this class of submodules and the class of purely small submodule. Firstly, we have to point out that there is no direct relationship between the two classes. In fact, this conclusion is based on the absence of a relationship between Prime and pure submodules that are considered in the basis for defining P-small submodule and purely small submodule.

In order to prove the next proposition we need the following lemma which is appeared in **[**4, Theorem (2.3.11), P.28**]**.

Lemma (3.1): If a module M is finitely generated, then every proper submodule of M is contained in a maximal submodule of M.

Proposition (3.2): Let M be a finitely generated R-module. If N is a P-small submodule of M, then N is purely small.

Proof: Let L be a proper pure submodule of M with N+L=M. Since M is finitely generated, then by Lemma (3.1), L is contained in a maximal (hence in a prime) submodule of M say K, i.e $L \subseteq K \subset M$. This implies that N+K=M, therefore N is not P-small which is a contradiction. Therefore N is a purely small submodule.

Note (3.3): If we replace the condition finitely generated in Proposition (3.2) by a non-zero multiplication, then it is also true, by using Theorem (2.5)(i) in [12] instead of Lemma (3.1).

Recall that an R-module M is called Noetherian if every submodule of M is finitely generated [4, Theorem (6.1.2)(II), P.147].

Corollary (3.4): If M is a Noetherian module, then every P-small submodule is purely small.

Proof: Since M is Noetherian, then M is finitely generated, and the result follows by Proposition (3.2).

It is known that a proper submodule N of a torsion free module M is pure if and only if N is prime submodule of M with (N:M)=0 [11]. So we have the following theorem.

Theorem (3.5): Let M be a torsion free R-module, then N is purely small submodule of M if and only if N is P-small with (N:M)=0.

Proof: Suppose that N is purely small, and let L be a prime submodule of M. We have to show that N+L≠M. Suppose the converse, i.e N+L=M, and since M is torsion free with (N:M)=0, then by Proposition (1.3) in [11], L is pure submodule of M. But N is a purely small submodule, so that L=M which is a contradiction since L is prime which is L is proper in M, therefore N+L≠M, that means N is P-small. Conversely, assume that N is P-small submodule of M with N+L=M, where L is a pure submodule of M. If L≠M, and since M is torsion free, then by Proposition (1.3) in [11], we have L is prime. But N+L=M, we obtain that N is not P-small which is a contradiction, thus L=M, and hence N is purely small.

Recall that a module M is called prime if ann(M)=ann(N) for all submodule N of M [11]. We have the following theorem.

Theorem (3.6): A submodule N of a prime module M is purely small if and only if N is P-small provided that ann(M)=(K:M) for every prime submodule K of M.

Proof: Assume that N is purely small submodule of M, and consider N+K=M, where K is prime submodule of M. Since M is a prime module and ann(M)=(K:M), then by Proposition (1.3) in [11], we obtain K is pure. Since N is purely small, therefore K=M. But this is a contradiction, since K is prime which is proper in M, thus N+K≠M, that is N is P-small submodule. For the converse, suppose that N is a P-small with ann(M)=(K:M) for every prime submodule K of M, and let N+K=M, where K is a pure submodule. Since M is a prime module, then by Proposition (1.3) in [11], K is a prime submodule. But N+K=M, therefore N is not P-small, hence K=M. This mean N is purely small.

The next example shows that the condition "ann(M)=(N:M)" in Theorem (3.6) cannot be dropped.

Example (3.7): Consider the Z-module Z. From Example (2.2)(3), we note that 2Z is purely small submodule. On the other hand, 2Z is not P-small since there exists a submodule 3Z of Z with 2Z+3Z=Z. In fact Z is a prime module but ann(Z)≠(2Z:Z).

4. Purely Hollow Modules

This section is devoted to introduce the concept of purely hollow modules. We examine the main properties of this class of modules, we also, discuss its relationship with the class of hollow modules.

Definition (4.1): An R-module M is called purely hollow if every proper submodule of M is purely small. A ring R is called purely hollow if every proper ideal of R is purely small.

