 Iraqi Journal of Science, 2019, Vol. 60, No. 3, pp: 633-637
DOI: 10.24996/ijs.2019.60.3.21

ISSN: 0067-2904

Essential Second Modules

Ghaleb Ahmed
Department of Mathematics, College of Education for Pure Science/ Ibn-Al-Haitham University of Baghdad, Iraq

Abstract
M is viewed as a right module over an arbitrary ring R with identity. The essential second modules is defined in this paper. We call M is essential second when for any $a \in R$, either $Ma = 0$ or $Ma \leq_e M$. Number of conclusions are gained and some connections between these modules and other related modules are studied.

Keywords: essential second modules, endo essential second modules, coprime modules

المقاسات الثنائية الاساسية

غالب احمد
قسم الرياضيات ، كلية التربية للعلوم الصرفة / ابن الهيثم ، جامعة بغداد ، بغداد ، العراق

الخلاصة
ليكن M مقاسا ايمن على حلقة R ذات محايد. في هذا البحث قدمت المقاسات الاساسية الثانية. نطلق على مقاسا M بانه اساسي ثاني اذا كان لكل $a \in R$ فانه اما $Ma = 0$ او $Ma \leq_e M$. عدد من النتائج اعطيت وتم دراسة العلاقات بين هذه المقاسات والمقاسات الاخرى.

1. Introduction

R is denoted a ring has identity and M is studied as a left S- right R-bimodule where $S = \text{End}_R(M)$. $0 \neq M$ is coprime (sometimes second) if all $0 \neq a \in R$ then either $Ma = M$ or $Ma = 0$ [1]. The idea of small coprime was presented and studied in [3] as a new type of the concept coprime. M is called small coprime if for each $0 \neq a \in R$ then either $Ma = M$ or $Ma \ll M$. N is small submodule in M abbreviated $N \ll M$ in case any submodule K of M with $N + K = M$ implies $K = M$ [2]. Accordingly the notion of essential second as another type of coprime modules. A non-zero R-module M is essential second when whole $a \in R$, either $Ma = 0$ or $Ma \leq_e M$. N of M is essential submodule in M abbreviated $N \leq_e M$ if $N \cap K \neq 0$ for each $0 \neq K \neq M$ [2]. Other studies within [4-14] is related topics.

The paper consists of two parts. Within part two, we investigate the essential second idea and we supply examples and needful features of this concept. We give a characterization of essential second modules (Thearem 2.4). The direct sum of essential second modules is discussed (Proposition 2.8). Among other results we look for many relationships between essential second modules and related modules such as (Proposition 2.9 and Proposition 2.13). Finally we define endo essential second modules and give basic properties about this modules (Remarks and Examples 2.17).

2. Essential Second Modules

Main facts of this type of modules are introduced.

Email: ghaleb.a.h@ihcoedu.uobaghdad.edu.iq

Definition 2.1. A non-zero R-module M is essential second when each $a \in R$, either $Ma = 0$ or $Ma \leq_e M$.

Remarks and Examples 2.2

(1) Obviously that every simple module is a coprime module and hence it is an essential second module while the opposite is not run. M is simple when $M \neq <0>$ and it has no submodules except $<0>$ and M [2]. For example, $\mathbb{Z}$ as $\mathbb{Z}$-module is essential second since for each $a \in \mathbb{Z}$, then $\mathbb{Z}a = 0$ or $\mathbb{Z}a \leq_e \mathbb{Z}$ while $\mathbb{Z}$ is not coprime because $0 \neq \mathbb{Z}a \neq \mathbb{Z}$ for each $0 \neq a \neq 1, -1$. Further $\mathbb{Z}$ is not simple.

(2) Clearly that every divisible module is faithful coprime module and hence it is a faithful essential second module where M over a domain R, is divisible when $Ma = M$ for each $0 \neq a \in R$ [2]. Also M is faithful whenever $ann_R(M) = \{ a \in R: ma = 0 \text{ for any } m \in M\} = 0$ [2].

