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Extended Eigenvalues and Eigenoperators of Some Weighted Shift Operators

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Abstract
A complex number μ is called an extended eigenvalue for an operator $T \in B(H)$ on a Hilbert space H if there exists a nonzero operator $X \in B(H)$ such that: $TX = \mu XT$, such X is called an extended eigenoperator corresponding to. The goal of this paper is to calculate extended eigenvalues and extended eigenoperators for the weighted unilateral (Forward and Backward) shift operators. We also find an extended eigenvalues for weighted bilateral shift operator. Moreover, the closedness of extended eigenvalues for the weighted unilateral (Forward and Backward) shift operators under multiplication is proven.

Keywords: Extended eigenvalues, Extended Eigenoperators, weighted bilateral shift operator, weighted unilateral shift operator.

القيم الذاتية والمؤثرات الذاتية الموسعة لبعض مؤثرات التحويل الموزونة

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الخلاصة
العدد المعقد μ يدعى قيمة ذاتية موسعة للمؤثر $T \in B(H)$ على فضاء هلبرت اذا وجد مؤثر غير صفري $X \in B(H)$ حيث ان $TX = \mu XT$. بحيث ان X يدعى مؤثر موسع للمؤثر T المقابل لـ μ. الهدف من هذا البحث هو حساب القيم الذاتية الموسعة والمؤثرات الذاتية الموسعة لعامل النقل بأتجاه واحد الى (الامام و الخلف) تحت وزن معين , كذلك ايجاد القيم الذاتية الموسعة لعامل النقل ذو الاتجاهين تحت وزن معين , اضافة الى ذلك سوف نوضح انغلاق القيم الذاتية الموسعة لعاملي النقل باتجاه واحد (للأمام والى الخلف) تحت عملية الضرب .

1. Introduction

Let H be the complex Hilbert space and B(H) be the algebra of all continuous linear operators on H . A complex number μ is called an extended eigenvalue for an operator $T \in B(H)$ if there exists a non- zero operator $X \in B(H)$ such that:

$$TX = \mu XT . \tag{1.1}$$

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The operator X is called an extended eigenoperator for an operator T corresponding to μ . Let E(T) be the set of all an extended eigenvalues and $E_{\mu}(T)$ be the set of all an extended eigenoperators that is :

$$E(T) = \{ \mu \in \mathbb{C} : TX = \mu XT \,;\, 0 \neq X \in B(H)\} \quad (1.2)$$

$$E_{\mu}(T) = \{0 \neq X \in B(H) \,; TX = \mu XT \} \quad (1.3)$$

This definition was given first by [1] to generalize Lomonosov theorem. Alan Lambert and Srdjan Petrovic [1] considered the integral Volterra operator on the space $L^2(0,1)$. Alan Lambert [2] explained an extended eigenvalue for an operator T is a scalar μ for which the operator equation $TX = \mu XT$ has a non-zero solution. Several scenarios are investigated where the existence of non-unimodular extended eigenvalues leads to invariant or hyper invariant subspaces.

Biswas and S. Petrovic [3] gave a description the relationship between the extended eigenvalues of an operator T and its powers. H. Alkanjo and G. Cassier [4] described the sets of extended eigenvalues and extended eigenoperators for the product of a positive and self adjoint operator which are both injective. They also treated the case of normal operators.

L.K. Shaakir and Anas A. Hijab [6] calculated the set of all extended eigenvalues of unilateral shift operator on the L^2 space . F. Leon and S. Petovic [7] gave a full solution to the problem of computing the extended eigenvalues for those composition operators c_{φ} induced on the Hardy space $H^2(\mathbb{D})$ by linear fractional transformations φ of the unit disk.

The goal of this paper is to calculate extended eigenvalues and an eigenoperators for weighted unilateral (Forward and Backward) shift operators. Also, we find extended eigenvalues for weighted bilateral shift operator.

2. Primaries

In this section, we present the most important definitions and important characteristics that we need in the next sections. To begin with, we mention the most important basic definitions about our topic.

