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2-prime submodules of modules

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Abstract:
Let R be a commutative ring with unity. And let E be a unitary R-module. This paper introduces the notion of 2-prime submodules as a generalized concept of 2-prime ideal, where proper submodule H of module F over a ring R is said to be 2-prime if rx ∈ H, for r∈R and x∈F implies that x ∈ H or r^2 ∈ [H: F]. we prove many properties for this kind of submodules, Let H is a submodule of module F over a ring R then H is a 2-prime submodule if and only if [$N_{\dot{F}}(r)$] is a 2-prime submodule of E, where r∈R. Also, we prove that if F is a non-zero multiplication module, then [K: F] ⊈[H: F] for every submodule k of F such that H⊊K. Furthermore, we will study the basic properties of this kind of submodules.

Keywords: prime ideal, prime submodules, 2-prime ideal, primary submodule.

المقاسات الجزئية شبه الاولية من النمط –٢–

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الخلاصة:
لتكن Rحلقه ابداليه ذات عنصر محايد وليكن Eمقاساً احادياً على R. هذا البحث يُعطي تعريف مفهوم المقاسات الجزئية الاولية من النمط–٢– كتعميم لمفهوم المثالي الاولي من النمط –٢– حيث يقال للمقاس الجزئيH من المقاس F المعرف على الحلقة R أنه مقاس جزئي اولي من النمط ٢ اذا كان $rx \ni$Hحيث $r \ni R$ و $x \ni F$ فأن $x \ni H$ او $r^2 \ni [H:F]$. لقد برهنا عدة خصائص لهذا النوع من المقاسات الجزئية, على سبيل المثال : اذا كان H مقاس جزئي من المقاس E على الحلقهR , فأن H هو مقاس جزئي اولي من النمط ٢ اذا وفقط اذا كان [$N_{\dot{F}}(r)$] كان مقاس جزئي اولي في F من النمط ٢ حيث $r \ni R$. كذلك برهنا اذا كان F مقاس جدائي غير صفري فأن [K: F] ⊉[H: F] لأي مقاس جزئي K من المقاس F بحيث ان H⊋K كذلك درسنا الخصائص الرئيسة لهذا النوع من المقاسات الجزئية.

1. Introduction

Let R be a commutative ring with unity, an ideal P of a ring R is prime if for all elements a, b ∈R, $ab \in p$ implies that either a∈p or b∈p [1, Def (2.8), p4], as a generalization of the prime ideal, [2] introduced prime submodule where a proper submodule H of module F over a ring R is said to be prime if $rx \in H$, for $r \in H$, and $x \in F$, then either x∈H or $r \in$[H: F] W.Messirdi introduced in [3] 2-prime ideals where a proper ideal I of a ring R is 2-prime ideal if for all x,y∈R such that xy∈I then either x^2or y^2 lies in I. This paper is devoted to studying a generalization of 2-prime ideals. A proper submodule of H of module F over a ring R is said

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to be 2-prime submodule, if rx∈H, where r∈R, x∈F then either x∈H or r^2 ∈[H: F]. This definition appeared in [4] and it is called a 2-primary submodule, however, in our work, it is convenient to call it a 2-prime submodule. We prove many properties for this kind of submodules such as if H is a submodule of module F over a ring R then H is a 2-prime submodule if and only if $[N_{\dot{F}}(r)]$ is a 2-prime submodule of E.

2. Basic result for a 2-prime submodule.

The concept of n-primary submodules was introduced by [4]. Let H be a proper submodule of an R-module F and n∈ Z_+. H is called n-primary submodule, if whenever r∈R, x∈F, rx ∈H implies x∈H or r^n ∈[H: F] where [H: F] ={r∈R, rF⊆H}, we shall study the case when n=2 for this kind of submodules.

Definition (2.1): Let H be a proper submodule of an R-module F, H is called 2-prime submodule, if whenever r∈R, x∈F, rx ∈F, implies x∈H or r^2 ∈[H: F].