Proof: Presume M is divisible over a domain R means $Ma = M$ for each $0 \neq a \in R$. Thus M is coprime and hence it is essential second since $Ma = M \leq_e M$ for each $0 \neq a \in R$. On the other side $Ma \neq 0$ for each $0 \neq a \in R$ this means $ann_R(M) = 0$ so M is faithful.

(3) Clearly every uniform module is essential second while the converse is not true where $0 \neq M$ is uniform when any non-zero submodule is essential in M [2]. For example, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is essential second since for each $a \in \mathbb{Z}$, $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)a$ is either $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \leq_e \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $<\bar{0}> \oplus <\bar{0}>$ but $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is not uniform.

(4) The essential second and small coprime concepts do not imply from each one to another. The $\mathbb{Z}$-module $\mathbb{Z}$ is essential second while it is not small coprime since $\mathbb{Z}a \neq \mathbb{Z}$ for each $-1, 1 \neq a \in \mathbb{Z}$ and $\mathbb{Z}a$ is not small of $\mathbb{Z}$ for each $0 \neq a \in \mathbb{Z}$. Further, $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is small coprime since for each $a \in \mathbb{Z}$, $(\mathbb{Z}_4 \oplus \mathbb{Z}_2)a$ is either $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ or $<\bar{2}> \oplus <\bar{0}>$ or $<\bar{0}> \oplus <\bar{0}>$ while $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is not essential second because $<\bar{2}> \oplus <\bar{0}>$ is not essential in $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.

(5) The essential second and S-coprime concepts are different where M is S-coprime if $Ma \ll M$ where $0 \neq a \in R$ implies that $Ma = 0$ [14]. For example, $M = \mathbb{Z}_4$ as $\mathbb{Z}$-module is essential second since for each $a \in \mathbb{Z}$, then Ma is one in three cases, $\mathbb{Z}_4$, $<\bar{2}>$ or $<\bar{0}>$ while $\mathbb{Z}_4$ is not S-coprime because $<\bar{2}> \ll \mathbb{Z}_4$. furthermore $M = \mathbb{Z}_6$ as $\mathbb{Z}$-module is S-coprime because for each $a \in \mathbb{Z}$, we have Ma one in four cases, $\mathbb{Z}_6$, $<\bar{2}>$, $<\bar{3}>$ or $<\bar{0}>$ where $\mathbb{Z}_6$, $<\bar{2}>$, $<\bar{3}>$ are not small while $\mathbb{Z}_6$ is not essential second since $M2 = <\bar{2}>$ is not essential in $\mathbb{Z}_6$.

(6) The essentially coprime and essential second concepts are not the same where M is an essentially coprime module when each $a \in R$ $Ma = M$ or $ann_M(a) \leq_e M$[3]. via the following examples. Let us discuss the module $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module is essentially coprime as follows

If $a = 0 \Rightarrow (\mathbb{Z}_4 \oplus \mathbb{Z}_2)a \neq \mathbb{Z}_4 \oplus \mathbb{Z}_2$ while $ann_{\mathbb{Z}_4 \oplus \mathbb{Z}_2}(a) = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \leq_e \mathbb{Z}_4 \oplus \mathbb{Z}_2$

If $a = 1 \Rightarrow (\mathbb{Z}_4 \oplus \mathbb{Z}_2)a = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \leq_e \mathbb{Z}_4 \oplus \mathbb{Z}_2$