Examples and Remarks (4.2):

1. It is clear that every hollow module is purely hollow.

2. The converse of (1) is not true in general, for example: the set of rational numbers Q is not hollow Z-module [14]. While it is purely hollow; in fact, the only proper pure submodule of Q is (0), therefore every proper submodule of Q is purely small submodule, thus Q is purely hollow. We conclude that the class of purely hollow modules contains properly the class of hollow modules.

3. Z_{p^∞} is purely hollow Z-module, see Example (2.2)(5). Note that Z_{p^∞} is also hollow module, see Example (3.2)(5) in [14].

4. By the same argument of (2), Z is a purely hollow Z-module. While clearly Z is not hollow, since there exist proper submodules not small in Z, see Example (2.2)(3).

5. Semisimple module cannot be purely hollow. In fact, in the semisimple module say M, every submodule N of M is direct summand, and by Example (2.2)(6), N is not purely small.

6. Every pure simple module is purely hollow module. This deduce directly by Example (2.2)(5).

Proposition (4.3): Every purely hollow module is indecomposable.

Proof: Let M be an R-module, and suppose that M is decomposable, then there exist proper submodules A and B with M=A⊕B. Since each direct summand is pure, therefore both of A and B are pure. Since M is purely hollow, then every proper submodule of M is purely small. This implies that either A=M or B=M. In each case we obtain a contradiction since both of A and B are proper. Thus M is indecomposable.

Proposition (4.4): For every purely hollow module M, if M/N is finitely generated for every proper submodule N of M, then M is finitely generated.

Proof: Let M be a purely hollow module, and N⫋M with M/N is finitely generated, then:
M/N=R(x_1+N)+R(x_2+N)+…..+R(x_n+N) ………(*)
where $x_i \in$M for each i, where i=1,2,…,n. Now, we claim that:
M=Rx_1+R$x_{2+....}$Rx_n
In order to verify that, assume that m∈M, then m+N∈ M/N. Since M/N is finitely generated, so by (*):
m+N = r_2(x_2+N)+r_2(x_2+N)+…r_n(x_n+N)
=r_1x_1+ r_2x_2+...... r_nx_n+ N.
This implies that:
m= r_1x_1+ r_2x_2+...... r_nx_n+ n for some n∈N
Thus:
M= (Rx_1+ Rx_2+...... Rx_n)+N
But M is purely hollow, then N is a purely small submodule of M which implies that
M= Rx_1+ Rx_2+...... Rx_n
Thus M is finitely generated.

The following theorem shows that in the class of F-regular module purely hollow and hollow modules are coincide.

Proposition (4.5): Let M be an F-regular module, then M is hollow if and only if M is a purely hollow module.

Proof: The necessity is obvious. For the converse, suppose that M is a purely hollow module, then every proper submodule N of M is purely small. But M is F-regular, so by Proposition (2.8), N≪M, hence M is a hollow module.

Proposition (4.6): If M is a pure simple module, then M is hollow if and only if M is a purely hollow module.

Proof: The necessity is clear. For the converse; suppose that M is a purely hollow module, then every proper submodule N of M is purely small. But M is pure simple, so that by Remark (1.2)(4), N≪M, hence M is a hollow module.

Now, we give another characterization of purely hollow module.

Theorem (4.7): An R-module M is purely hollow if and only if every proper pure submodule of M is small.

Proof: Suppose that M is purely hollow, and let N be a proper pure submodule of M. Suppose there exists a proper submodule W of M with N+W=M……..(*)
Since M is purely hollow and N is pure, so we can rewrite (*) as W+N=M. By assumption we have W$\ll_{pu}$M, so that N=M which is a contradiction, since N is proper, thus N+W≠M, that is N is small. Conversely, assume that N is a proper submodule of M such that N+L=M, where L is pure. If L≠M, then L≪M. This implies that N=M, which is a contradiction since N is proper. Thus L=M, and hence N$\ll_{pu}$M.

Ahmed [15] introduced a prime hollow module as a module in which each prime submodule is small. There is no direct implication between this kind of modules and purely hollow. However, under certain conditions we obtain the following result.

Proposition (4.8): Let M be a finitely generated module. If M is prime hollow then M is a purely hollow module.