Definition (2.1) [8] **:** Let $\{\alpha_n\}$ be a sequence of numbers such that $\alpha_n > 0$ for all $n \in \mathbb{N}$ and let $\lim_{n\to+\infty} \alpha_n = \alpha_+$, for $x \in l^2(N)$, define the weighted unilateral forward shift operator $A_{\alpha} \in B(l^2(\mathbb{N}))$ by:

$$A_{\alpha}(x_1, x_2, x_3, \dots) = (0, \alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \dots) \quad (2.1)$$

for all $x \in l^2(\mathbb{N})$ under the base in $l^2(\mathbb{N})$ that is: $A_{\alpha} e_n = \alpha_n e_{n+1}$ (2.2)

The definition of the weighted unilateral backward shift operator $B_{\alpha} \in B(l^2(\mathbb{N}))$ is given by: $B_{\alpha}(x_1, x_2, x_3, \dots) = (\alpha_1 x_2, \alpha_2 x_3, \alpha_3 x_4, \dots)$ (2.3)

for all $x \in l^2(\mathbb{N})$. By some simple calculations, we get: $B_{\alpha}^* = A_{\alpha}$. (2.4)

Definition (2.2) [8]: Let $\{\alpha_n\}$ be a sequence of numbers such that $\alpha_n > 0$ for all $n \in \mathbb{Z}$ and let $\lim_{n\to-\infty} \alpha_n = \alpha_-$ and $\lim_{n\to+\infty} \alpha_n = \alpha_+$ for $x \in l^2(\mathbb{Z})$, the weighted bilateral shift operator $T_{\alpha} \in B\left(l^2(\mathbb{Z})\right)$ is defined by:

$$T_{\alpha}(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, \alpha_{-2} x_{-2}, \alpha_{-1} x_{-1}, \alpha_0 x_0, \dots) \quad (2.5)$$

for all $x \in l^2(\mathbb{Z})$. under the base in $l^2(\mathbb{Z})$ that is: $T_{\alpha} e_n = \alpha_n e_{n+1}$. (2.6)

The adjoint of T_{α} is:

$$T_{\alpha}^*(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, \alpha_{-1} x_0, \alpha_0 x_1, \alpha_1 x_2, \dots) \quad (2.7)$$

Definition (2.3) [8] **:** If $B: l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$, then the bilateral shift operator is defined by:

$$B(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, x_{-2}, x_{-1}, x_0, \dots) . \quad (2.8)$$

It is given under the base in$l^2(\mathbb{Z})$ by:

$$Be_n = e_{n+1} \quad (2.9)$$

The next theorem appeared in [3], we gave a second proof that was easier from our point of view.

Theorem (2.4): If $\mu \in E(T)$ then $\mu^n \in E(T^n)$ for all $n \in \mathbb{N}$.

Proof: We will prove by mathematical induction,
Since $\mu \in E(T)$ then there is anon zero operator $X \in B(H)$ such that TX = μXT.
To prove the statement is true for $n = 2$, assume that there is anon zero operator $X \in B(H)$ is an extended eigenoperator for T corresponding to μ
Such that: $T^2X = T(TX) = T(\mu XT) = \mu(TX)T = \mu(\mu XT)T = \mu^2 XT^2$.
So $T^2X = \mu^2XT^2$, This means that $\mu^2 \in E(T^2)$.
Now, assume that the statement is true for $n-1$ this means $T^{n-1}X = \mu^{n-1}XT^{n-1}$
To prove the statement is true for n:

$$T^nX = T(T^{n-1}X) = T(\mu^{n-1}XT^{n-1}) = \mu^{n-1}(TX)T^{n-1}$$
$$= \mu^{n-1}(\mu XT)T^{n-1} = \mu^n XT^n$$

So that$T^nX = \mu^nXT^n$. Therefore, $\mu^n \in E(T^n)$. □

3. Main work

In this section, we calculate extended eigenvalues for weighted unilateral (Forward and Backward) shift operators .We also find the extended eigenvalues for weighted bilateral shift operator. Further, we explain that an extended eigenvalues for the weighted unilateral (Forward and Backward) shift operators are closed under multiplication.