If $a = 2 \Rightarrow (\mathbb{Z}_4 \oplus \mathbb{Z}_2)a = <\bar{2}> \oplus <\bar{0}> \neq \mathbb{Z}_4 \oplus \mathbb{Z}_2$ but $ann_{\mathbb{Z}_4 \oplus \mathbb{Z}_2}(a) = \{(\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{2}, \bar{0}), (\bar{2}, \bar{1})\} \leq_e \mathbb{Z}_4 \oplus \mathbb{Z}_2$. For each $r \in \mathbb{Z}$, if we continue by this way we obtain the same results. This means $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ is essentially coprime as $\mathbb{Z}$-module but not essential second as reported in example (3). As well $\mathbb{Z}$ as $\mathbb{Z}$-module is essential second as we mentioned before in example (1). But $\mathbb{Z}$ as $\mathbb{Z}$-module is not essentially coprime because if we take $a = 2 \in \mathbb{Z}$ implies $\mathbb{Z}2 \neq \mathbb{Z}$ and $ann_{\mathbb{Z}}(2) = 0$ is not essential in $\mathbb{Z}$. Further since $\mathbb{Z}$ as $\mathbb{Z}$-module is a uniform, this means if M is uniform, then M may be not essentially coprime. Can be compared with example (3).

(7) Every essentially coprime torsion free module over a domain is faithful essential second. M is called torsion free over a domain R when $ann_R(m) = \{ a \in R: ma = 0 \} = 0$ for every non-zero element $m \in M$ [2].

Proof. Assume M is an essentially coprime over a domain R, each $a \in R$, $Ma = M$ or $ann_M(a) \leq_e M$. But M is torsion free implies $ann_M(a) = 0$ is not essential in M so $Ma = M$. This means that M is divisible, so via example (2), M is faithful essential second.

(8) Homomorphic image of essential second module need not be essential second. The natural epimorphism $\mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{\mathbb{Z}_6} = \mathbb{Z}_6$ as $\mathbb{Z}$-modules we have $\mathbb{Z}$ is essential second but $\mathbb{Z}_6$ is not essential second as discussed in example (4).

Lemma 2.3 [2] Consider N and K as submodules M where $K \subseteq N$. $K \leq_e N$ iff $K \leq_e N \leq_e M$.

Theorem 2.4 Statements are tantamount

(1) M is essential second.
(2) Each I of R, $MI = 0$ or $MI \leq_e M$.

Proof: (1) $\Rightarrow$ (2) Study I as an ideal of R and $MI \neq 0$, so there is $a \in I$, $Ma \neq 0$ implies $Ma \leq_e M$ since M is essential second. Via lemma (2.3), we have $MI \leq_e M$ as desired.

(2) $\Rightarrow$ (1) clear.

Proposition 2.5 The essential second property under an isomorphism is moved.

Proof: Discuss, N as modules over R where N is essential second with $f : N \to M$ is an isomorphism. To prove for each $a \in R$, either $Ma = 0$ or $Ma \leq_e M$. Since N is essential second, we have either $Ma = f(N)a = f(Na) = f(0) = 0$ or $Ma = f(N)a = f(Na) \leq_e f(N) = M$ as required.

Examples 2.6

(1) Whenever M is essential second with N a submodule in M, implies N need not essential second. Consider $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ as $\mathbb{Z}$-module is essential second since for each $\in \mathbb{Z}$, $(\mathbb{Z}_4 \oplus \mathbb{Z}_4)\,a$ is either $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ or $< \bar{2} > \oplus < \bar{2} >$ or $< \bar{0} > \oplus < \bar{0} >$. Now take the submodule N generated by $(\bar{1}, \bar{0})$ and $(\bar{0}, \bar{2})$. So we have $N \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$ is not an essential second as $\mathbb{Z}$-module as we mentioned in example (3).

(2) If each proper submodule in M is essential second, then M need not be essential second. For example $\mathbb{Z}_6$ over $\mathbb{Z}$ in which every proper submodule is simple module, so they are coprime and hence essential second by example (1), while $\mathbb{Z}_6$ itself not essential second as shown in example(4).

(3) Let M and $\mathbb{Z}_n$ be an essential second module over $\mathbb{Z}$ with $0 \neq M \neq \mathbb{Z}_n$ for each $n \in \mathbb{Z}$ with $n \notin ann_{\mathbb{Z}}(M)$. Yield $M \oplus \mathbb{Z}_n$ is not essential second over $\mathbb{Z}$.

