ISSN: 0067-2904

Solving Linear and Nonlinear Fractional Differential Equations Using Bees Algorithm

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Received: 22/2/2022 Accepted: 12/7/2022 Published: 30/3/2023

Abstract
A numerical algorithm for solving linear and non-linear fractional differential equations is proposed based on the Bees algorithm and Chebyshev polynomials. The proposed algorithm was applied to a set of numerical examples. Faster results are obtained compared to the wavelet methods.

Keywords: fractional differential equations, Bees algorithm, Chebyshev polynomials, Caputo derivative, Numerical solution, Metaheuristic algorithm.

حل المعادلات التفاضلية الكسرية الخطية وغير الخطية باستخدام خوارزمية النحل

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الخلاصة
تم اقتراح خوارزمية عددية لحل معادلات تفاضلية كسرية خطية وغير خطية بالاعتماد على خوارزمية النحل ومتعددات حدود جيبيشيف، تم تطبيق الخوارزمية المقترحة على مجموعة من الأمثلة العددية، تم الحصول على نتائج سريعة مقارنة بطرق المويجات.

1. Introduction

Fractional Ordinary Differential Equations (FODEs) are one of the interesting branches of Mathematics due to the presence of several physical and chemical applications [1]. The interest of mathematicians at the present time is focused on how to develop and increase the efficiency of solutions to FODEs. Since FODEs have different degrees of complexity, an exact solution may be difficult to find. Numerical methods are a vital tool for finding the approximated solutions to FODEs [2].

Recently, the wavelet methods have been relied on to solve FODEs, by finding an operational matrix based on the Caputo derivative of the fractional differentiation and one of

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the mother wavelets, for example, Haar wavelet, Legendre wavelet, Chebyshev wavelet, Jacobi wavelet and Bernoulli wavelet [3-7].

Although these methods have succeeded in solving FODEs, they may require expensive computational efforts; because the accuracy of the solutions depends on the amplitude of the operational matrix. These methods are used to find the coefficients of polynomials.

In this paper, an algorithm is proposed to solve linear and nonlinear FODEs. The coefficients of the polynomials are calculated without the need for an operational matrix. The Bees algorithm is implemented with the use of Chebyshev polynomials to solve linear and nonlinear FODEs. This gives a general approach to finding the solution to FODEs.

The solutions of the FODEs are firstly obtained by transforming the FODEs into a constrained optimization problem. Then, it is transformed into an unconstrained optimization problem by using the penalty function and then solving it using the Bees algorithm.

Caputo's derivative of a polynomial is subject to a non-iterative rule, which reduces the computational effort. The use of any polynomial may give closed results to the Chebyshev polynomial because the efficiency of the solution depends on the Bees algorithm.

In the next section, a set of basic concepts of the paper is presented that includes important definitions in addition to the Bees algorithm, it followed by the proposed algorithm. Then the application of the proposed algorithm to a set of numerical examples is provided. Finally, the conclusions are drawn.

2. Preliminary

In this section, a set of fundamental concepts are presented. For example, fractional derivative, linear and nonlinear FODEs, Chebyshev polynomials and the Bees algorithm.

2.1. Fractional Derivative

The fractional derivative of order (α) is expressed by the Caputo concept as follows [8]:

$$D^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(\alpha)}(t)}{(t-t)^{\alpha+1-n}}\, dt, & \alpha \in (n-1, n), \quad n \in N \\ \frac{d^{\alpha}}{dt^{\alpha}} f(t), & \alpha = n \in N \end{cases} \tag{1}$$

where n is a smallest integer such that $n > \alpha$.

The Caputo derivative satisfies the following properties:

$$D^{\alpha} C = 0, \tag{2}$$

$$D^{\alpha} X^{j} = \begin{cases} 0, & j \in N \cup \{0\}\ and\ \ j < \lceil \alpha \rceil \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} X^{j-\alpha}, & j \in N \cup \{0\}\ and\ j \geq \lceil \alpha \rceil \ \ or\ \ j \notin N\ and\ j > \lfloor \alpha \rfloor \end{cases} \tag{3}$$

$$D^{\alpha}\big(\beta f(t) + \mu g(t)\big) = \beta D^{\alpha} f(t) + \mu D^{\alpha} g(t), \tag{4}$$

where β ,μ and C are constants, ⌈ ⌉ , ⌊ ⌋ represents the smallest, largest integer function, respectively.

