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Some Cases on Cozy Partitions

Ali A. Alabdali*, Ammar S. Mahmood
Department of Mathematics, College of Education for Pure Sciences, University of Mosul, Mosul, Iraq

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Abstract
The cozy partitions achieved more creativity by emerging with many topics in representation theory and mathematical relations. We find the precise number of cozy tableaux in the case (ω_1, ω_2) with any number of ω_1 and ω_2. Specifically, we use the MATLAB programme that coincided with the mathematical solution in giving precision to these numbers in this case.

Keywords: Cozy tableau; Hecke algebra; Specht module.

بعض حالات التجزئات اللطيفة

علي عبدالقادر العبدلي*, عمار صديق محمود
قسم الرياضيات, كلية التربية للعلوم الصرفة, جامعة الموصل, الموصل, العراق

الخلاصة
حققت التجزئات اللطيفة مزيدًا من الإبداع من خلال الظهور مع العديد من الموضوعات في نظرية التمثيل والعلاقات الرياضية. نجد العدد الدقيق للالواح اللطيفة في الحالة (ω_1, ω_2) لأي عدد من ω_1 و ω_2. على وجه التحديد ، نستخدم برنامج ماتلاب الذي تزامن مع الحل الرياضي في إعطاء الدقة لهذه الأعداد في هذه الحالة.

1. Introduction and statement of results

The notion of Young tableaux was introduced by Alfred Young [1]. This was one of the most important subjects that plays the main role in the representation theory of symmetric groups and combinatorics. Kang et al. in [2] show a clear relation between cozy and standard tableaux of shape ω. That cozy tableaux of shape ω have a one to one correspondence with standard tableaux of shape ω. Also, Kang et al. established an explicit isomorphic between cozy and standard tableaux of shape ω. These relations are commonly used for connecting to Hecke algebras and Specht modules; (see [2], [3], [4], [5], [6], and [7]). To the best of our knowledge, there is no rule for counting the number of cozy ω-tableaux. In this paper, we determine the number of cozy tableaux of shape ω. We give a general rule by dealing with the case when $\omega = (\omega_1, \omega_2)$ to enumerate cozy tableaux and leave the other cases for future work. We can now state our main result.

Theorem 1

*Email: ali.alabdali@uomosul.edu.iq

If the t tableau of shape ω is of the form $(\omega_1,\ \omega_2)$, then the following procedures can be used to enumerate the exact number of cozy tableaux:

i. $\mathrm{cozy}_{(\omega_1,\,\omega_2)} = \sum_{\mathrm{t}_1=\omega_2}^{\omega_1} \mathrm{cozy}_{(\mathrm{t}_1,\,\omega_2-1)}$.

ii. If $\begin{cases} \omega_2 - 1 > 0, & \text{continue,} \\ \omega_2 - 1 \le 0, & \text{stop.} \end{cases}$

iii. When $\omega_2 - 1 > 0$, then $\sum_{t_1=\omega_2}^{\omega_1} cozy_{(t_1,\,\omega_2-1)} = l_s(\ cozy_{(s,\,\omega_2-2\,)}\ )$ where

$$l_s = \begin{cases} \omega_1 - \omega_2 + 1, & if\ \ s = \omega_2 - 1 \ \text{ or } \omega_2, \\ \omega_1 - s + 1, & if\ s = \omega_2 + 1, \omega_2 + 2, \cdots, \omega_1. \end{cases}$$

iv. If $\begin{cases} \omega_2 - 2 > 0, & \text{continue,} \\ \omega_2 - 2 \le 0, & \text{stop.} \end{cases}$

v. When $\omega_2 - 2 > 0$, then $l_s\left(\mathrm{cozy}_{(s,\,\omega_2-2\,)}\right) =$

$$\begin{cases} \sum_{s=\omega_2}^{\omega_1} l_s\left(\mathrm{cozy}_{(\omega_2-2,\,\omega_2-3\,)} + \mathrm{cozy}_{(\omega_2-1,\,\omega_2-3)}\right), \\ \left(\ \sum_{s=\omega_2-1}^{\omega_1} l_s - \sum_{g=\omega_2-1}^{d} l_g\ \right)\left(\mathrm{cozy}_{(d+1,\,\omega_2-3)}\right), & if\ d = \omega_2 - 1, \omega_2, \cdots, \omega_1 - 1. \end{cases}$$