Proof. Take $a = n \in \mathbb{Z}$ such that $n \notin ann_R(M)$ implies $< 0 > \oplus < \bar{0} > \neq (M \oplus \mathbb{Z}_n)a = Ma \oplus < \bar{0} >$ is not essential in $M \oplus \mathbb{Z}_n$ because $(Ma \oplus < \bar{0} >) \cap (< 0 > \oplus \mathbb{Z}_n) = < 0 > \oplus < \bar{0} >$. Thus $M \oplus \mathbb{Z}_n$ is not essential second.

(4) $\mathbb{Z}_n \oplus \mathbb{Z}_m$ as $\mathbb{Z}$-module is not essential second for each $n \neq m$ since if $a = n$ then $(\mathbb{Z}_n \oplus \mathbb{Z}_m)a = 0 \oplus \mathbb{Z}_m \neq 0 \oplus 0$ and $0 \oplus \mathbb{Z}_m$ is a direct summand of $\mathbb{Z}_n \oplus \mathbb{Z}_m$ that is $0 \oplus \mathbb{Z}_m$ is not essential in $\mathbb{Z}_n \oplus \mathbb{Z}_m$. Similarly if $a = m$ then $(\mathbb{Z}_n \oplus \mathbb{Z}_m)a = \mathbb{Z}_n \oplus 0$ is not essential in $\mathbb{Z}_n \oplus \mathbb{Z}_m$.

Lemma 2.7 [2] In case $M = \oplus_\wedge M_\alpha$ is a direct sum of modules M_α $(\alpha \in \wedge)$ with $N_\alpha \leq_e M_\alpha$ for each $\alpha \in \wedge$ then $\oplus_\wedge N_\alpha \leq_e M$.

Proposition 2.8 M is essential second iff $M \oplus M$ essential second.

Proof: Assume M is an essential second module over$R$. For each$a \in R$, either $Ma = 0$ implies $(M \oplus M)a = Ma \oplus Ma = < \bar{0} > \oplus < \bar{0} >$ or $Ma \leq_e M$. Via result (2.7),$(M \oplus M)a = Ma \oplus Ma \leq_e M \oplus M$. For theother side. Suppose M is not essential second so there is $a \in R$ such that $Ma \neq 0$ and Ma is not essential in M implies $Ma \cap N = 0$ for a submodule N of M. This implies $(Ma \oplus Ma) \cap (N \oplus N) = < \bar{0} > \oplus < \bar{0} >$ and $(M \oplus M)a = Ma \oplus Ma \neq < \bar{0} > \oplus < \bar{0} >$ so $Ma \oplus Ma$ is not essential in $M \oplus M$ that is $M \oplus M$ is not essential second which is contradicts the hypothesis and hence M is essential second.

Remind N is pure in M when $NI = MI \cap N$ for any I of R [16]. M is called regular when every submodule is pure [16].

Proposition 2.9 Every pure submodule of an essential second module inherits essential second property.

Proof: Consider N as pure in essential second M over R. For each ideal I of R, $MI = 0$ implies $NI = MI \cap N = 0$ or $MI \leq_e M$ implies $NI = MI \cap N \leq_e M \cap N = N$ as required.

Corollary 2.10: Every summand submodule of an essential second module inherits essential second property.

Corollary 2.11 Each submodule of a regular essential second module is also essential second.

Corollary 2.12 Any submodule of a semisimple essential second module is also essential second.

Examples 2.13

(1) The condition in the corollaries (2.9), (2.10) that the module M must be essential second is necessary. Let $M = \mathbb{Z}_{24}$ be as $\mathbb{Z}$-module and $N = 4\mathbb{Z}_{24}$ is a summand of M where M and N is not essential second as $\mathbb{Z}$-modules. If we take $a = 2$ then $0 \neq Na$ is not essential in N because $N2 \cap N3 = 0$ implies N is not essential second $\mathbb{Z}$-module. Further if $a = 8$ implies $0 \neq Ma$ is not essential in M because $M8 \cap M12 = 0$ and hence M is not essential second $\mathbb{Z}$-module.

