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Generalized Permuting 3-Derivations of Prime Rings

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Abstract
This work generalizes Park and Jung's results by introducing the concept of generalized permuting 3-derivation on Lie ideal.

Keywords: Permuting 3-derivation, Lie ideal, prime ring, commuting, centralizing.

تعميم المشتقات الثلاثية التبادليه للحلقات الاولية على مثالي لي

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الخلاصة
هذا البحث يعمم نتائج park و Jung وذللك بتقديم مفهوم تعميم المشتقات الثلاثيه التبادليه على مثالي لي.

Introduction

Throughout this paper, R will represent an associative ring, and $Z(R)$ will be its center. Let $x, y \in R$, the commutator $xy - yx$ will be denoted by $[x, y]$ [1]. A ring R is said to be prime ring if $aRb = (0)$ implies that $a = 0$ or $b = 0$ such that $a, b \in R$ [2]. An additive mapping d from a ring R into R is called a derivation of R if $d(xy) = d(x)\,y + x\,d(y)$ for all $x, y \in R$[1] . In 1987 the concept of a symmetric bi-derivation has been introduced by Maksa in [3], by a bi-derivation we mean a bi-additive map $d : R \times R \to R$ is such that if $(xy, z) = d(x, z)y + xd(y, z)$,$d(x, yz) = d(x, y)z + yd(x, z)$. In 1989 J. Vukman [4,5] investigated symmetric bi-derivations on prime and semiprime rings.

A ring R is said to be n-torsion-free where $n \neq 0$ is an integer if whenever $na = 0$ with $a \in R$, then $a = 0$ [2].

Let S be a nonempty subset of R. Then a map $f : R \to R$ is said to be commuting (resp. centralizing) on S if $[f(x), x] = 0$ (resp. $[f(x), x] \in Z(R)$) for all $x \in S$ [1]. An additive subgroup $U \subseteq R$ is called a Lie ideal of R if whenever $u \in U, r \in R$ and $[U, r] \in U$ [2]. A Lie ideal U of R is called a squar closed lie ideal of R if $u^2 \in U$, for all $u \in U$ [6]. A squar closed Lie ideal of R such that $U \not\subseteq Z(R)$ is called an admissible Lie ideal of R [7]. In 2007, Park and Jung's introduced the concept of permuting 3-derivation and they are studied this concept as centerilizing and commuting [1]. The history of commuting and centralizing mapping goes back to 1955, Divinsky [8]. Posner initiated several aspects of a study of commuting and centralizing derivations on prime ring [9]. In this paper we introduce the concept of generalized permuting 3-derivation and study the commuting and centeralizing of this concept and commutativity of Lie ideal under certain conditions.

Preliminaries

The following lemmas are basic to get the main results

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Lemma(2.1) [10]

Let U be a Lie ideal of a prime ring R and $[t,U] \subseteq Z(R)$, then either $t \in Z(R)$ or $U \subseteq Z(R)$.

Lemma(2.2) [10]

Let U be a Lie ideal of a prime ring R such that $u^2 = 0$, for all $u \in U$ then $U = 0$.

Lemma(2.3) [11]

Let R be is 2-torsion free semiprime and U is commutative Lie ideal, then U contiand in $Z(R)$.

Lemma (2.4) [12]

Let R be prime ring of $char. \neq 2$ and U be a nonzero an admissible Lie ideal I of R, then U contains a nonzero ideal of R.

Lemma(2.5) [13]

Let R be a prime ring of $char. \neq 2$ and U be a Lie ideal of R with $U \not\subset Z(R)$, if a,b ∈ R and $aUb = 0$, then either $a = 0$ or $b = 0$.

Definition (2.6) [1]

A map $d: R \times R \times R \to R$ is said to be permuting if the equation $d(x_1, x_2, x_3) = d(x_{\Pi(1)}, x_{\Pi(2)}, x_{\Pi(3)})$ holds for all $x_1, x_2, x_3 \in$ R and for every permution { Π(1), Π(2), Π(3)} .

Definition (2.7)[1]

A 3-derivation map $d: R \times R \times R \to R$ is said to be permuting 3-derivation if the following equations are identical :

$d(xw, y, z) = d(x, y, z)w + x\,d(w, y, z)$,

$d(x, yw, z) = d(x, y, z)w + y\,d(x, w, z)$ and

$d(x, y, zw) = d(x, y, z)w + z\,d(x, y, w)$.

