DOI: 10.24996/ijs.2023.64.1.29

ISSN: 0067-2904

Utilization of Dynamical System on Enfolding Semi-group

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Received: 19/2/2022 Accepted: 16/6/2022 Published: 30/1/2023

Abstract

In this paper, we use the definition of the action on the set of semi-group of the structure of this research .We introduce the concepts of K-system which is a triple (K,X,φ) such that X is a Hausdorff compact space called phase space, K is a semi-group of transformations with a continuous action φ of K on X. We study and proof some theoretical properties related with that system. We also introduce the concept of Enfolding semi-group $\mathcal{E}(K,X)$,and we prove that it is a compact right topological semi-group. In addition, we study the left and right ideals in the Enfolding semi-group. By using the dynamical system, we reflect various properties concerning with its structure for the Enfolding semi-group. Furthermore, we describe the connections between the algebraic and topological properties of the Enfolding semi-group space.

Keywords: Action function, Dynamical System, K- System, Enfolding semi-group, Ellis semi-group.

استخدام النظام الديناميكي في المنطوي شبه الزمرة

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الخلاصة:

في هذا البحث تم استخدام تعريف التمثيل المعرف على شبه الزمرة في البنية الرئيسية في هذا البحث. تم تقديم مفهوم النظام$-K$ ويعبر عنه بالشكل (K,X,φ) حيث ان X عبارة فضاء متراص هاوزدوروف يسمى بال phase space وان K شبه زمرة و φ هي دالة التمثيل المستمرة المعرفة بين شبة الزمرة K والفضاء X وتمت دراسة وبرهنة بعض الخواص المرتبطة لهذا النظام. أيضا تم طرح مفهوم المنطوي شبه الزمرة $\mathcal{E}(K,X)$ وتم البرهنة انه فضاء تبولوجي يميني متراص شبة زمرة وتم دراسة المثاليات اليمينة واليسارية له. وباستخدام النظام الديناميكي مع الفضاء المنطوي شبة الزمرة تم برهنة العديد من النظريات والخصائص وتم وصف العلاقة الجبرية والتبولوجية في الفضاء المنطوي شبة الزمرة.

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1. Introduction:

Consider a system that is defined on a compact phase space X and a semi-group of K transitions. Measuring of the system is the complexity of the algebraic and topological structure of the Enfolding semi-group on the system. For example, if all transformations in the system consist of contractions that towards to one fixed point of X, then the stationary designation of that point is the only additional transformation in the Enfolding semi-group that was not already in K. Historically, the Enfolding semi-group is the closure of the set of continuous functions on a phase space X that was used to study the dynamical system. For more details see Anima Nagar [1] and Khatam AD. Zaghir [2]. We showed that product topology D^D with composition is a right topological semi-group iff left translations is continuous then prove closure of semi-group is a semi-group.

Preliminaries:

Let K be a semi-group given with Hausdorff topology. For$k,r \in K$ with productkr, we write $\lambda_r(k) = rk = \rho_k(r)$ the functions λ_r and ρ_k on K are called left translations and right translations, respectively. The function λ_r is a left continuous. The semigroup K is called the left topological and the function ρ_k is a right continuous. The semi-group K is called the right topological if all ρ_k and λ_r are continuous, then the semi-group K is semi topological as given [3]. The Enfolding semi-group $\mathcal{E}(K,X)$ is a compact right topological semi-group that is proved by Tychonoff's theorem. The Enfolding semi-group has been established to be an essential tool in the abstract theory of topological dynamical systems. We look to the algebraic and topological properties of the Enfolding semi-group. The ideal is a non-empty set $I \subseteq \mathcal{E}(K,X)$ which is both left and right ideal where $\mathcal{E}(K,X) \cdot I \subseteq I$ is a left ideal and $I \cdot \mathcal{E}(K,X) \subseteq I$ is a right ideal. In this paper, Zorn's lemma [4] is also used throughout this paper and in proofs of some of our results. This is if S is a partially ordered set and if every totally ordered subset of S has an upper bound, then S contains a maximal element.[4]. The left ideal I of Enfolding semi-group is minimal iff I is closed in $\mathcal{E}(K,X)$ and I does not contain any other proper subset left ideal. The set of Enfolding $\mathcal{E}(K,X)$ contains an idempotent. If $I.J \subseteq \mathcal{E}(K,X)$ be a minimal ideal in $\mathcal{E}(K,X)$ and $u^2 = u \in I$ be an idempotent. Then there exists $v \in J$ is a unique idempotent with $uv = u$ and $vu = v$.