Remarks and Examples (2.2):

1. Every prime submodule is a 2-prime submodule.

Proof: Let H be a prime submodule of an R-module F and let r∈R, x∈F such that rx ∈H, since H is a prime submodule, either x∈H or r ∈[H: F], hence r^2 ∈[H: F] and therefore H is a 2-prime submodule.

2. The converse of (1) is not true in general for example the submodule 4Z in Z-module Z is a 2-prime submodule, however, it is not prime submodule.

3. The converse of (1) is true, if $[N_{\dot{R}}F]$ is the semi-prime ideal of R which means if a submodule N of an R-module is 2-prime and $[N\colon F]$is semiprime ideal, then N is prime.

Proof: Let $\mathrm{rx} \in \mathrm{N}$, where∈ R, x∈ F. Since N is a 2-prime submodule then either x∈N or $\mathrm{r}^2 \in [\mathrm{N}\colon \mathrm{F}]$, so r∈ $\sqrt{[\mathrm{N}\colon \mathrm{F}]}$, but N is semiprime ideal. So r∈ [N: F], hence N is a prime submodule.

4. The Z-module $Z_{P\infty}$ has no 2-prime submodules.

Proof: Every proper submodule in $Z_{p\infty}$is of the form$\langle\frac{1}{p^n} + Z\rangle$where n∈ $Z_+$and by [5] if k is a proper submodule of $Z_{p\infty}$, then $[K\colon Z_{p\infty}] = (0)$. Now, let N⊊ $Z_{p\infty}$ and thus N=$\langle\frac{1}{p^i} + Z\rangle$, where i ∈Z₊, It's clear that $p\frac{1}{p^{i+1}} + Z = \frac{1}{p^{i+1}} + Z$ ∉N and $p^2 \notin [N\colon Z_{p\infty}] = (0)$. Then, N is not a 2-prime submodule.

5. Every 2-prime submodule is a primary submodule where a proper submodule H of an R-module F is primary, whenever $\mathrm{rx} \in \mathrm{H}$ for r∈R and x∈F then either x∈H or $r^n \in [H_{\dot{R}} F]$ for some n∈Z_+ [6].

6. The converse of (4) is not true as the following example shows: Consider Z as Z-module let H=8Z is a submodule of Z. H is primary, but $2 \cdot 4$ ∈H and $2^2 \notin [8Z\colon Z] = 8Z$.So that H is not a 2-prime submodule.

Note: prime submodule $\underset{\not\leftarrow}{\rightarrow}$2-prime submodule $\underset{\not\leftarrow}{\rightarrow}$primary submodule.

7. Let F be a module on integral domain R. Then $\tau(F)$ is a 2-prime submodule if $\tau(F) \neq F$ where $\tau(F)$ is called torsion submodule defined $\tau(F) = \{x \in F\colon \exists r \in R,\ r\neq 0 \text{ such that } rx = 0\}$[7].

Proof: By [[8], remark and examples (1.2).P6] $\tau(F)$ is prime when$\tau(F) \neq F$.

Thus, by (1) $\tau(F)$ is a 2-prime submodule.

8. Let F be a torsion-free module over an integral domain R. Then every pure submodule of F is 2-prime where module F over an integral domain R is torsion-free if $\tau(F) = 0$. And a submodule K of F is pure if $IF \cap K = IK$ for every ideal I of R.

Proof: clear from [[8], remark, and examples (1.2), P6], and by (1)]

Proposition (2.3): If N is a 2-prime submodule of an R-module F then [N: F] is the 2-prime ideal of R.

Proof: Let $a, b \in R$ such that $ab \in [N: F]$, assume $a \notin [N: F]$ thus there exists $x \in F$ and $ax \notin N$ but $ab \in [N: F]$, so $abx \in N$, but N is 2-prime submodule and $ax \notin N$ thus $b^2 \in [N: F]$ Therefore, [N: F] is the 2-prime ideal.

Remark (2.4): The converse of proposition (2.3) is not true in general for example: Let $F = Z \oplus Z$ as Z-module and let $N = (0) \oplus 8Z$. Then $[N: F] = (0)$ which is 2-prime ideal. But, N is not a 2-prime submodule since $2(0,4) \in N$, but $(0, 4) \notin N$ and $2^2 \notin [N: F] = (0)$

Now, we give a characterization for 2-prime submodules.