Definition (2.8) [1]

A map $\delta_d : R \to R$ which is defined by $\delta_d(x) = d(x, x, x)$ for all $\in R$, where d is permuting map is called the trace of d.

Theorem (2.9) [14]

Let R be a 6-torsion free prime ring and U be an admissible Lie ideal of R. Suppose that there exists a permuting 3-derivation $d : U \times U \times U \to R$ such that the trace δ_d of d is commuting on U. Then $d = 0$ on R × $R \times R$.

Theorem (2.10) [14]

Let R be a 6-torsion free prime ring and U be an admissible Lie ideal of R. Suppose that there exists a permuting 3-derivation $d : U \times U \times U \to R$ such that the trace δ_d of d is centralizing on U. Then δ_d is commuting on R × $R \times R$.

Now, we introduce the concept of generalized permuting 3-derivation to get our main results.

Definition (2.11)

Let U be a Lie ideal of R. A 3-additive map $F: U \times U \times U \to R$ is said to be a generalized 3-derivation if there exists a 3 – derivation $d: U \times U \times U \to R$ such that :

$F(xw, y, z) = F(x, y, z)w + x\,d(w, y, z)$

$F(x, yw, z) = F(x, y, z)w + y\,d(x, w, z)$

$F(x, y, zw) = F(x, y, z)w + z\,d(x, y, w)$, for all $, y, z, w \in U$.

Definition (2.12)

Let U be a Lie ideal of R. A generalized 3-derivation map $F: U \times U \times U \to R$ is said to be a generalized permuting 3-derivation if there exist a permuting 3-derivation $d : U \times U \times U \to R$ such that the equations in definition (2.11)).(2.10) are equal to each other.

Example (2.13)

Let S be a commutative ring and $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in S\}$ with usual addition and multiplication is a ring. Now $U = \{\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in S\}$ is a Lie ideal of R. Define $F : U \times U \times U \to R$ such that

$$F\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & e \\ 0 & h \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & bdh \end{pmatrix}, \text{ for all } \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & e \\ 0 & h \end{pmatrix} \in U .$$

Then by definition (2.11) F is generalized permuting 3-derivation since there exists a permuting 3-derivation $d: U \times U \times U \to R$ defined by

$\mathrm{d}\left(\begin{pmatrix}0 & a\\ 0 & b\end{pmatrix},\begin{pmatrix}0 & e\\ 0 & h\end{pmatrix},\begin{pmatrix}0 & c\\ 0 & d\end{pmatrix}\right)=\begin{pmatrix}0 & ace\\ 0 & 0\end{pmatrix}$, for all $\begin{pmatrix}0 & a\\ 0 & b\end{pmatrix},\begin{pmatrix}0 & c\\ 0 & d\end{pmatrix},\begin{pmatrix}0 & e\\ 0 & h\end{pmatrix}\in U.$

Main Results

We begin with following lemmas which are basic to get the main result.

Lemma (3.1)

Let U be a Lie ideal of R and δ_F be the trace of permuting 3-additive map $F: U\times U\times U\rightarrow R$. Then

$[\delta_F(x+2y),x+2y]+[\delta_F(x),x]=2[\delta_F(\mathrm{x+y}),\mathrm{x+y}]+6[F(x,x,y),y]+18[F(x,y\,y),y]$ $+12[\delta_F(y),y]+6[F(x,y,y),x]+6[\delta_F(y),x]+2[\delta_F(y),y]$, for all $x,y\in U$.

proof:

$[F(x+y,x+y,x+y),x+y]$

$=[F(x,x+y,x+y)+F(y,x+y,x+y),x+y]$

$=[F(x,x,x+y)+F(x,y,x+y)+F(y,x,x+y)+F(y,y,x+y),x+y]$

$=[F(x,x,x)+F(x,x,y)+F(x,y,x)+F(x,y,y)+F(y,x,x)+F(y,x,y)$ $+F(y,y,x)+F(y,y,y),x+y]$