2.2. Linear and nonlinear FODEs

A linear FODE of order α is provided as follows [4]:

$$D^{\alpha}y(t)=\sum_{i=0}^{s-1}a_i(t)y^{(\kappa_i)}(t)+a_s(t)y(t)+g(t),t\in[a,b] \quad (5)$$

subject to the initial conditions

$$y^{(i)}(t_0)=d_i,\quad i=0,1,\dots.,n \quad (6)$$

or boundary conditions

$$\begin{aligned}y^{(r)}(a)&=p_i,\quad r=0,1,\dots<n\\ y^{(l)}(b)&=q_i,\quad l=0,1,\dots<n\end{aligned} \quad (7)$$

where n is the number of initial and boundary conditions, $a_j(t)$ for j $=0,1,\dots,$s are real function coefficients, $n<\alpha\leq n+1,\ 0<\kappa_1<\kappa_2<\cdots<\kappa_s<\alpha$, $r+l=n$, and D^{α}denotes the fractional derivative of order α.

A non-linear FODE of order α is expressed by [4],[9]:

$$D^{\alpha}y(t)=F(t,y(t),y^{(\kappa_0)},\dots\dots.,y^{(\kappa_{s-1})}). \quad (8)$$

subject to initial or boundary conditions in Eq.(6) and Eq.(7).

2.3. Chebyshev Polynomial and Approximation

The Chebyshev polynomials are one of the functions that are defined on the interval $[-1,1]$ and fulfil the orthogonality property. Furthermore, it is a good approximation of the functions defined on the real interval $[a,b]$, and it has the following recursive form [10], [11]:

$$C_0(t)=1$$
$$C_1(t)=t \quad (9)$$
$$C_n(t)=2tC_{n-1}(t)-C_{n-2}(t),\ \textit{for all } n>1.$$

A given function $y(t)$ defined on $[a,b]$ can be approximated by Chebyshev polynomials as follows:

$$y(t)=\sum_{i=0}^{\infty}\alpha_i\,C_i(t)\approx Y(t)=\sum_{i=0}^{N}\alpha_i\,C_i(t) \quad (10)$$

where $\alpha_i,\ i=0,1,\dots$ are the unknown coefficient of approximation, N is the degree of approximation.

2.4. The Bees algorithm

The Bees algorithm is one of the metaheuristic algorithms that was first introduced in 2005 by Pham et al., through their study of the social behavior of a group of honey Bees. The metaheuristic algorithms guarantee to reach the optimal solution in the complex unconstrained optimization problem, as it consists of two important elements that enhance the optimal solution, namely exploration (local search) and exploitation (global search). The pseudo-code of the Bees algorithm is described as follows [12], [13], [14]:

1- Creation of scout Bees (s) is randomly within the search space $[Vmn,Vmx]$.

2- Calculate the fitness function for each scout bee.

3- Determine the sites of the smallest value of the fitness function called elite sites of scout Bees (el).

4- Enroll Bees around chosen sites ($s1$) and evaluate the fitness function.
5- Determine the best bee in each site (b) and harness the rest of the Bees ($s2$) to search randomly in the search space.
6- If all stopping criteria are satisfied, give the best site. Otherwise, go to step (3).

3. The Proposed Algorithm

The proposed method is based on transforming the FODE into an unconstrained optimization problem by a fitness function. The fitness function represents the error arising from the Chebyshev polynomials approximation. After substituting the truncated series in both the FODE and the initial or boundary conditions, a certain error is obtained. Consequently, the error is reduced to the lowest value. Therefore, the optimal coefficient values for Chebyshev polynomials are extracted, which are represented by Eq. (10).