Theorem (2.5): Let N be a proper submodule of an R-module F. The following three statements are equivalent:

1. N is a 2-prime submodule,
2. $a^2 \in [N \dot{:}_R F]$, $a \in R$ if and only if $a^2 \in [N \dot{:}_R (c)]$ for every $c \in F$.
3. $a^2 \in [N \dot{:}_R F]$, $a \in R$ if and only if $a^2 \in [N: K]$ for every submodule K of F such that N<K.

Proof: $(\mathbf{1}) \Rightarrow (\mathbf{2})$ let $c \in F \backslash N$ if $a^2 \in [N: F]$ then it is clear that $a^2 \in [N \dot{:}_R (c)]$ then $a(ac) \in N$ Since N is a 2-prime submodule. Then either $ac \in N$ or $a^2 \in [N: F]$. If $a^2 \in [N: F]$, then there is nothing to do. If $ac \in N$. as N is 2-prime submodule and $c \notin N$, so $a^2 \in [N: F]$, hence we get the result.

$(\mathbf{2}) \Longrightarrow (\mathbf{3})$ If $a^2 \in [H: F]$, then $a^2 \in [N: K]$, if $a^2 \in [N: K]$ where $N \subsetneq K$. Thus $a^2 K \subseteq N$. Since $N \subsetneq K$ and $x \notin N$ and so $a^2 x \in N$, i.e. $a^2 \in [N: (x)]$. it follows $a^2 \in [N: F]$ (by condition 2).

(3) ⟹(1) let $am \in N$ and suppose $m \notin N$, put $K=N+<m>$, so $k \supsetneq N$. Then $a^2 = a^2 (N+<m>) = a^2 N + a^2 <m> \subseteq N$ therefore $a^2 \in [N: K]$. Hence by condition (3), $a^2 \in [N: F]$ and thus N is a 2-prime submodule.

By using theorem (2.5), we have the following result.

Corollary (2.6): Let F be an R-module and N is a proper submodule. The following statements are equivalent:-

1. (0) is a 2-prime submodule,
2. $a^2 \in \text{ann}F$ $a \in R$ if and only if $a^2 \in \text{ann}(c)$, Where $c \notin N$,
3. $a^2 \in \text{ann}F$, $a \in R$ if and only if $a^2 \in \text{ann}(c)$, where K is a submodule of F such that $N \subsetneq K$.

Proposition (2.7): Let H be a proper submodule of module F over a ring R. Then H is a 2-prime submodule if and only if $[H \dot{:}_F (r)]$ is a 2-prime submodule for every $r \in R$.

Proof: (⟹)let $ax \in [H \dot{:}_F (r)]$, so $axr \in H$. Since H is a 2-prime submodule, Then either $xr \in H$ or $a^2 \in [H: F]$. If $xr \in H$, then $x \in [H \dot{:}_F (r)]$ and if $a^2 \in [H: F]$, hence $a^2 F \subseteq H$. So $a^2 Fr \subseteq Hr \subseteq H$. i.e. $a^2 Fr \subseteq N$, so $a^2 F \subseteq [H \dot{:}_F (r): F]$. Thus, $[H \dot{:}_F (r)]$ is a 2-prime submodule.

(⟸)Now, let $ax \in H, x \in F, a \in R$, so $axr \in Hr \subseteq H$ and thus $axr \in H$. Therefore $ax \in [N \dot{:}_F (r)]$ but $[H \dot{:}_F (r)]$ is a 2-prime submodule, so either $x \in [H \dot{:}_F (r)]$ or $a^2 \in [H: (r)): F]$. If $x \in [H: (r)]$ take r=1 then $x \in H$ and if $a^2 \in [(H: (1)): F] = [a^2: F]$ therefore, H is a 2-prime submodule.