vi. If $\begin{cases} \omega_2 - 3 > 0, & \text{continue,} \\ \omega_2 - 3 \le 0, & \text{stop.} \end{cases}$

vii. When $\omega_2 - 3 > 0$, then we have three answers

$$\begin{cases} (\omega_1 - (\omega_2 - 3) + 1) \sum_{s=\omega_2-1}^{\omega_1} l_s\left(\mathrm{cozy}_{(t_2,\,\omega_2-3\,)}\right), & if\ t_2 = \omega_2 - 3\ \text{ or } \omega_2 - 2, \\ (\omega_1 - (\omega_2 - 3)) \sum_{s=\omega_2-1}^{\omega_1} l_s\left(\mathrm{cozy}_{(\omega_2-1,\,\omega_2-3\,)}\right), \\ \left((\omega_1 - (\omega_2 - 3) + t_3) \sum_{s=\omega_2-1}^{\omega_1} l_s - (\omega_1 - (\omega_2 - 3) + (2 - t_3) \sum_{g=\omega_2-1}^{d} l_g\ \right)\left(\mathrm{cozy}_{(d+1,\,\omega_2-4)}\right), \end{cases}$$

where $t_3 = -1, -2, -3, \ldots$

viii. If $\begin{cases} \omega_2 - 4 > 0, & \text{continue,} \\ \omega_2 - 4 \le 0, & \text{stop.} \end{cases}$

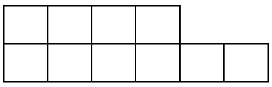
So on.

2. Preliminaries

Let r be a positive integer. A sequence $\omega = (\omega_1,\ \omega_2,\ \cdots,\ \omega_n)$ of non-negative integers is said to be a composition of r such that $\sum_i \omega_i = r$, where $i = 1, 2, \cdots, n$. A partition is a composition ω if $\omega_i \ge \omega_{i+1}$ for all $i = 1, 2, \cdots, n$. Young's diagram of a composition ω is the subset of $N \times N$ as follows:

$$[\omega] = \{(\delta, \theta) \colon 1 \le \theta \le \omega_\delta \text{ and } \delta \ge 1\}.$$

We denote by YD Young's diagram. For example, if $\omega = (4, 6)$ then YD is



A bijection $t \colon [\omega] \longrightarrow \{1, 2, \cdots, r\}$ is said to be ω-tableau if ω is a composition of r. If the entries in ω-tableau t increase from left to right in each row (i.e. $t(\delta, \theta) < t(\delta, \theta + 1)$), then t is called row standard. whilst ω-tableau t is called standard if ω is a partition, t is a row standard and the entries in t increase from top to bottom in each column (i.e. $t(\delta, \theta) < t(\delta + 1, \theta)$). We denote by RS (respectively S) row standard (respectively standard) ω-tableau. For example, if $\omega = (3, 1)$ then the following are RS

1	2	3
4		

1	3	4
2		

2	3	4
1		

It is clear that the only first two tableaux are S. The standard ω-tableau t^{ω} is called unique if $t^{\omega}(\delta,\theta+1)=t^{\omega}(\delta,\theta)+1, \forall(\delta,\theta)\in[\omega]$. For example, if $\pi=(3,2)$ and $\varphi=(4,1)$, then

$$t^{\pi}= \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array}$$

$$t^{\varphi}= \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & & & \\ \cline{1-1}\end{array}$$

According to Kang et al. in [2], a semi cozy is an ω-tablaeu $t:[\omega]\longrightarrow\{1,2,\cdots,r\}$ if $1\leq t(\delta,\theta)\leq t^{\omega}(\delta,\theta), \forall(\delta,\theta)\in[\omega]$. Also, the cozy is defined in [2] as a semi cozy ω-tableau if it satisfies:

1. t is a row standard.

2. $t(\delta,\theta)+\theta\leq t(\delta+1,\theta), \forall(\delta,\theta)\in[\omega]$.