The proposed algorithm is described in the following steps:
1- Selection of parameters value: scout Bee (s), elite sites (el), selected site (b), number of coefficients (nVr), search space ($[Vmn, Vmx]$), maximum number of iteration (MI), tolerance (Tol).
2- Create an approximate function $Y(t)$ as follows:

$$Y(t) = \sum_{i=0}^{nVr} \alpha_i\, C_i(t), t \in [a,b] \tag{11}$$

3- Evaluate the error function ($Error(t)$) by adding the substitution of the function $Y(t)$ in the FODE (Eq. 5 or 8) and the substitution of the initial or boundary conditions in the function $Y(t)$.
4- Calculate the fitness function as follows:

$$FF = \sum_{i=0}^{n} \frac{\sqrt{(Error(t_i))^2}}{n} + \sum_{i=1}^{nIVP} \left|y^{(i)}(a) - d_i\right| \tag{12}$$

or

$$FF = \sum_{i=0}^{n} \frac{\sqrt{(Error(t_i))^2}}{n} + \sum_{i=1}^{nBVP} \left(\left|y^{(i)}(a) - p_i\right| + \left|y^{(i)}(b) - q_i\right|\right) \tag{13}$$

where $t_0 = a, t_1 = a + h, \dots, t_n = a + nh = b,\ h > 0,\ nIVP,\ nBVP$ are the number of initial, boundary conditions respectively.
5- Apply Bees algorithm (sec. 2.4) to minimize the fitness function .
6- If $FF < Tol,\ RMSE < Tol$ or the maximum number of iterations holds stop and give the coefficients, otherwise go to step (5).

4. Numerical Examples

The proposed algorithm is implemented by using MATLAB 2020a, HP laptop, Intel(R) Core (TM) i7-4500U CPU @ 1.80GHz 2.40 GHz, RAM 16 GB. The algorithm was run 10 times to verify the reliability of the results. Parameter values are set in all examples as follows: $[Vmn, Vmx] = [-1,1]$, $[a, b] = [0,1]$, $Tol = 10^{-3}$, $MI = 100$, $s = 20$, $el = 4$, $b = 10$. The value of the fitness function, the number of iterations, the consumed time and the value are shown in Table (1). Figure (1) illustrates the approximate solutions corresponding to the exact solution and the absolute error. One can see in Figure (1), that satisfactory accuracy is obtained in most of the tests, and the error rate is between 10^{-3} and 10^{-5}. Exact solutions are also obtained in Example 4. The convergence of the proposed algorithm for all examples is shown in Figure (2).

Example 1. Consider the following linear FODE [15]:

$$(D^2 - 2D + D^{0.5} + 1)y(t) = f(t),$$

where $f(t) = t^3 + 6t - 6t^2 + \frac{16}{5\sqrt{\pi}} t^{2.5}$,

with initial conditions (I.Cs) $y(0) = y'(0) = 0$ and the exact solution $y = t^3$.

Let $nVr = 5$ and $y(t) \approx Y(t) = \sum_{i=1}^{nVr} \alpha_i C_i(t)$, then the error function is obtained as follows:

$Error(t) = |(D^2 - 2D + D^{0.5} + 1)Y(t) - f(t)|$, and the fitness function is calculated as follows:

$$FF = \sum_{i=0}^{n} \frac{\sqrt{(Error(t_i))^2}}{n} + |Y(0)| + |Y'(0)|$$

By applying the Bees algorithm, we get the following numerical solution:

$$Y(t) = 0.753423\, C_1(t) + 0.2501649\, C_3(t) + O(t^4)$$

Example 2. Suppose the linear FODE [16]:

$$(D^{1.5} - t^{1.5})y(t) = g(t),$$

where $g(t) = 4\sqrt{\frac{t}{\pi}} - t^{3.5}$, with boundary conditions $y(0) = 0$ and $y\,(1) = 1$. The exact solution is $y = t^2$.

Let $nVr = 3$ then we have: $Error(t) = |(D^{1.5} - t^{1.5})Y(t) - g(t)|$, and the fitness function is calculated as:

$$FF = \sum_{i=0}^{n} \frac{\sqrt{(Error(t_i))^2}}{n} + |Y(0)| + |Y(1) - 1|$$

Apply the proposed algorithm, we get:

$$Y(t) = 0.5068 - 0.01076\, C_1(t) + 0.5041007\, C_2(t) + O(t^3)$$

Example 3. Assume the nonlinear FODE [17]:

$$(D^{2.2} + D^{1.25} + D^{0.75})y(t) + |y(t)|^3 = g(t),$$

where $g(t) = \frac{2t^{0.8}}{\Gamma(1.8)} + \frac{2t^{1.75}}{\Gamma(2.25)} + \frac{2t^{2.25}}{\Gamma(3.25)} + \frac{t^9}{27}$, with I.Cs $y(0) = y'(0) = y''(0) = 0$, and the exact solution is $y = \frac{t^3}{3}$.