Proposition (2.8): Let F, $\acute{F}$ be R-modules and $\theta: F \longrightarrow \acute{F}$ be an epimorphism and L is a submodule of F such that $\ker\theta \leq L$. If N is a 2-prime submodule of F such that $L \leq N$, then $\theta(N)$ is a 2-prime submodule of $\acute{F}$ such that $\theta(L) \leq \theta(N)$.

Proof: First we claim that $\theta(N)$ is a proper submodule of $\acute{F}$. If not i.e. $\theta(N) = \acute{F}$, thus for every $a \in F$ there exists $n \in N$ such that $\theta(n) = \theta(a)$ so $a - n \in \ker\theta \leq L \subseteq N$, hence $a \in N$ therefore, N=F, but this is a contradiction. Thus, $\theta(n) \leq \acute{F}$

Now, let $r \in R$ and $\acute{x} \in \acute{F}$ such that $r\acute{x} \in \theta(N)$, but θ is an epimorphism and thus there exists $x \in F$ such that $\theta(x) = \acute{x}$. So $r\acute{x} = r\theta(x) = \theta(rx) \in \theta(n)$ then there exists $y \in N$ such that $\theta(y) = \theta(rx)$. i.e. $rx - y \in \ker\theta \subseteq L \leq N$, so $rx \in N$ but N is a 2-prime submodule of F and thus either $x \in N$ or $r^2 \in [N: F]$ and therefore either $\theta(x) = \acute{x} \in \theta(N)$ or $r^2 \in [\theta(N): \acute{F}]$, hence $\theta(n)$ is the 2-prime submodule of $\acute{F}$. It's clear that $\theta(L) \leq \theta(N)$.

Corollary (2.9): Let N, H be submodules of an R-module F such that $H \leq N$ and N is 2-prime of F, $\frac{N}{H}$ is a 2-prime submodule of $\frac{F}{H}$.

Proof: Let $\pi: F \longrightarrow \frac{F}{H}$ be an R-homomorphism since N is 2-prime of F then by proposition (2.8),$\pi(N)$ is a 2-prime submodule of, $\frac{F}{H}$.

Corollary (2.10): Let F be an R-module and $H \leq N \leq L \leq F$ such that $\frac{L}{H}$ is a 2-prime submodule of $\frac{F}{H}$, then $\frac{L}{N}$ is a 2-prime submodule of $\frac{F}{N}$.

Proof: $\theta: \frac{F}{H} \longrightarrow \frac{F}{N}$ be the map defined by $\theta(x + H) = x + N$, $\forall\, x \in F$. Clear that θ is an epimorphism, since $\frac{L}{H}$ is a 2-prime submodule of $\frac{F}{H}$ then $\theta(\frac{L}{H})$ is 2-prime in $\frac{F}{H}$. That means $\frac{L}{N}$ is a 2-prime submodule of $\frac{F}{H}$.

Corollary (2.11): If T, Y are two submodules of module F such that T is 2-prime of F then T is 2-prime in Y.

Proof: Letr $\in$ R, a$\in$Y such that ra$\in$T since T is 2-prime in F. Then either a $\in$X or $r^2 \in [T: F]$ i.e. $r^2F \subseteq T$ since Y$\subseteq$F. Then $r^2 \in [T: Y]$ therefore, T is 2-prime in Y.

Corollary (2.12): Let F be an R-module and $H < N < F$. If N is a direct summand of F and H is a 2-prime submodule of F. Then H is a 2-prime submodule of N.

Proof: since N is the direct summand of F, $N \oplus L = F$ for L<F but H is a 2-prime submodule of F, then $H \oplus (0)$ is 2-prime of N$\oplus$L=F by (proposition 3.12).

Thus, H is a 2-prime submodule of N.

Remark (2.13): If T and Y are two submodules such that $T \leq Y$ and Y is a 2-prime submodule of module F, then T is not 2-prime of F, for example: Consider Z_{12} as a Z-module and T= $\{\overline{0}, \overline{6}\}$, $Y = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}$.Y is a 2-prime submodule of Z_{12} since $\overline{2} \cdot \overline{3} = \overline{6} \in Y, \overline{3} \in Y$, but $2^2 \notin [Y\colon Z_{12}] = 3Z$.but T is not a 2-prime of Z_{12}since $\overline{2} \cdot \overline{3} \in T, \overline{2}, \overline{3} \notin T$, and $2^2, 3^2 \notin [T\colon Z_{12}]$.