$[F(x,x,x),x]+[F(x,x,x),y]+[F(x,x,y),x]+[F(x,x,y),y]=$

$[F(x,y,x),x]+[F(x,y,x),y]+[F(x,y,y),x]+[F(x,y,y),y]+$

$[F(y,x,x),x]+[F(y,x,x),y]+[F(y,x,y),x]+[F(y,x,y),y]$

$+[F(y,y,x),x]+[F(y,y,x),y]+[F(y,y,y),x]+[F(y,y,y),y]$

$[\delta_F(x),x]+[\delta_F(x),y]+[\delta_F(y),x]+3[F(x,x,y),x]+3[F(x,y,y),y]=$

$+3[F(x,x,y),y]+3[F(x,y,y),x]+[\delta_F(y),y]$, for all x , y $\in$ U. …(1)

Replace x by $(-x)$ in equation (1) and comparing the results, we get

$[\delta_F(x+y),x+y]+[\delta_F(-x+y),-x+y]=2[\delta_F(x),x]+6[F(x,x,y),y]$

$+6[F(x,y,y),x]+2[\delta_F(y),y]$ for all $x,y\in U$. …(2)

Replace x by x + y in equation (2) and use equation (1) and (2) to get

$[\delta_F(x+2y),x+2y]+[\delta_F(x),x]=2[\delta_F(x+y),x+y]+6[F(x+y,x+y,y),y]$

$+6[F(x+y,y,y),x+y]+2[\delta_F(y),y]$

$=2[\delta_F(x+y),x+y]+6[F(x,x,y),y]+18[F(x,y,y),y]+12[\delta_F(y),y]+6[F(x,y,y),x]+$ $6[\delta_F(y),x]+2[\delta_F(y),y]$.

Proposition (3.2)

Let U be a Lie ideal of a 6-torsion free ring R and δ_F be the trace of permuting 3-derivation map $F: U\times U\times U\rightarrow R$. Then

(1) If δ_F is commuting on U, $3[F(x,y,y),y]+[\delta_F(y),x]=0$, for all $x,y\in U$.

(2) If δ_F is centralizing on U, $3[F(x,y,y),y)+[\delta_F(y),x]\in Z(R)$, for all x, y ∈ U.

Lemma (3.3)

Let U be a square closed Lie ideal of a 2-torsion free prime ring R such that $[x^2,y]=0$, for all $x,y\in U$. then either $U\subseteq Z(R)$ or $U=0$.

Proof:

Since $[x^2,y]=0$, for all $x,y\in U$, Then this means $[x^2,U]=0$, for all $x\in U$.

By Lemma (2.1) ,we get $\mathrm{x}^2\in Z(R)$, for all $x\in U$ or $U\subseteq Z(R)$.

If $\mathrm{x}^2\in Z(R)$, for all $x\in U$, then $[x^2,r]=0$, for all $x\in U$. **(1)**

Replace x by $x+y$ in equation (1) and use it to get

$0=[xy+yx,r]$, for all $x,y\in U, r\in R$. **(2)**

Replace y by $2\mathrm{y}^2$ in equation(2) and use equation (1) , we get

$0=2[x\,\mathrm{y}^2+y^2x,r]=2[x,r]\,\mathrm{y}^2+2x\,[\mathrm{y}^2,r]+2[y^2,r]x+2y^2[x,r]$

$=4[x,r]\,y^2$

Since R is 2-torsion free, then $0=[x,r]y^2$

Since R is prime, we get either $U\subseteq Z(R)$ or y^2=0 , for all $y\in U$.

By Lemma (2.2) , we get $U=0$ this is contraducition.

Theorem (3.4)

Let R be a 6- torsion free prime ring and U be an admissible Lie ideal of R. Suppose that there exists a generalized permuting 3- derivation $F: U\times U\times U\rightarrow R$ associated with permuting 3-

derivation d such that the traces δ_F of F and δ_d of d are commuting on U, then $F = 0$ on $R \times R \times R$.

proof :

Since δ_F is commuting on , then $[\delta_F(x), x] = 0$, for all $x \in U$. **(1)**

By using proposision (3.1), we get

$0 = [\delta_F(y), x] + 3[F(x,y,y), y]$, for all $x, y \in U$. **(2)**

Putting $2yx$ instead of x in equation (2)

$0 = [\delta_F(y), 2yx] + 3[F(2yx,y,y), y]$

$= 2[\delta_F(y), y]x + 2y[\delta_F(y), x] + 6[F(y,y,y)x + y\,d(x,y,y), y]$

Since F is commuting on U, then the last equation reduced to

$2y[\delta_F(y), x] + 6\delta_F(y)[x,y] + 6y[d(x,y,y), y] = 0$

Since R is 6- torsion free , the last equation becomes

$y[\delta_F(y), x] + 3\delta_F(y)[x,y] + 3y[d(x,y,y), y] = 0$. **(3)**

Multiply equation (2) by y from left and compare the result with equation (3) and by applying Theorem (2.9) , we get

$3\delta_F(y)[x,y] - 3y[F(x,y,y), y] = 0$, for all $, y \in U$. **(4)**

Replace x by $2xz, z \in U$ in equation (4) and by using equation (4) and theorem (2.9) we get

$6(\delta_F(y)x - yF(x,y,y))[z,y] = 0$.