Definition 1.1 [4]**:** Let K be semi-group and let X be a non-empty set. The action of K on X (we simply call that K acts on X) is a function $\varphi : K \times X \to X$, $(k, t) \to kt$ such that $(kr)t = k(rt)$ for all $t \in X$ and $k, r \in K$.If K has an identity say e, then $e \cdot t = t$ for all $t \in X$.

Definition 1.2 [5]**:** A Dynamical System will be denoted by a pair (K, X), where X is a topological space acting on semi-group K on X. The set $Kt = \{kt : t \in X\}$, for all $t \in X$, is called the orbit of t and $\Gamma_{(t)}$ will be denoted the set of the orbit closure of Kt.

In fact, one can say that a Topological Dynamics is the study of orbits for all points in X.

Definition 1.3 [6]**:** A K-System is a triple (K,X,φ) such that X is a Hausdorff compact space, is called phase space, and $\varphi: K \times X \to X$ is a continuous action of K on X, we write $\varphi(k,t) = k \cdot t = \varphi^k(t)$.

A subset A of X in a K-system is invariant if $KA = \{ta \mid a \in A, t \in K\} \subseteq A$.

Definition 1.4 [7]**:**A homomorphism from two K-systems (K,X) and (K,Z) is a continuous surjection function,$\pi: X \longrightarrow Z$ satisfying $\pi(ka) = k\pi(a)$ for all $k \in K$ and $a \in X$. In addition, if π is one to one, then it is called an isomorphism of K-systems.

Next, we present our main definition which will be used throughout this work.

Definition 1.5: Let X be a compact Hausdorff topological space, and X^X denotes the collection of all continuous functions from X to itself, which is a semi-group under the composition, provided with the product topology, or the topology point wise convergence. Let $T = \{f: X \to X\}$ be a contained subset in X^X, then the closure of set T is called Enfolding semi-group of T which is denoted by $\mathcal{E}(K,X)$, given with the topology of pointwise convergent.
The motivation for studying the Enfolding semi-group is to understand the algebraic properties of a K-system.
One reason for the rarity and difficulty of examples of Enfolding semi-groups is that these objects are usually non- metrizable.

Example 1.6 [2]**:** For each $n \in \mathbb{N}$,
let $K_n = \{(r,\theta) = \left(\frac{1}{2^n}, \frac{2x\pi}{2^n}(mod\ 2\pi)\right):\ x = 0, 1, 2, \ldots\ldots.\}$ and
$K = \bigcup_{n\in\mathbb{N}} K_n \cup \{(0,0)\}$ which is a countable a subspace of R^2.
Define $f: K \to K$ as $f(r,\theta) = (r, \theta + 2\pi r(mod 2\pi))$.
For any $s \in \mathbb{N}$,

$$f^s\left(\frac{1}{2^n}, \theta\right) = \left(\frac{1}{2^n}, \theta + \frac{2s\pi}{2^n}(mod 2\pi)\right).$$

Let s be a $2-$adic integer. Suppose $s = 2^{s_1} + 2^{s_2} + \cdots\cdots + 2^{s_r}$, then $f^s\left(\frac{1}{2^n}, \theta\right) = \left(\frac{1}{2^n}, \theta + 2\pi\left(\frac{2^{s_1}+2^{s_2}+\cdots+2^{s_r}}{2^n}\right)(mod\ 2\pi)\right) = \left(\frac{1}{2^n}, \theta + 2\pi\left(\frac{1}{2^{n-s_1}} + \frac{1}{2^{n-s_2}} + \cdots\cdots + \frac{1}{2^{n-s_r}}\right)(mod\ 2\pi)\right)$.
Let $a = \cdots\cdots 10101 = 1 + 4 + 16 + \ldots\ldots$ be a $2-adic$ integer. Then for the function f_a which is defined as $f_a(r,\theta) = (r, \theta + 2x\pi a_n(mod(2\pi))$, where $a_{2x} = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\cdots + \frac{1}{2^{2x-2}} + \frac{1}{2^{2x}}$ and $a_{2x+1} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots\cdots + \frac{1}{2^{2x-1}} + \frac{1}{2^{2x+1}}$. we see that f_a will be a member of $\mathcal{E}(K)$ corresponding to a.