Remark (2.14): The 2-prime submodule is not transitive which means if $H \leq N \leq F$ and H is a 2-prime submodule of N and N is a 2-prime submodule, then H does not a 2-prime submodule of F, as the example in Remark (2.12) in the Z-module Z_{12} and $\{\overline{0}, \overline{3}, \overline{6}, \overline{9}\}$ is a 2-prime submodule of Z_{12}, however, $\{\overline{0}, \overline{6}\}$ is not 2-prime submodule of Z_{12}

3 The other result about 2-prime submodules.

In this section, we give the relation between 2-prime submodules and maximal submodule, primary. Also, we study the 2-prime submodules in the module of fractions.

Definition (3.1) [9]: Let F be an R-module and H be a proper submodule of F. H is called maximal if for every submodule L of F such that L $\subseteq$ H then either H=L or F=L.

Lemma (3.2) [10]: If H is a maximal submodule of F, then $[H:_R F]$is maximal ideal in R.

Proposition (3.3) [10]: Let H a submodule of F, if [H: F] maximal ideal in R, then H is prime submodule in F.

Corollary (3.4): Let H be a submodule of F. If [H: F] maximal ideal in R, then H is a 2-prime submodule of F.

Proof: By Proposition 3.3 and by Remarks and Examples (2.2) (1).

Corollary (3.5): If K is a proper submodule of an R-module F such that H $\subseteq$ K and the ideal [H: F] is maximal in R then K is a 2-prime submodule of F.

Proof: It's clear that $[H\colon F] \subseteq [K\colon F]$ since H$\subseteq$K. But [K: F] is proper in R and K is proper submodule in F. Also, [H: F] is maximal in R then [K: F] = [H: F] and by corollary (3.4) then K is 2-prime submodule of F.

Corollary (3.6): Every maximal submodule is a 2-prime submodule.

Proof: By [8, corollary (2.5), P14] and by remarks and examples (2.2) (1).