The aim of the following theorem is to calculate extended eigenvalues for the weighted unilateral forward shift operator A_α ,where $\alpha = \sup\{\alpha_1, \alpha_2, \dots\}$.

Theorem (3.1): If A_α is the weighted unilateral forward shift operator on $l^2(\mathbb{N})$, then $E(A_\alpha) = \{\mu \in \mathbb{C} : |\mu| \geq \max\{1, \alpha\} ; \mu \neq 0\}$.

Proof: Suppose that $|\mu| \geq \sup\{\alpha_1, \alpha_2, \dots\}$ and let $D_{\frac{1}{\mu}}$ be the diagonal operator such that $D_{\frac{1}{\mu}}(x_1, x_2, x_3, \dots) = (x_1, \frac{1}{\mu}x_2, \frac{1}{\mu^2}x_3, \dots)$ for all $x \in l^2(\mathbb{N})$. Claim that:
$A_\alpha D_{\frac{1}{\mu}} = \mu D_{\frac{1}{\mu}} A_\alpha$. To prove this claim, we start with the left side that means, $A_\alpha D_{\frac{1}{\mu}}(x_1, x_2, x_3, \dots) = A_\alpha\left(x_1, \frac{1}{\mu}x_2, \frac{1}{\mu^2}x_3, \dots\right)$

$$= \left(0, \alpha_1 x_1, \alpha_2 \frac{1}{\mu} x_2, \alpha_3 \frac{1}{\mu^2} x_3, \dots\right)$$

It is clear that the right side is given as follows:

$$\mu D_{\frac{1}{\mu}} A_\alpha(x_1, x_2, x_3, \dots) = \mu D_{\frac{1}{\mu}}(0, \alpha_1x_1, \alpha_2x_2, \alpha_3x_3, \dots)$$
$$= \mu\left(0, \frac{1}{\mu}\alpha_1x_1, \frac{1}{\mu^2}\alpha_2x_2, \frac{1}{\mu^3}\alpha_3x_3, \dots\right)$$
$$= \left(0, \alpha_1x_1, \alpha_2\frac{1}{\mu}x_2, \alpha_3\frac{1}{\mu^2}x_3, \dots\right)$$

Thus, $A_\alpha D_{\frac{1}{\mu}} = \mu D_{\frac{1}{\mu}} A_\alpha$ $\mu \in E(A_\alpha)$.
Therefore, $\{\mu \in \mathbb{C} : |\mu| \geq \max\{1, \alpha\}\} \subseteq E(A_\alpha)$.
Now suppose that $|\mu| < \sup\{\alpha_1, \alpha_2, \dots\}$ such that $\mu \in E(A_\alpha)$ then there exists $0 \neq X \in B(H)$ such that $A_\alpha X = \mu XA_\alpha$

$$A_\alpha e_n = (0,0,0, \dots, \alpha_n, 0,0, \dots) = \alpha_n(0,0,0, \dots, 1,0, \dots)$$

$= \alpha_n e_{n+1}$. Therefore, $e_{n+1} = \frac{1}{\alpha_n} A_\alpha e_n$.
Assume that $\alpha_0 = \inf\{\alpha_1, \alpha_2, \alpha_3, \dots\}$,
Thus, $\|Xe_{n+1}\| = \left\|X \frac{1}{\alpha_n} A_\alpha e_n\right\| = \frac{1}{|\alpha_n|} \|XA_\alpha e_n\| = \frac{1}{|\alpha_n|} \left\|\frac{1}{\mu} A_\alpha X e_n\right\|$

$$\geq \frac{1}{|\alpha_n|} \frac{1}{|\mu|} |\alpha_0| \|Xe_n\|$$

$$\vdots$$

$$\geq \frac{1}{|\alpha_n||\alpha_{n-1}|\dots|\alpha_2||\alpha_1||\mu|^{n+1}} |\alpha_0||\alpha_0| \dots |\alpha_0| \|Xe_1\| \longrightarrow \infty \quad as \quad n \longrightarrow \infty$$

So $\|Xe_{n+1}\| \longrightarrow \infty$ .This is a contradiction. Since X is bounded operator.
Thus, $E(A_\alpha) = \{ \mu \in \mathbb{C} : |\mu| \geq \max\{1, \alpha\}\}$.