Suppose $nVr = 4$,then we have:

$Error(t) = |(D^{2.2} + D^{1.25} + D^{0.75})Y(t) + |Y(t)|^3 - g(t)|$, and the fitness function is obtained by:

$$FF = \sum_{i=0}^{n} \frac{\sqrt{(Error(t_i))^2}}{n} + |Y(0)| + |Y'(0)| + |Y''(0)|$$

Apply the Bees algorithm:

$$Y(t) = -0.0025 + 0.2539\, C_1(t) - 0.0018\, C_2(t) + 0.0821\, C_3(t) + O(t^4)$$

Example 4. Let the following nonlinear FODE [18]:

$D^\alpha u(t) + (1 + t^2)\big(y(t)\big)^2 = g(t), 0 < \alpha \le 1$

where $g(t) = \frac{t^{(1-\alpha)}}{(1-\alpha)\Gamma(1-\alpha)}(1 + t^2)(1 + t)^2$, with the initial conditions $y(0) = 1$ and analytical solution is $y = 1 + t$.

Suppose $nVr = 2, \alpha = 0.2$ then we have:

$Error(t) = \left| D^\alpha Y(t) + (1 + t^2)\big(Y(t)\big)^2 - g(t) \right|$, so that

$$FF = \sum_{i=0}^{n} \frac{\sqrt{(Error(t_i))^2}}{n} + |Y(0) - 1|$$

By using the proposed algorithm, we get the exact solution as follows:

$$Y(t) = 1 + C_1(t) + O(t^2) = 1 + t.$$

Example 5. Assume the following nonlinear FODE [18]:

$$(D^4 + D^{3.5})y(t) + y^3(t) = t^9$$

with the boundary conditions $y(0) = y''(0) = 0,\ y(1) = 1, y'''(0) = 6$, and $y = t^3$ is the analytical solution.

Let $nVr = 4$,then we have: $Error(t) = |(D^4 + D^{3.5})Y(t) + Y^3(t) - t^9|$,so that

$$FF = \sum_{i=0}^{n} \frac{\sqrt{(Error(t_i))^2}}{n} + |Y(0)| + |Y'(0)| + |Y(1) - 1| + |Y'''(0) - 6|$$

By utilize the Bees algorithm, we have

$$Y(t) = -0.002024 + 0.75447\, C_1(t) - 0.00122\ C_2(t) + 0.2503\ C_3(t) + O(t^4)$$

Example 6. Consider the nonlinear FODE [19]:

$$(D^3 + D^{2.5})y(t) + y^2(t) = t^4.$$

with the boundary conditions $y(0) = 0, y''(0) = 2, y(1) = 1$ and exact solution $y = t^2$.

Let $nVr = 4$,then we have:

$Error(t) = |(D^3 + D^{2.5})Y(t) + Y^2(t) - t^4|$, so that

$$FF = \sum_{i=0}^{n} \frac{\sqrt{(Error(t_i))^2}}{n} + |Y(0)| + |Y''(0) - 2| + |Y(1) - 1|$$

By utilize the Bees algorithm, we get

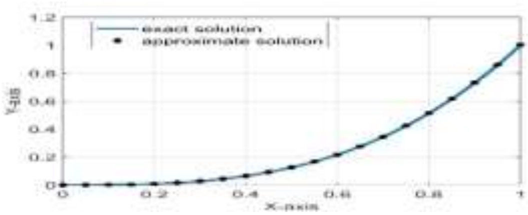
$$Y(t) = 0.499804 - 0.00025\, C_1(t) + 0.499939\ C_2(t) - 0.00019\ C_3(t) + O(t^4)$$

Table 1: This table shows the results of all examples

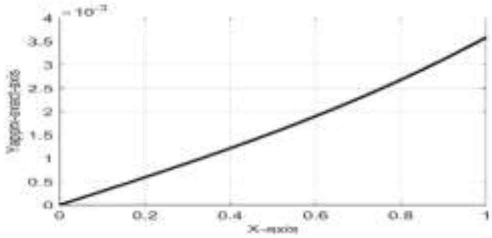
Example	Fitness function	Number of iterations	Consumed time (s)	RMSE
1	8.4193e-04	29	39.12	1.63e-03
2	5.4421e-03	23	11.46	7.64e-04
3	2.3190e-03	17	12.88	9.43e-04
4	6.7704e-17	2	02.76	0.00e-00
5	1.4787e-03	31	17.74	6.72e-04
6	2.8814e-03	30	23.91	2.07e-04