Note that any $t(1,\theta)=\theta$. For example, the following tableau is cozy:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 4 & & \\ \cline{1-2}\end{array}$$

while this one is not cozy:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & & \\ \cline{1-2}\end{array}$$

3. Cases of $\boldsymbol{\omega}$-tableau $(\boldsymbol{\omega_1},\boldsymbol{\omega_2})$ and Proof of Theorem 1

3.1. Case $(\omega_1,0)$

We have used Matlab programming to study all the cases (ω_1,ω_2) incrementally to seek what behaviour they exhibit in order to determine the exact number of cozy ω-tableaux. In this case, we note that each of the examples in Table 1 has given only one result, as follows:

Table 1: Number of cozy tableaux of shape $(\omega_1,0)$.

(1, 0)	**(2, 0)**	**(3, 0)**	**(4, 0)**	$\cdots$
$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 3 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}$	$\cdots$

It is clear from Table 1 that the number of cozy tableaux in every case of ω-tableau of shape $(\omega_1,0)$ is equal to one, that means $cozy_{(\omega_1,0)}=1$.

3.2. Case $(\omega_1,1)$

This case as shown in Table 2 is essential for interpreting the cozy tableaux description, particularly the outcomes that will emerge later, and it runs like the below:

Table 2: Number of cozy tableaux of shape $(\omega_1, 1)$.

(1, 1)	(2, 1)		(3, 1)		(4, 1)		⋯
$\begin{matrix}1\\2\end{matrix}$	$\begin{matrix}1&2\\2&\end{matrix}$	$\begin{matrix}1&2\\3&\end{matrix}$	$\begin{matrix}1&2&3\\2&&\end{matrix}$	$\begin{matrix}1&2&3\\3&&\end{matrix}$	$\begin{matrix}1&2&3&4\\2&&&\end{matrix}$	$\begin{matrix}1&2&3&4\\3&&&\end{matrix}$	⋯
			$\begin{matrix}1&2&3\\4&&\end{matrix}$		$\begin{matrix}1&2&3&4\\4&&&\end{matrix}$	$\begin{matrix}1&2&3&4\\5&&&\end{matrix}$	

The number of cozy tableaux in every case of ω-tableau of shape $(\omega_1, 1)$ is $cozy_{(\omega_1,1)} = \omega_1$, as shown in Table 2. Table 3 shows the number of cozy tableaux in each shape of ω-tableau based on the cases of $(\omega_1, 0)$ as given in Table 1, and we proceed similarly in the next cases of ω-tableaux.

Table 3: Number of cozy tableaux for each shape of $(\omega_1, 1)$.

$(\omega_1, 1)$	(1, 0)	(2, 0)	(3, 0)	(4, 0)	⋯
(1, 1)	1				
(2, 1)	1	1			
(3, 1)	1	1	1		
(4, 1)	1	1	1	1	
⋮	⋮	⋮	⋮	⋮	⋮

3.3. Case $(\omega_1, 2)$

In Table 4, the number of cozy tableaux of shape $(\omega_1, 2)$ based on Table 1 is expressed as the Pascal triangle and the numbers are repetition of two cases of $(\omega_1, 0)$, as shown below:

Table 4: Number of cozy tableaux for each shape of $(\omega_1, 2)$.

$(\omega_1, 2)$	(1, 0)	(2, 0)	(3, 0)	(4, 0)	⋯	⋯	⋯
(2, 2)	1	1					
(3, 2)	2	2	1				
(4, 2)	3	3	2	1			
(5,2)	4	4	3	2	1		
(6, 2)	5	5	4	3	2	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Total	$2(cozy_{(\omega_1-1,\ 1)})$				$\binom{\omega_1 - 1}{\omega_1 - 3}$		

3.4. Case $(\omega_1, 3)$

We now continue with the cases in Table 5, where there are several surprises in obtaining the general rule for each case separately, as given below:

Table 5: Number of cozy tableaux for each shape of $(\omega_1, 3)$.

	A				B			
$(\omega_1, 3)$	(1, 0)	(2, 0)	(3, 0)	(4, 0)	⋯	⋯	⋯	⋯
(3, 3)	2	2	1					
(4, 3)	5	5	3	1				
(5, 3)	9	9	6	3	1			
(6, 3)	14	14	10	6	3	1		
(7, 3)	20	20	15	10	6	3	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Total	$2(cozy_{(\omega_1-1,\ 2)})$				$\binom{\omega_1}{\omega_1 - 3}$			

3.5. Case $(\omega_1, 4)$

In Table 6, the column $(3, 0)$ is completely different from the other cases in the latter tables, as we notice below:

Table 6: Number of cozy tableaux for each shape of $(\omega_1, 4)$.