The aim of the following theorem is to calculate extended eigenvalues for the weighted unilateral backward shift operator B_α .

Theorem (3.2): If B_α is the weighted unilateral backward shift operator on $l^2(\mathbb{N})$ then $E(B_\alpha) = \{ \mu \in \mathbb{C} : 0 < |\mu| \leq 1\}$.
Proof: Suppose that $0 < |\mu| \leq 1$ and let D_μ be the diagonal operator such that $D_\mu(x_1, x_2, x_3, \dots) = (x_1, \mu x_2, \mu^2 x_3, \dots)$ for all $x \in l^2(\mathbb{N})$. Claim that:
$B_\alpha D_\mu = \mu D_\mu B_\alpha$ .To prove this claim, we start with the left side that means
$B_\alpha D_\mu(x_1, x_2, x_3, \dots) = B_\alpha(x_1, \mu x_2, \mu^2 x_3, \dots) = (\alpha_1 \mu x_2, \alpha_2 \mu^2 x_3, \dots)$
It is clear that the right side is given as follows:
$\mu D_\mu B_\alpha(x_1, x_2, x_3, \dots) = \mu D_\mu(\alpha_1 x_2, \alpha_2 x_3, \dots) = \mu(\alpha_1 x_2, \alpha_2 \mu x_3, \dots)$
$= (\alpha_1 \mu x_2, \alpha_2 \mu^2 x_3, \dots)$
Thus, $B_\alpha D_\mu = \mu D_\mu B_\alpha$. Note that D_μ nonzero operator, that is $\mu \in E(B_\alpha)$.
Now let $|\mu| > 1$ and let D_μ be the diagonal operator such that $B_\alpha D_\mu = \mu D_\mu B_\alpha$.
If we take the left side,
$B_\alpha D_\mu(x_1, x_2, x_3, \dots) = B_\alpha(x_1, \mu x_2, \mu^2 x_3, \dots) = (\alpha_1 \mu x_2, \alpha_2 \mu^2 x_3, \dots)$.
But the vector $(\alpha_1 \mu x_2, \alpha_2 \mu^2 x_3, \dots)$ does not belong to $l^2(\mathbb{N})$.
Therefore, $E(B_\alpha) = \{ \mu \in \mathbb{C} : 0 < |\mu| \leq 1\}$.

The goal of the following theorem is to calculate an extended eigenvalues for the bilateral shift operator B .

Theorem (3.3): If B is the bilateral shift operator on $l^2(\mathbb{Z})$ then $\mu \in E(B)$ such that $\{\mu \in \mathbb{C} : |\mu| = 1\}$.
Proof: Suppose that $|\mu| = 1$ and let D_μ be the diagonal operator such that $D_\mu(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, \mu^{-2} x_{-1}, \mu^{-1} x_0,, x_1, \mu x_2, \dots)$ for all $x \in l^2(\mathbb{Z})$. Claim that $BD_\mu = \mu D_\mu B$. To prove this claim.
The left side, $BD_\mu(\dots, x_{-1}, x_0, x_1, \dots)$
$= B(\dots, \mu^{-2} x_{-1}, \mu^{-1} x_0,, x_1, \mu x_2, \dots) = (\dots, \mu^{-2} x_{-2}, \mu^{-1} x_{-1}, x_0,, \mu x_1, \dots)$
It is clear that the right side is given as follows:
$\mu D_\mu B(\dots, x_{-1}, x_0, x_1, \dots) = \mu D_\mu(\dots, x_{-2}, x_{-1}, x_0, \dots)$
$= \mu(\dots, \mu^{-3} x_{-2}, \mu^{-2} x_{-1}, \mu^{-1} x_0, x_1, \dots)$
$= (\dots, \mu^{-2} x_{-2}, \mu^{-1} x_{-1}, x_0, \mu x_1, \dots)$
Thus, $BD_\mu = \mu D_\mu B$.
Note that D_μ is a nonzero operator, that is $\mu \in E(B)$.