5. Conclusions

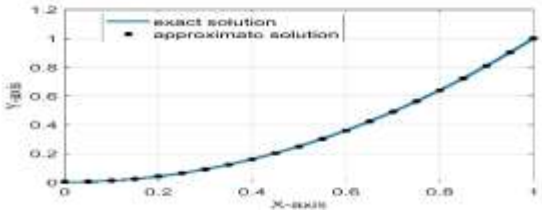
Recently, interest has been paid to numerical methods based on an operational matrix that was derived according to the Caputo concept and using one of the mother wavelets. Although these methods give good results, solving non-linear FODEs may take considerable computational time and effort. Therefore, an algorithm is applied to multiple types of FODEs, as it was applied to a set of numerical examples. The algorithm is characterized by the speed in reaching the solution, in contrast to the use of operational matrices.



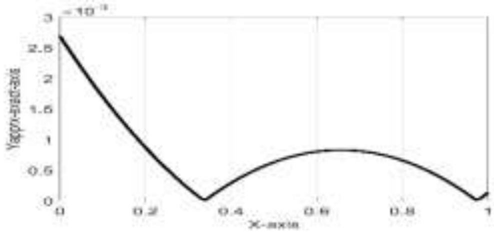
(a) Plotting of Ex1.



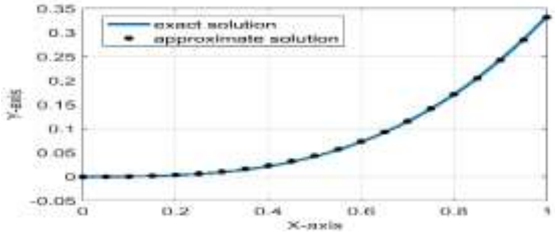
(b) Absolute error of Ex1.



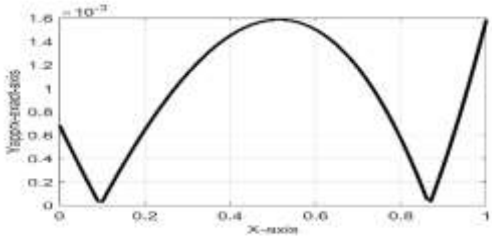
(c) Plotting of Ex2.



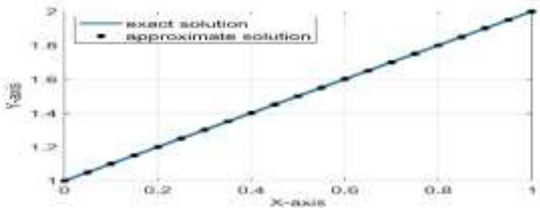
(d) Absolute error of Ex2.



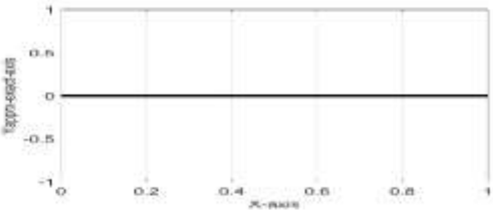
(e) Plotting of Ex3.



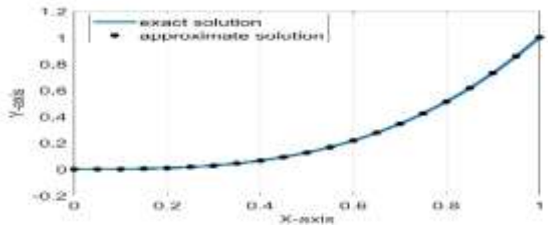
(f) Absolute error of Ex3.



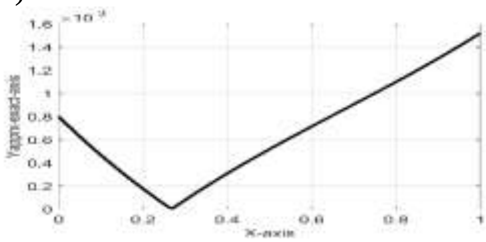
(g) Plotting of Ex4.