	A		C	B					
$(\omega_1, 4)$	**(1, 0)**	**(2, 0)**	**(3, 0)**	**(4, 0)**	…	…	…	…	…
(4, 4)	5	5	3	1					
(5, 4)	14	14	9	4	1				
(6, 4)	28	28	19	10	4	1			
(7, 4)	48	48	34	20	10	4	1		
(8, 4)	75	75	55	35	20	10	4	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Total	$2(cozy_{(\omega_1-1,\ 3)})$		$\binom{\omega_1}{\omega_1-3}-1$	$\binom{\omega_1}{\omega_1-4}$					

3.6. Case $(\omega_1, 5)$

In Table 7, one can easily see that the last values completely changed our method of research due to the presence of several variables that forced us to find a more smooth and acceptable method. Because if we continue with the last cases upwards, we will inevitably face unexpected values and therefore probably could not control them, which led us to the proof of Theorem 1. Therefore, through the previous results, we put the appropriate method to find the exact number of cozy tableaux by the steps in Theorem 1.

Table 7: Number of cozy tableaux for each shape of $(\omega_1, 5)$.

	A		D	C	B					
$(\omega_1, 5)$	**(1, 0)**	**(2, 0)**	**(3, 0)**	**(4, 0)**	**(5, 0)**	…	…	…	…	…
(5, 5)	14	14	9	4	1					
(6, 5)	42	42	28	14	5	1				
(7, 5)	90	90	62	34	15	5	1			
(8, 5)	165	165	117	69	35	15	5	1		
(9, 5)	275	275	200	125	70	35	15	5	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Total	$2(cozy_{(\omega_1-1,\ 4)})$		$\sum_{t=5}^{\omega_1}(\binom{\omega_1}{\omega_1-3}-1)_{cozy_{(t,\ 4)}}$	$\binom{\omega_1}{\omega_1-4}-1$	$\binom{\omega_1}{\omega_1-5}$					

3.7. Case $(\omega_1, 6)$

Finally, in Table 8, it is easy to see that the last values are completely different from previous tables due to the presence of several variables especially the new column that forced us to find a more smooth and acceptable method. Because if we continue with the last cases upwards, we will inevitably face unexpected values and therefore probably could not control them. As in column E the total is $\frac{1}{2}\ A + B + C + D$, where A, B, C and D are the values in the last table. So that led us to the proof of Theorem 1 as well. Therefore, through the previous results, we put the appropriate method to find the exact number of cozy tableaux by the steps in Theorem 1.

Table 8: Number of cozy tableaux for each shape of $(\omega_1, 6)$.

		A	E	D	C			B			
$(\omega_1, 6$	(1, 0)	(2, 0)	(3, 0)	(4, 0)	(5, 0)	(6, 0)	…	…	…	…	…
(6, 6)	42	42	28	14	5	1					
(7, 6)	132	132	90	48	20	6	1				
(8, 6)	297	297	207	117	55	21	6	1			
(9, 6)	572	572	407	242	125	56	21	6	1		
(10, 6)	1001	1001	726	451	251	126	56	21	6	1	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Total	$2(cozy_{(\omega_1-1,}$		$(\frac{1}{2} A + B + C + D)_{cozy(\omega_1,5)}$	$\sum_{t=6}^{\omega_1}(\binom{\omega_1}{\omega_1 - 4} - 1)_{cozy(t,\ 5)}$	$\binom{\omega_1}{\omega_1 - 5} - 1$	$\binom{\omega_1}{\omega_1 - 6}$					

4. Example

In this example, the ω-tableau has the shape $(7, 4)$, as illustrated in Table 6. We determine the precise number of cozy tableaux. We have the following steps and results based on Theorem 1.

$$cozy_{(7,4)} = \left(cozy_{(4,3)} + cozy_{(5,3)} + cozy_{(6,3)} + cozy_{(7,3)}\right) \quad (1)$$

Now each cozy tableau in Eq. (1) has the following form

$$cozy_{(4,3)} = \left(cozy_{(3,2)} + cozy_{(4,2)}\right).$$
$$cozy_{(5,3)} = \left(cozy_{(3,2)} + cozy_{(4,2)} + cozy_{(5,2)}\right).$$
$$cozy_{(6,3)} = \left(cozy_{(3,2)} + cozy_{(4,2)} + cozy_{(5,2)} + cozy_{(6,2)}\right).$$
$$cozy_{(7,3)} = \left(cozy_{(3,2)} + cozy_{(4,2)} + cozy_{(5,2)} + cozy_{(6,2)} + cozy_{(7,2)}\right).$$