In the following theorems, we will show that the set of extended eigenvalues for weighted unilateral (Forward and Backward) shift operators are closed under multiplication .

Theorem (3.4): If A_α is the weighted unilateral Forward shift operator on $l^2(\mathbb{N})$,then $E(A_\alpha)$

is closed under multiplication operation that is if
$\mu \in E(A_\alpha)$ and $\lambda \in E(A_\alpha)$ then $\mu\lambda \in E(A_\alpha)$.

Proof : Suppose that $D_{\frac{1}{\mu}}$ *and* $D_{\frac{1}{\lambda}}$ are two diagonal operators for A_α which are defined in theorem (2.1) which are extended eigenoperators for A_α corresponding to μ , λ, respectively, such that:

$A_\alpha D_{\frac{1}{\mu}} = \mu D_{\frac{1}{\mu}} A_\alpha$ and $A_\alpha D_{\frac{1}{\lambda}} = \lambda D_{\frac{1}{\lambda}} A_\alpha$.

Now, $A_\alpha D_{\frac{1}{\mu}} D_{\frac{1}{\lambda}} = \mu D_{\frac{1}{\mu}} A_\alpha D_{\frac{1}{\lambda}} = \mu D_{\frac{1}{\mu}} \lambda D_{\frac{1}{\lambda}} A_\alpha = \mu\lambda D_{\frac{1}{\mu}} D_{\frac{1}{\lambda}} A_\alpha$.

Thus, $\mu\lambda \in E(A_\alpha)$. Finally, to show that $D_{\frac{1}{\mu}} D_{\frac{1}{\lambda}}$ nonzero operator:

For all $x \in l^2(\mathbb{N})$ then $D_{\frac{1}{\mu}} D_{\frac{1}{\lambda}} (x_1, x_2, x_3, \dots) = D_{\frac{1}{\mu}} \left(x_1, \frac{1}{\lambda} x_2, \frac{1}{\lambda^2} x_3, \dots\right)$

$$= \left(x_1, \frac{1}{\lambda\mu} x_2, \frac{1}{\lambda^2\mu^2} x_3, \dots\right) \neq 0$$

Thus , $D_{\frac{1}{\mu}} D_{\frac{1}{\lambda}}$ nonzero operator.

Theorem (3.5): If B_α is the weighted unilateral backward shift operator on $l^2(\mathbb{N})$ then, $E(B_\alpha)$ is closed under multiplication that is if $\mu \in E(B_\alpha)$ and $\lambda \in E(B_\alpha)$,then $\mu\lambda \in E(B_\alpha)$.

Proof: Suppose that D_μ *and* D_λ are two diagonal operators for B_α which defined in Theorem (2.2) which are extended eigenoperators for B_α corresponding to μ , λ, respectively, such that : $B_\alpha D_\mu = \mu D_\mu B_\alpha$ and $B_\alpha D_\lambda = \lambda D_\lambda B_\alpha$.

Now $B_\alpha D_\mu D_\lambda = \mu D_\mu B_\alpha D_\lambda = \mu D_\mu \lambda D_\lambda B_\alpha = \mu\lambda D_\mu D_\lambda B_\alpha$

Thus, $\mu\lambda \in E(B_\alpha)$.

Finally, to show that $D_\mu D_\lambda$ nonzero operator for all $x \in l^2(\mathbb{N})$, then $D_\mu D_\lambda (x_1, x_2, x_3, \dots)$

$= D_\mu (x_1, \lambda x_2, \lambda^2 x_3, \dots)$

$= (x_1, \mu\lambda x_2, \mu^2\lambda^2 x_3, \dots) \neq 0$

Thus, $D_\mu D_\lambda$ nonzero operator.

Conclusions

The goal of this paper is to calculate extended eigenvalues and eigenoperators for weighted unilateral (Forward and Backward) shift operators. We also find extended eigenvalues for weighted bilateral shift operators.

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