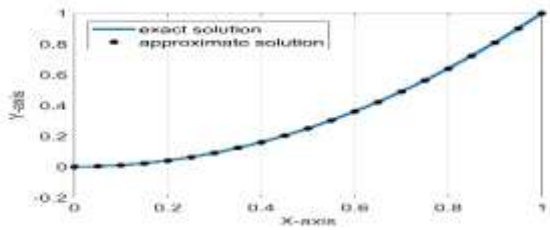
(h) Absolute error of Ex4.



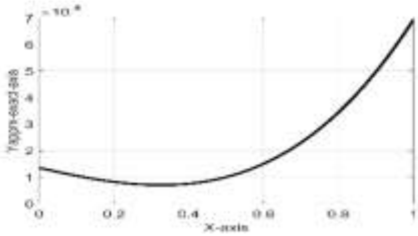
(i) Plotting of Ex5.



(j) Absolute error of Ex5.

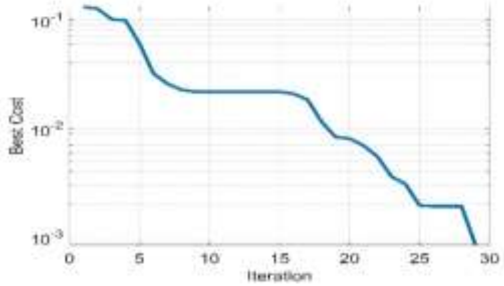


(k) Plotting of Ex6.

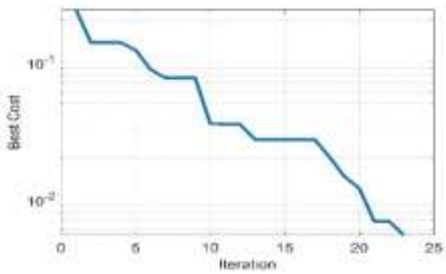


(l) Absolute error of Ex6.

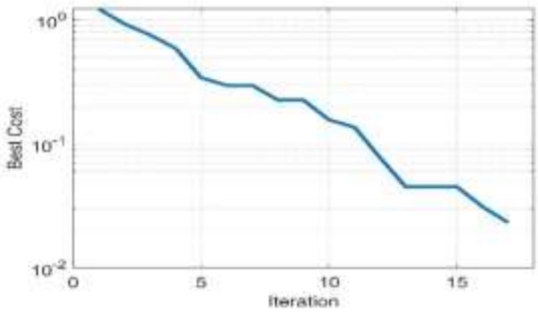
Figure 1: This figure shows the comparison between the exact solution and the approximated solution. (a),(c),(e),(g),(i),(k) .Plot of example(1) to (6), and (b),(d),(f),(h),(j),(l) absolute error of example(1) to (6).



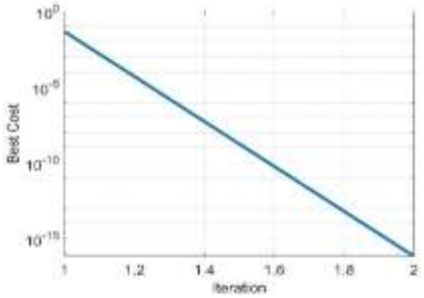
(a) Convergence of Ex1.



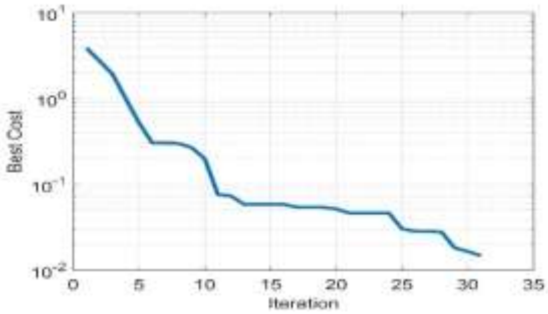
(b) Convergence of Ex2.



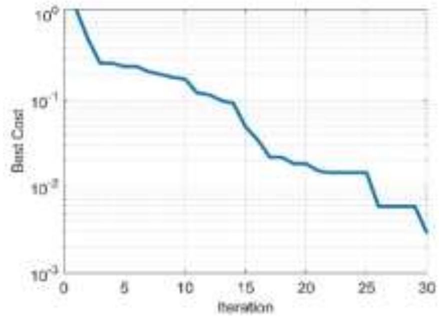
(c) Convergence of Ex3.



(d) Convergence of Ex4.



(e) Convergence of Ex5.