Then using the latter forms, the Eq. (1) become as follows

$$cozy_{(7,4)} = 4cozy_{(3,2)} + 4cozy_{(4,2)} + 3cozy_{(5,2)} + 2cozy_{(6,2)} + cozy_{(7,2)}. \quad (2)$$

Similarly each cozy tableau in Eq. (2) has the following form

$$4(cozy_{(3,2)}) = 4\left(cozy_{(2,1)} + cozy_{(3,1)}\right).$$
$$4(cozy_{(4,2)}) = 4\left(cozy_{(2,1)} + cozy_{(3,1)} + cozy_{(4,1)}\right).$$
$$3(cozy_{(5,2)}) = 3(cozy_{(2,1)} + cozy_{(3,1)} + cozy_{(4,1)} + cozy_{(5,1)}).$$
$$2(cozy_{(6,2)}) = 2\left(cozy_{(2,1)} + cozy_{(3,1)} + cozy_{(4,1)} + cozy_{(5,1)} + cozy_{(6,1)}\right).$$
$$cozy_{(7,2)} = \left(cozy_{(2,1)} + cozy_{(3,1)} + cozy_{(4,1)} + cozy_{(5,1)} + cozy_{(6,1)} + cozy_{(7,1)}\right).$$

We obtain Eq. (3) by following the previous steps from Eq. (2) as follows

$$cozy_{(7,4)} = 14cozy_{(2,1)} + 14cozy_{(3,1)} + 10cozy_{(4,1)} + 6cozy_{(5,1)} + 3cozy_{(6,1)} + cozy_{(7,1)}. \quad (3)$$

Finally, we have each cozy tableau in Eq. (3) as the following forms

$$14cozy_{(2,1)} = 14\left(cozy_{(1,0)} + cozy_{(2,0)}\right).$$
$$14cozy_{(3,1)} = 14\left(cozy_{(1,0)} + cozy_{(2,0)} + cozy_{(3,0)}\right).$$
$$10cozy_{(4,1)} = 10(cozy_{(1,0)} + cozy_{(2,0)} + cozy_{(3,0)} + cozy_{(4,0)}).$$
$$6cozy_{(5,1)} = 6\left(cozy_{(1,0)} + cozy_{(2,0)} + cozy_{(3,0)} + cozy_{(4,0)} + cozy_{(5,0)}\right).$$
$$3cozy_{(6,1)} = 3\left(cozy_{(1,0)} + cozy_{(2,0)} + cozy_{(3,0)} + cozy_{(4,0)} + cozy_{(5,0)} + cozy_{(6,0)}\right).$$
$$cozy_{(7,1)} = cozy_{(1,0)} + cozy_{(2,0)} + cozy_{(3,0)} + cozy_{(4,0)} + cozy_{(5,0)} + cozy_{(6,0)} + cozy_{(7,0)}.$$

Now the final form of counting the exact number of cozy tableaux of shape $(7,4)$ is obtained by Eq. (4) from following the latter forms in Eq. (3)

$$cozy_{(7,4)} = 48cozy_{(1,0)} + 48cozy_{(2,0)} + 34cozy_{(3,0)} + 20cozy_{(4,0)} + 10cozy_{(5,0)} + 4cozy_{(6,0)} + cozy_{(7,0)}. \quad (4)$$

Therefore, the precise number of cozy tableaux of the shape $(7,4)$ based on Table 1 is $cozy_{(7,4)} = 165$ as shown in Table 6.

5. Conclusion

In conclusion, we investigate the exact number of cozy tableaux of the shape $(\omega_1,\ \omega_2)$ by Theorem 1. We can see that in every new case of $(\omega_1,\ \omega_2)$ where $\omega_2 > 3$, we obtain a new column that differs from the new columns in the previous cases. However, the total form of the new column depends on the previous cases. As we get the total form of the new (gray) column in Table 8 depending on the values of columns A, B, C and D in Table 7. Therefore, to avoid that, the MATLAB programme was used to solve this problem.

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