(2) A submodule of a semisimple module is not needful essential second. Discuss $M=\mathbb{Z}_6$ as $\mathbb{Z}$-module is semisimple then $M\oplus M$ as $\mathbb{Z}$-module is semisimple because every sum of semisimple modules is semisimple [2]. We see that $N=M2\oplus M$ is not essential second $\mathbb{Z}$-module because if we take $a=2$ then $aN\neq 0$ and $aN=M2\oplus M2$ is not essential in N since $(M2\oplus M2\ )\cap(0\oplus M3)=0\oplus 0$.

Recall that M is multiplication when each submodule N of M, then $N = MI$ for I of R. we able to take $I=\ (N:_R\ M)$ [17]. M is torsion when $ann_R(m)\neq 0$ for all $m\in M$ [17].

Proposition 2.14 Every multiplication essential second module is either torsion or uniform.

Proof: Assume M is essential second over R.Then for each $a\in R$, either $Ma=0$ or $Ma\leq_e M$. In case $Ma=0$ then for each $m\in M$, $ann_R(m)\neq 0$ and hence M is torsion or $Ma\leq_e M$ for each $a\in R$ and since M is multiplication then each $0\neq N$ of M, $N=MI$ for I of R so there exists $a\in I$ such that $0\neq Ma\leq_e M$ for $a\in R$, hence $Ma\leq_e N\leq_e M$. Thus M is uniform.

Corollary 2.15 Every cyclic essential second module is either torsion or uniform.

Examples 2.16 Investigate M as essential second

(1) $ann_R(M)$ may not be a prime ideal of R. See $\mathbb{Z}_4$ as $\mathbb{Z}$-module is essential second as we recorded in example (4) but $ann_{\mathbb{Z}}(\mathbb{Z}_4)=\mathbb{Z}4$ which is not prime in $\mathbb{Z}$.

(2) Generally, $ann_R(M)\neq ann_R(\frac{M}{N})$ for some submodule N of M. For example, $\mathbb{Z}_4\oplus\mathbb{Z}_4$ as $\mathbb{Z}$-module is essential second as we have seen in in example 2.6(1). Take $N=<\bar{2}>\oplus<\bar{2}>$ implies $ann_{\mathbb{Z}}(\mathbb{Z}_4\oplus\mathbb{Z}_4)=\mathbb{Z}4\neq ann_R\left(\frac{M}{N}\right)=[N:_R M]=[<\bar{2}>\oplus<\bar{2}>:_{\mathbb{Z}}\mathbb{Z}_4\oplus\mathbb{Z}_4]=\mathbb{Z}2$.

Definition 2.17 $0\neq M$ is an endo essential second module when every $f\in End(M)$ either $f(M)=0$ or $f(M)\leq_e M$.

Remarks and Examples 2.18

(1) Every endo essential second module is essential second.

Proof. Assume M is endo essential second over R then for every $f\in End(M)$ either $f(M)=0$ or $f(M)\leq_e M$. postulate $a\in R$ and define $f_a: M\to M$ by $f_a(M)=ma$ for any $m\in M$. It is clear f_a is well-dfined and $f_a\in End(M)$. Then $Ma=f_a(M)=Imf_a=0$ or $Ma=f_a(M)=Imf_a\leq_e M$ as required.

(2) The reverse of (1) is not hold broadly. $\mathbb{Z}_4\oplus\mathbb{Z}_4$ as $\mathbb{Z}$-module is essential second as we mentioned in example 2.6(1) but $\mathbb{Z}_4\oplus\mathbb{Z}_4$ is not endo essential second module since we have $f\in End(\mathbb{Z}_4\oplus\mathbb{Z}_4)$ by $f(x,y)=(x,0)$ implies that $f(\mathbb{Z}_4\oplus\mathbb{Z}_4)=\mathbb{Z}_4\oplus 0$ is not essential in $\mathbb{Z}_4\oplus\mathbb{Z}_4$.