Proof: In order to prove M is purely hollow, we will apply Theorem (4.7). Assume that N is a proper pure submodule of M and L≤M with N+L=M, we have to show that L=M. Suppose that L≠M. Since M is finitely generated, then by Lemma (3.1), L is contained in a maximal (hence in a prime) submodule of M say K, i.e L⊆K⊂M. This implies that N+K=M. But M is prime hollow, therefore K=M which is a contradiction since K is proper. Thus L=M, hence N is small. By theorem (4.7), we deduce that M is purely hollow.

Corollary (4.9): Let M be multiplication with prime annihilator. If M is prime hollow module then M is purely hollow.

Proof: Since M is multiplication with prime annihilator, then M is finitely generated [10, Corollary (3.6), P.56]. So the result follows by Proposition (4.8).

An R-module M is said to be lifting if for every submodule N of M, there exists a submodule K of N such that M=K⊕K′, where K′≤M and N⋂K′≪K′. Equivalently: M is lifting if for each submodule N of M there exists a direct summand K of M such that K≤N, and N/K≪M/K [6, P.6]. The extend of hollow module into purely hollow module implies that purely hollow loses some properties that is satisfied in hollow modules, such as "every hollow module is lifting" [16, Remark (1.1.7), P.9]. Next example shows that there is a purely hollow module which is not lifting .

Example (4.10): Consider the Z-module Q. We notice in Example (4.2)(2), that this module is purely hollow, while it is not lifting [16, P.10].

However, under certain conditions we have the following.

Proposition (4.11): Every F- regular purely hollow module is lifting.

Proof: Let M has the above properties. By Proposition (4.5), M is hollow, hence M is lifting [16, Remark (1.1.7), P.9].

Garib [6] introduced a generalization of lifting modules, and he called it purely lifting modules, where an R-module M is called purely lifting if for every submodule N of M, there exists a pure submodule K of M, such that K≤N and N/K≪M/K. In the following definition we also use the purity property to introduce a different generalization.

Definition (4.12): An R-module M is called nearly lifting if for every proper pure submodule N of M, there exists a direct summand K of M such that K≤N and N/K$\ll_{pu}$M/K.

Example (4.13): The Z-module Q is nearly lifting, where Q is the set of all rational numbers, since the only proper pure submodule in Q is N=(0), so that trivially there exists a direct summand K=(0) of M such that K≤N and N/K$\ll_{pu}$M/K. Note that Q is neither lifting [16, P.10] nor purely lifting [6, Example (2.2.2)(2), P.38].

Remark (4 .14): It is clear that every lifting module is nearly lifting. To show that, let N be a proper pure submodule of M, that is N≤M. Since M is lifting, then there exists a direct summand K of M such that N/K≪M/K [16, P.6], hence N/K$\ll_{pu}$M/K.

Compare the following proposition with [16, Remark (1.1.7), P.9].

Proposition (4.15): Every purely hollow module is nearly lifting.

Proof: Let M be a proper purely hollow module, and N is a proper pure submodule of M, so that there exists a direct summand K=(0) of M, K≤N. Since N$\ll_{pu}$M, then N/K$\ll_{pu}$M/K, therefore M is nearly lifting.

As a sequel of this paper, we have the following conclusions.

Conclusions: In the literature, many generalizations have been done in the category of module. In this work, the class of small submodules and the class of hollow modules are generalized to new classes. They are called the class of purely small submodules and the class of purely hollow modules, respectively. Some properties of these classes are studied and investigated. For example, it has been seen that if N is multiplication with pure annihilator of an R-module M and K$\ll_{pu}$N, then H$\ll_{pu}$M, where H and K are submodules of M with H≤K≤N≤M. Furthermore, relationship of purely small submodules with P-small submodules have also been studied in details, for example it has been proved that a submodule N of a prime module M is purely small if and only if M is P-small provided that ann(M)=(K:M) for every prime submodule K of M. Finally, significant results about purely hollow modules with, hollow modules, F-regular modules, pure simple modules, and lifting modules have been given. Furthermore, a new generalization of lifting module has been considered in the last section of this paper.

Acknowledgment: The authors of this paper would like to thank the referees for their valuable suggestions and helpful comments.

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