Remark 1.7: If we adjoined the identity mapping on X, then the Enfolding semi-group $\mathcal{E}(K,X)$ is called the Ellis semi-group.
The next theorem helps to prove that the Enfolding semi-group is a right topological semi-group.

Theorem 1.8: Let (D,τ) be a topological space and V be the product topology on D. Then $(D^D, \circ, V)$ is a right-topological semi-group. Moreover, for each $h \in D^D$, h is continuous if and only if λ_h is continuous where $\lambda_h(g) = h \circ g = \rho_g(h)$.

Proof: Suppose that $h \in D^D$ and $< g_i >_{i\in I}$ is a net convergence to g in the product topology D^D. Since that a net $< x_i >_{i\in I}$ is convergent to x in D iff $< f_i(x) >_{i\in I}$ is convergent to $f(y)$ for every $y \in D$.This implies that $< g_i(h(x)) >$ converges to $g(h(x))$ in D. Therefore, $< g_i \circ h >_{i\in I}$ convergent to $g \circ h$ in D^D, hence $g_i \circ h$ is continuous. This shows ρ_h is continuous. Hence, the set of all function from $D \to D$ is a right topological semi-group. For the second part. Suppose that h is-continuous. Hence for every given net $< g_i >_{i\in I}$ convergence to g in the product topology of D^D. Then, $< h(g_i(x)) >_{i\in I}$ is convergent to $h(g(x))$ for every $x \in D$. This implies that $< h \circ g_i >_{i\in I}$ is convergent to $(h \circ g)(x)$. Hence $h \circ g_i$ is continuous that means we show that λ_h is continuous.

Conversely, we suppose λ_h is continuous. Let $< x_i >_{i\in I}$ be a net convergent to x in D.
Define $g_i : D \longrightarrow D$ such that $g_i(x) = x_i \ for\ all\ i \in I$.
And $g : D \longrightarrow D$ such that $g(x) = x$.
Therefore, $< g_i >_{i\in I}$ converge to g in D^D so $< h \circ g_i >_{i\in I}$ converges to $h \circ g$. This means $< h(x_i) >_{i\in I}$ converges to $h(x)$ .Therefore, h is continuous.
The next two results prove that the Enfolding is a right- topological semi-group.

Proposition 1.9: On the right -topological semi-groupD^D, let K be a subset of the topological center of D^D. If K is a semi-group, then $\overline{K}$ is a semi-group.

Proof: Let m , $n \in \overline{K}$. Let U be any neighborhood of mn. By Theorem 1.8 since ρ_n is continuous, then there exists a neighborhood V of m such that $\rho_n(V) = Vn \subseteq$ U.
Since $m \in \overline{K}$, so $V \cap K \neq \emptyset$. Let $m_1 \in V \cap K$, hence $\lambda_{m_1}(y) = m_1 n = \rho_n(m) \in U$. Note that $m_1 \in K \subseteq$ center D^D, which implies that $\lambda_{m_1}$is continuous .
Moreover, there is a neighborhood W of n with $\lambda_{m_1}(W) \subseteq U$. Since $n \in \overline{K}$, then $W \cap K \neq \emptyset$.
Let $n_1 \in W$ and $\lambda_{m_1}(n) \in \lambda_{m_1}(W) \subseteq U$.

$$\Longrightarrow m_1 n_1 \in K$$
$$\Longrightarrow \ U \cap K \neq \emptyset.$$

Hence, $mn \in \overline{K}$.

Lemma 1.10: The Enfolding semi-group $\mathcal{E}(K,X)$ is a compact right- topological semi-group.