Corollary (3.7): Let F is an R-module and I is maximal ideal if $F \neq IF$, then IF is a 2-prime submodule of F.
Proof: By [8, cor. (2.6), p14], IF is prime and by Remarks and Examples (2.2) (1), IF is a 2-prime submodule.
Proposition (3.8): Let H be a proper submodule of module F over Ring R such that $[K: F] \subseteq [H: F]$ for every submodule K of F with $H \subsetneq K$. Then H is 2-prime of F if and only if the ideal [H: F] is 2-prime ideal in R.
Proof:($\Leftarrow$) suppose [H: F] is 2-prime ideal in R and let $rx \in H$ where $r \in R, x \in F$ and $x \notin H$. $H + \langle x \rangle = L$ Contain H properly, thus $[L: F] \nsubseteq [H: F]$therefore there exists$t \in [L: F]$ and $t \notin [H: F]$. Hence, $tF \subseteq L$ and $tF \nsubseteq H$. But, $rtF \subseteq H$ i.e $rt \in [H: F]$. Since [H: F] is the 2-prime ideal of R and $t \in [H: F]$. Thus, $r^2 \in [H: F]$ therefore, H is a 2-prime submodule of F.
($\Rightarrow$)(By proposition 2.3).
Remark (3.9): [8, Remark (2.15), P18] If $E \neq 0$ is multiplication module and N is a proper submodule of E, then $[K: E] \nsubseteq [H: E]$ for every submodule K of E such that $H \subsetneq K$.
Corollary (3.10): Let H be a proper submodule of a multiplication module E, then H is a 2-prime in E if and only if the ideal [H: E] is 2-prime in R.
Proof: It's obvious from Proposition (3.9) and Remark (3.10)
Proposition (3.11): Let E_1 and E_2 be two R-modules. Then N_1 *and* N_2 are 2-prime submodules of E_1 and E_2, respectively if and only if $N_1 \oplus N_2$ is 2-prime submodule $E_1 \oplus E_2$.
Proof:($\Rightarrow$) Let $r \in R$ and $x=(x_1, x_2) \in E_1 \oplus E_2$ where$x_1 \in E_1$and $x_2 \in E_2$ such that $rx \in N_1 \oplus N_2$. Thus $(rx_1, rx2) = (n_1, n_2)$ for some$n_1 \in N_1$,$n_2 \in N_2$ this implies that $rx_1 = n_1 \in N_1$ and $rx_2 = n_2 \in N_2$. But each of N_1 and N_2 is 2-prime submodules of E_1 and E_2 respectively. And therefore either $x_1 \in N_1$ or $r^2 \in [N_1: E_1]$ and either$x_2 \in N_2$ or $r^2 \in [N_2: E_1]$. Hence either $x=(x_1, x_2) \in N_2 \oplus N_2$ or $r^2 \in [N_1 + N_2: E_1 + E_2]$
($\Leftarrow$) Let $\rho_1: E_1 \oplus E_2 \longrightarrow E_1$ be the natural projection and suppose that H is the 2-prime submodule of E. By proposition (2.8) $\rho_1(H)$ is the 2-prime submodule of E_1, i.e. H_1 is a 2-prime submodule of E_1. Similarly, H_2 is a 2-prime submodule of E_2.
Proposition (3.12): Let N be a 2-prime submodule of an R-module E such that$S^{-1}(N) \neq S^{-1}(E)$. Then $S^{-1}N$ is a 2-prime submodule of, $S^{-1}E$ [where $S^{-1}E = \{\frac{x}{s}$: where $x \in E$ and $s \in S\}$ (S is a multiplicative subset of R)].
Proof: First, notice that $S^{-1}(N)$ is proper of $S^{-1}(E)$. Now, let $r \in R$ and $\frac{r}{s} \in S^{-1}(R)$ and $\frac{m}{\acute{s}} \in S^{-1}(E)$ such that $\frac{r}{s} \cdot \frac{m}{\acute{s}} \in S^{-1}(N)$ and thus $\exists\ n \in N$ and $t \in S$ such that $\frac{rm}{s\acute{s}} = \frac{n}{t}$ this means that $\exists\ w \in S$ such that $wtrm \in N$. So $r(wtm) \in N$. But N is a 2-prime submodule. Then, either $wtm \in N$ or $r^2 \in [N: E]$. Thus, $\frac{wtm}{wt\acute{s}} \in N_s$ but $\frac{wtm}{wt\acute{s}} = \frac{m}{\acute{s}} \in N_s$ $\because r^2 \in [N: E]$ then $\frac{r^2}{s} \in [S^{-1}(N): S^{-1}(E)]$ then $\frac{1}{s} \cdot \frac{r^2}{s} = \frac{r^2}{s^2} \in [S^{-1}(N): S^{-1}(E)]$.
Proposition (3.13): Let E be an R-module, S multiplicative subset of R, if W is 2-prime submodule in $S^{-1}(E)$, then $\varphi^{-1}(W)$ is a 2-prime submodule of E.
Proof: since $W \neq S^{-1}(E)$ so $S \cap [\varphi^{-1}(W): E] = \Phi \Rightarrow \varphi^{-1}(W)$=E. Let $r \in R, m \in E$ such that $rm \in \varphi^{-1}(W) \Rightarrow \varphi(rm) = r\varphi(m) = rm \in W$ but W is 2-prime in $S^{-1}(E)$ then either $\frac{m}{1} \in W$ or $\frac{r^2}{1} \in [W: S^{-1}(E)]$then $\varphi^{-1}(m) = m \in W$ or$r^2 \in [\varphi^{-1}(W): E] \therefore \varphi^{-1}(W)$is 2-prime ideal in E.
Definition (3.14): An R-module M is called Noetherian if the set of all sub-modules of M is Noetherian. [7].
Definition (3.15): An R-module E is faithful if its annihilator is zero [7].
Definition (3.16): An R-module E is called multiplication, if for every submodule H of E, $H = (H: E)E$ where $(H: E) = \{r \in R: rE \subseteq H\}$ [11]

Proposition (3.17): Let F be a faithful multiplication Noetherian R-module. Then, F satisfies d.c.c on 2-prime submodules.