Proposition 2.19 Every multiplication essential second module is endo essential second.

Proof: Assume M is multiplication essential second over R and $f\in End(M)$ then $f(M)=MI$ for I of R. But $MI=0$ or $MI\leq_e M$ implies $f(M)=0$ or $f(M)\leq_e M$ as desired.

Recall M is a scalar module when all $f\in End(M)$ there is $a\in R$ with $f(m)=ma$ for any $m\in M$ [18].

Proposition 2.20 Every scalar essential second module is endo essential second.

Proof: is similar to Proposition (2.19).

References

1. Yassemi S. **2001**. The dual notion of prime submodules. *Arch. Math. (Brno)*, **37**(4): 273–278.
2. Dungh, N. V., Huynh, D. V., Smith P. F. and Wisbauer, R. **2008.** *Extending Modules*. Pitman Research Notes in Math Serie, longman, Harlow.
3. Yaseen S.M. **2003**. Coquasi-Dedekind Modules. Ph.D Thesis, University of Baghdad, Baghdad, Iraq.
4. GhalebAhmed. **2016**. Pure Rickart Modules and Their Generalization, *International Journal of Mathematics Trend and Technology*, **30**(2): 82-95.
5. Ghaleb Ahmed. **2017**. Coclosed Rickart Modules, *Ibn Al-Haitham Journal for Pure and Applied Science*, IHSCICONF, Special Issue.
6. Ghaleb Ahmed. **2017**. Dual Pure Rickart Modules and Their Generalization, *International Journal of Science and Research*, **6**(2): 882-886.
7. Ghaleb Ahmed. **2017**. Goldie Pure Rickart Modules and Duality, *International Journal of Science and Research*, **6**(2): 917-921.

8. Hadi, I.M., Ghaleb Ahmed. **2012**. Modules with ⊛(⊛'or ⊛'') Condition, *The Bulletin of Society for Mathematical Services and Standards*, (4): 12-21.

9. Hadi I.M., Ghaleb Ahmed. **2012**. Strongly (Comletely) Hollow Submodules I, *Ibn Al-Haitham Journal for Pure and Applied Science* **3**(25): 393-403.

10. Hadi I.M., Ghaleb Ahmed. **2013**. Strongly (Completely) Hollow Submodules II, *Ibn Al-Haitham Journal for Pure and Applied Science*, **1**(26): 292- 302.

11. Hadi I.M., Ghaleb Ahmed. **2016**. t-Regular Modules, *International Journal of Advanced Scientific and Technical Research*, **6**(1): 128-137.

12. Hadi I.M., Ghaleb Ahmed. **2016**. Almost Lifiting Modules, *International Journal of Math Trend and Technology*, **35(**3): 177-186.

13. Nuhad S. AL-Mothafar, Ghaleb Ahmed. **2015**. 2-Regular Modules, *Ibn Al-Haitham Journal for Pure and Applied Science*, **28**(2): 184-192.

14. Nuhad S. AL-Mothafar, Ghaleb Ahmed. **2015**. 2-Regular Modules II, *Ibn Al-Haitham Journal for Pure and Applied Science*, **28**(3): 235-244.

15. Hadi I.M. and Kahalf R.I. **2011**. S-coprime Modules, *Journal of Basrah Researches (Science)* **37** (4): 78-86.

16. Yaseen S.M. **1993**. F-Regular Modules. M.Sc. Thesis, University of Baghdad, Baghdad, Iraq.

17. El-Bast Z. A. and Smith P. F. **1988**. Multiplication modules, *Commutative In Algebra*, (16): 755-779.

18. Shihab B.N. **2004**. Scalar Reflexive Modules. Ph.D. Thesis, University of Baghdad, Baghdad, Iraq.