Since R is 6-torsion free, the last equation reduced to

$(\delta_F(y)x - yF(x,y,y))[z,y] = 0$, for all $x, y, z \in U$. **(5)**

Putting $2tz$ instead of $z, t \in U$ in equation (5) , then

$0 = 2(\delta_F(y)x - yF(x,y,y))t[z,y] + 2(\delta_F(y)x - yF(x,y,y))[t,y]z$. **(6)**

Use equation (5) in equation (6) and since R is 6-torsion free, then

$0 = (\delta_F(y)x - yF(x,y,y))t[z,y]$, for all $x, y, z, t \in U$.

Apply Lemma (2.5) on the last equation to get

either $(\delta_F(y)x - yF(x,y,y)) = 0$ or $[z,y] = 0$.

If $[z,y] = 0$, for all $z, y \in U$, then U is commutative and by Lemma (2.3) and this contradication with the hypothesis .

That is, $(\delta_F(y)x - yF(x,y,y)) = 0$, for all $x, y \in U$. **(7)**

Putting $y = 2y^2$ in equation (7) and since R is 6- torsion free and by Theorem (2.9), we get

$\delta_F(y)y^3x - y\delta_F(y)xy^2 = 0$, for all $x, y \in U$.

Since δ_F is commuting on , then the last equation reduced to

$0 = \delta_F(y)y^3x - \delta_F(y)yxy^2$

$= \delta_F(y)y[y^2, x]$ **(8)**

Putting $2zt$ instead of x in equation(8) and use equation (8) we get

$0 = \delta_F(y)yz[y^2, t]$, for all $y, z, t \in U$.

By Lemma (2.5) and since R is 6-torsion free , we get either $\delta_F(y)y = 0$ or $[y^2, t] = 0$.

By Lemma (2. 5) and since $0 \neq U \nsubseteq Z(R)$, then by Lemma (2.5) we have

$\delta_F(y)(y)y = 0$, for all $\in U$. **(9)**

Since δ_F is commuting on U, and by using equation (9) we get

$y\delta_F(y) = 0$, for all $y \in U$.

Multiply equation (7) by y from left and use the last equation to get

$y^2F(x,y,y) = 0$, for all $x, y \in U$. **(10)**

Substitute equation (9) in equation (5) and by Theorem (2.9) to get

$0 = 3\delta_F(y)xy - yx\delta_F(y)$

Substitute equation (7) in the last equation, we get

$0 = 3yF(x,y,y)y - yx\delta_F(y)$

Multiply the last equation by y from left and by using equation (10) , we get

$0 = 3y^2x\delta_F(y), for\ all\ x, y \in U.$

Since R is 6-torsion free prime and by Lemma (2.3), either $y^2 = 0$ or $\delta_F(y) = 0$.

By Lemma (2.1) and since U is nonzero , then $\delta_F(y) = 0$, for all $y \in U$. **(11)**

Linearize equation (11) on y we get

$0 = \delta_F(x) + \delta_F(y) + 3F(x,x,y) + 3F(x,y,y)$.