Proof: Since X is compact, then by Tychonoff's theorem the $X{\times}X{\times}X$ ... is compact. But, $T \subseteq X{\times}X{\times}X$ ... and $\mathcal{E}(K,X)$ is a closed subset of- compact-Hausdorff space. Therefore, $\mathcal{E}(K,X) = c\ell(T) \subseteq\ X{\times}X{\times}X$ ... is compact-Hausdorff space.

Remark 1.11: The algebraic structure of the Enfolding semi-group gives some important characterization of the dynamical system properties of the K-system (K,X).
1-The map $\Phi : \beta K \rightarrow \mathcal{E}(K,X)$ is both a semigroup homomorphism and a K-system.
2- The map $\psi : \mathcal{E}(K,X) \rightarrow X$ define as $t \mapsto tx$ is a K -system homomorphism for all $x \in X$.
3- The map $\Phi : \beta K \rightarrow \mathcal{E}(\beta K,K)$ is an isomorphism.
4- Let $\varphi : (K,X) \rightarrow (K,Z)$ be a homomorphism of K-system, then $\varphi(tx) = t\varphi(x)$ for all $x \in X$ and $t \in \beta K$.

2- New result by ideals in Enfolding semi-group $\mathcal{E}(K,X)$

Definition 2.1: Let (K,X,φ) be a K-system and $\mathcal{E}(K,X)$ be its Enfolding semi-group. Then for a non-;empty set $I \subseteq \mathcal{E}(K)$ is a left- ideal if $\mathcal{E}(K,X) \cdot I \subseteq I$; that means for $\ell \in I$ and $q \in \mathcal{E}(K,X)$ such that $q\ell \in I$. I is called right- ideal if $\ell q \in I$. Moreover, $I \in \mathcal{E}(K,X)$ is an ideal iff I is both right and left- ideal.

Definition 2.2: A left ideal I on Enfolding semi-group is minimal if and only if I is closed in $\mathcal{E}(K.X)$ and I does not- contain any other proper subset left-ideal.

Lemma 2.3: Let $\mathcal{E}(K,X)$ be an Enfolding semi-group in K-system (K,X). Then any left ideal in $\mathcal{E}(K,X)$ contains a minimal-ideal.

Proof: Let I be any left-ideal of $\mathcal{E}(K,X)$, and let $\mathcal{B} = \{J: J$ is a closed-left ideal of $\mathcal{E}(K,X)$ and $J \subseteq I\}$.
Applying -Zorn's lemma to $\mathcal{B}$, one can get a left ideal T minimal -among all closed left-ideal contained in I.
Lemma 2.4: Let X be a-compact topological space then the set of Enfolding $\mathcal{E}(K,X)$ contains an idempotent.

Proof: Let $\mathcal{H}$ be a minimal subset of $\mathcal{E}(K,X)$ defined by $\{A \subseteq \mathcal{E}(K,X),\ \mathcal{E}(K,X) \neq \emptyset,\ AA \subseteq A,\ A$ is compact$\}$. Since $\mathcal{E}(K,X)$ satisfies that properties, so $\mathcal{H} \neq \emptyset$.
We claim $\mathcal{H}$ has a minimal set of this kind. Let C be a chain in $\mathcal{H}$ which is a collection of a closed subset of $\mathcal{E}(K,X)$. This chain will satisfy the finite intersection property. Therefore, $\cap\, C \neq \emptyset$ is trivially compact. Hence, $\cap\, C \in \mathcal{H}$. By Zorn's lemma, let A be a minimal element of $\mathcal{H}$. We need to show that $y \cdot y = y,\ \forall\, y \in A$; that means $Ay = A$. Take any $u \in A$. $Au \subseteq A$ is compact since $Au = \rho_u\,(A)$ is a continuous image of compact space Let $\mathcal{B} = \{v: vu = u\}$. Then, $\mathcal{B} \neq \emptyset$ because $u \in A = Au$. Moreover, since $\mathcal{B} = A \cap \rho_u^{-1}[\{u\}]$,then $\mathcal{B}$ is closed, this implies $\mathcal{B}$ is compact. Given $x,z \in \mathcal{B}$, so $xz \in AA \subseteq A$ and $xzy = xy = y$. That implies $xz \in \mathcal{B}$ and ,thus $\mathcal{B} \in \mathcal{H}$.Since $\mathcal{B} \in \mathcal{H}$ and A is minimal, thus $\mathcal{B} = A$, hence $u \in \mathcal{B}$, so that $u \cdot u = u^2 = u$.