(f) Convergence of Ex6.

Figure 2: This figure shows the convergence of the proposed algorithm. (a),(b),(c),(d),(e),(f) convergence of example(1) to (6).

It can be noted from Figure (2) that for all examples, the convergent of the algorithm is reached with less than 35 iterations. This gives a promised results with acceptable errors.

References:

[1] F. Gao and C. Chi, "Improvement on conformable fractional derivative and its applications in fractional differential equations," *J. Funct. Spaces*, vol. 2020, 2020.

[2] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Topics in fractional differential equations*, vol. 27. Springer Science & Business Media, 2012.

[3] Y. Li and W. Zhao, "Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations," *Appl. Math. Comput.*, vol. 216, no. 8, pp. 2276–2285, 2010.

[4] A. Seçer, S. Altun, and M. Bayram, "Legendre wavelet operational matrix method for solving fractional differential equations in some special conditions," 2019.

[5] M. H. Atabakzadeh, M. H. Akrami, and G. H. Erjaee, "Chebyshev operational matrix method for solving multi-order fractional ordinary differential equations," *Appl. Math. Model.*, vol. 37, no. 20–21, pp. 8903–8911, 2013.

[6] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, "A new Jacobi operational matrix: an application for solving fractional differential equations," *Appl. Math. Model.*, vol. 36, no. 10, pp. 4931–4943, 2012.

[7] E. Keshavarz, Y. Ordokhani, and M. Razzaghi, “Bernoulli wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations,” *Appl. Math. Model.*, vol. 38, no. 24, pp. 6038–6051, 2014.

[8] A. Saadatmandi, “Bernstein operational matrix of fractional derivatives and its applications,” *Appl. Math. Model.*, vol. 38, no. 4, pp. 1365–1372, 2014.

[9] N. A. Khan, O. Ibrahim Khalaf, C. Andrés Tavera Romero, M. Sulaiman, and M. A. Bakar, “Application of Intelligent Paradigm through Neural Networks for Numerical Solution of Multiorder Fractional Differential Equations,” *Comput. Intell. Neurosci.*, vol. 2022, 2022.

[10] J. C. Mason and D. C. Handscomb, *Chebyshev polynomials*. CRC press, 2002.

[11] T. J. Rivlin, *Chebyshev polynomials*. Courier Dover Publications, 2020.

[12] X.-S. Yang, “Nature-Inspired Computation and Swarm Intelligence: Algorithms, Theory and Applications,” 2020.

[13] D. T. Pham, A. Ghanbarzadeh, E. Koc, S. Otri, S. Rahim, and M. Zaidi, “The bees algorithm,” *Tech. Note, Manuf. Eng. Centre, Cardiff Univ. UK*, 2005.

[14] D. T. Pham, A. Ghanbarzadeh, E. Koç, S. Otri, S. Rahim, and M. Zaidi, “The bees algorithm—a novel tool for complex optimisation problems,” in *Intelligent production machines and systems*, Elsevier, pp. 454–459, 2006.

[15] A. H. Bhrawy, M. M. Tharwat, and M. A. Alghamdi, “A new operational matrix of fractional integration for shifted Jacobi polynomials,” *Bull. Malays. Math. Sci. Soc*, vol. 37, no. 4, pp. 983–995, 2014.

[16] R. B. Albadarneha, I. M. Batihab, and M. Zurigatb, “Numerical solutions for linear fractional differential equations of order 1< α< 2 using finite difference method (ffdm),” *Int. J. Math. Comput. Sci*, vol. 16, no. 1, pp. 103–111, 2016.

[17] E. H. Doha, A. H. Bhrawy, D. Baleanu, and S. S. Ezz-Eldien, “On shifted Jacobi spectral approximations for solving fractional differential equations,” *Appl. Math. Comput.*, vol. 219, no. 15, pp. 8042–8056, 2013.

[18] E. Babolian, A. R. Vahidi, and A. Shoja, “An efficient method for nonlinear fractional differential equations: combination of the Adomian decomposition method and spectral method,” *Indian J. Pure Appl. Math.*, vol. 45, no. 6, pp. 1017–1028, 2014.

[19] A. Saadatmandi and M. Dehghan, “A new operational matrix for solving fractional-order differential equations,” *Comput. Math. with Appl.*, vol. 59, no. 3, pp. 1326–1336, 2010.