By equation (11) and Since R is 6-torsion free, the last equation can be reduced to
$F(x,x,y) + F(x,y,y) = 0$, for all $x,y \in U$. **(12)**
Again linearize equation (12) on y and since R is 6-torsion free , then $0 = F(x,y,z)$, for all $x,y \in U$.
Since U is an admissible Lie ideal, by Lemma (2.4) U contains a nonzero ideal I of U. Therefore, $(x,y,z) = 0$, for all $x,y,z \in I$. **(13)**
Replace x by rx in equation (13) to get
$0 = F(rx,y,z) = F(r,y,z)\,x + r\,d(x,y,z)$
By Theorem (2.9), the last equation reduce to
$0 = F(r,y,z)\,x = 0$, for all $,y \in I$, r$\in R$ since I is ideal and R is prime, then $F(r,y,z) = 0$, for all $r \in R,\ x,y,z \in I$. **(14)**
Replace y by $, s \in R$ in equation (14) to get
$0 = F(r,sy,z) = F(r,s,z)\,y + s\,d(r,y,z)$
By Theorem (2.9), the last equation reduce to
$0 = F(r,s,t)\,y$, for all $r,y \in I$, $s \in R$ and this implies that $(r,s,z) = 0$, for all $x,y,z \in I$. **(15)**
Replace z by $zt, t \in R$ in equation (15) to get
$0 = \mathrm{F}(\mathrm{r},\mathrm{s},\mathrm{tz}) = \mathrm{F}(\mathrm{r},\mathrm{s},\mathrm{t})\,\mathrm{y} + \mathrm{t}\,\mathrm{d}(\mathrm{r},\mathrm{s},\mathrm{z})$
By Theorem (2.9), the last equation reduce to
$0 = F(r,s,t)\,y$, for all $r,s,t \in R$ and this lead us to $F(r,s,t) = 0$, for all $r,s,t \in R$.
The following corollary is a special case of last theorem.

Corollary (3.5)

Let R be a non commutative 6- torsion free prime ring. Suppose that there exists a generalized permuting 3- derivation $F: R \times R \times R \to R$ associated with permuting 3-derivation d such that the trace $\delta_F$of F and δ_d of d are commuting on U, then $F = 0$.
The following theorem is a generalization of Theorem(2.9)

Theorem (3.6)

Let R be a 5!-torsion free prime ring and U be an admissible Lie ideal of R. Suppose that there exists a generalized permuting 3- derivation $F: U \times U \times U \to R$ such that the trace δ_F of F and the trace δ_d of d are centralizing on U . Then δ_F is commuting on .

Proof :

Since δ_F is centralizing on U, then $[\delta_F(x), x] \in Z(R)$, for all $x \in U$. **(1)**
By using proposition (3.1) , we get
$[\delta_F(y), x] + 3\,[F(x,y,y), y] \in Z(R)$ **(2)**
putting $x = 2y^2$ in equation (2) , then
$2\,[\delta_F(y), y^2] + 6\,[F(y^2,y,y), y]$
$= 2\,[\delta_F(y), y]\,y + 2\,y\,[\delta_F(y), y] + 6\,[\delta_F(y), y + y\,\delta_d(y)\ , y]$
$= 4\,[\delta_F(y), y]\,y + 6\,y[\delta_F(y), y] + 6\,y\,[\delta_d(y), y]$
$= 10\,[\delta_F(y), y]\,y + 6\,y[\delta_d(y), y] \in Z(R)$. **(3)**
By Theorem (2.9) and Theorem (2.10), equation (5) reduce to
$10\,[\delta_F(y), y]\,y \in Z(R)$.
That is , $0 = 10[\,[\delta_F(y), y]\,y, x] = 10\,[\delta_F(y), y]\,[x, y] + 10[\,[\delta_F(y), y], x]\,y$
$= 10\,[\delta_F(y), y]\,[y, x]$. **(4)**
Putting $x = 2xz$ in equation (4) , we get
$0 = 20\,[\delta_F(y), y]\,x\,[y,z] + 20\,[\delta_F(y), y]\,[y,x]\,z$
By using equation (4) and since R is 5!- torsion free , we get
$0 = [\delta_F(y), y]\,x\,[y,z]$, for all $, y \in U$.
By Lemma (2.5) and since U is not commutative we get δ_F is commuting .

Corollary (3.7)

Let R be a 5!-torsion free prime ring. Suppose that there exists a generalized permuting 3-derivation $F: R \times R \times R \to R$ such that the trace δ_F of F and the trace δ_d of d are centralizing on R . Then δ_F is commuting on.

Theorem (3.8)

Let R be a 5!- torsion free prime ring and U be a square closed Lie ideal of R. Suppose that there exist a nonzero generalized permuting 3-derivation $F : U \times U \times U \rightarrow R$ such that is the trace δ_F of F centralizing on U. Then $U \subseteq Z(R)$.

Proof :

Suppose that, $U \nsubseteq Z(R)$, by Theorem (3.6) δ_F is commuting and by Theorem (3.4) $F = 0$ and this contradiction with our hypothesis then $U \subseteq Z(R)$.

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