Remark 2.5: Let (K,X) be K-system and $I \subseteq \mathcal{E}(K,X)$ be a minimal-ideal in $\mathcal{E}(K,X)$, hence J is the set of idempotent of I is non-empty.
We note that to the result that is given as an exercise in [4].

Proposition 2.6: Let (K,X) be K-system and $I \subseteq \mathcal{E}(K,X)$ be a minimal ideal in $\mathcal{E}(K,X)$then:
1) For all $q \in I$ and $v \in J$ then $vq = q$.
2) For all $v \in J$ then Iv is a group with identity v.
3) The partition of I is $\{Iv: v \in J\}$.

Proof:
1) Let $q \in I$, and $v \in J$.In order to prove that $vq = q$, we have vI is an ideal subset of I. So $vI = I$, then there exists $p \in I$ with $vp = q$. This implies $vq = vvp = vp = q$.
2- Suppose $p \in Iv$, then there exists $q \in I$ with $qv = p$, and we have $pv = qvv = qv = p$, so v is both a left and right identity for Iv. Since I is an ideal, and pI is an ideal subset of I, then $pI = I$. Hence, there exists $r \in I$ with $pr = v$, and $p(rv) = (pr)v = vv = v$.
Note that $(rp)(rp) = r(pr)p$
$= r(vp)$
$= rp$
Hence, $(rv)p = wp$
$= (rp)v$
$= v$
This implies wv is a left and right inverse of q in Iv.
3) Let $q \in I$, and qI be an ideal subset of I, then $qI = I$. Suppose $S = \{p \in I \mid qp = q\} = L_P^{-1}(q)$ is non-empty closed sub semi-group of I, see proof Proposition 3.9 [1].
Hence, there exists an idempotent $u \in J$ with $qu = q$, so $q \in Iu$. Then $I = \cup\,\{Iv \mid v \in J\}$, and let $u.v \in J$ and $q \in Iv \cap Iu$. So that $q = qu = qv$ and there exists $p \in Iv$ with $pq = v$. This leads $t\, u = vu = (pq)u = p(qu) = pq = v$.
The next proposition shows that if we have Y be the set of all idempotent in $\mathcal{E}(K,X)$. One can define an equivalence-relation $\sim$ on Y as x~y iff $xy = y$ and $yx = x$. Then, we say x and y -are equivalent.

Proposition 2.7: Let (K,X) be a K-system. If $I,J \subseteq \mathcal{E}(K,X)$ be a minimal -ideal in $\mathcal{E}(K,X)$ and $u^2 = u \in I$ be an-idempotent. Then, there is $v \in J$ is a unique-idempotent with $uv = u \, and \, vu = v$.

Proof: Let $u^2 = u \in I$. And uJ is a closed ideal subset of I, which means $uJ = I$. Suppose $A = \{j \in J \mid uj = u\} \neq \emptyset$. Then $A = J \cap L_u^{-1}(u)$ is closed, and $A^2 \subseteq A$. Then there exists $v^2 = v \in A$, so $uv = u$.

Similarly, there exists $r^2 = r \in I$ with $vr = v$. We get $r = ur = uvr = uv = u$. In the same way, we can get $vu = v$.

Now suppose $\gamma^2 = \gamma \in J$ with $u\gamma = u \, and \, \gamma u = \gamma$. We need to show $v = \gamma$. Hence, $\gamma = \gamma u = \gamma uv = \gamma v$. This means $\gamma \in Jv \cap J\gamma$, we get $v = \gamma$, v is unique.

Conclusions

Studying Enfolding semi-group gives the same properties of the dynamic system . This work can be used by restrictions to study Ramsey theory, which is an important application in number theory.

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