Proof: Let $H_1 \supseteq H_2 \supseteq \cdots$ be descending chain of 2-prime submodules of F. Then, $[H_1: F] \supseteq [H_2: F] \supseteq \cdots (1)$.

On the other hand, by proposition (2.3), for each $i \in Z_+ [H_i: F]$ is the 2-prime ideal of R, hence (1) is a descending of 2-prime ideals of R. But F is a faithful multiplication Noetherian R-module, so R is a Noetherian [6, 5.3, P.767], hence R satisfies d.c.c on 2-prime, it follows that $\exists k \in Z_+$such that $[H_k: F] = [H_{k+1}: F] = \cdots$.Therefore, $[H_k: F]F = [H_{k+1}: F]F = \cdots$.Thus, $H_k = H_{k+1} = \cdots$.

We will give the difference between our work and the work in [4] and we give a conclusion for each of them.

The difference between the work in [4] and our work is that every prime submodule is a 1-primary, hence it is an n-primary submodule, [for each n$\in Z_+$]. It makes a difference when $n = 1, n = 2$.

Now, we will summarize the main result of [4] as follows:

1. Every n-primary submodule is a primary submodule.
2. If N is an n-primary submodule of an R-module F, then N is an n+1-primary submodule.
3. It is clear every prime submodule is a 1-primary, and hence it is an n-primary submodule, for each n$\in Z_+$.
4. If N is a 2-prime submodule of an R-module F, then it is not necessary that N is an n-primary submodule as the following example shows; Let F be the Z module Z. Let N =125Z. It is easy to show that N is a 3-primary submodule of F. However N is not 2-primary since 125=5.25∈N, but 25∉N and $5^2 = 25 \notin (N: Z)$.
5. If N is an n-primary submodule of an R-module F, then (N: F) is an n-primary ideal of R.
6. The converse of (5) is not true as the following example; Let F be the Z-module$Z \oplus Z$, let N=0⊕8Z then [N: M] = (0), which is an n-primary ideal, for each n$\in Z_+$. However, N is not an n-primary submodule, for any n$\in Z_+$, since 2(0, 4) ∈N, but (0, 4) ∉N and $2^n \notin$(N: F) = (0), for any n$\in Z_+$.
7. Let N submodule of an R-module F, such that for each submodule K of F, K⊃N and [K: F] ⊄[N: F]. Then N is n-primary submodule if and only if [N: F] is an n-primary ideal of R.
8. Let F be a multiplication R-module, N a submodule of F and n$\in Z_+$. Then N is an n-primary submodule of F if and only if [N: F] is an n-primary ideal of R.
9. Let N be a primary submodule of an R-module F, let n$\in Z_+$. Then, N is the n-primary submodule of F if and only if [N: F] is an n-primary ideal of R.
10. Let F be a multiplication R-module, and let I be an n-primary ideal of R. Then, IF is an n-primary submodule of F.

Conclusions

1. If N is a 2-prime submodule of an R-module F, Then [N: F] is the 2-prime ideal of R.
2. The converse of (1) is not true in general for example: Let $F = Z \oplus Z$ as Z-module and let $N = (0) \oplus 8Z$.Then [N: F] = (0) which is 2-prime ideal. But N is not a 2-prime submodule since$2(0,4) \in N$, but (0, 4) ∉N and$2^2 \notin [N: F] = (0)$.
3. Let H be a proper submodule of module F over a ring R. Then H is a 2-prime submodule if and only if$[H_{\dot{F}}(r)]$ is a 2-prime submodule, for every r∈R.
4. Let F, $\acute{F}$ be an R-modules and $\theta: F \longrightarrow \acute{F}$ be an epimorphism and L is a submodule of F such that $\ker\theta \leq L$. If N is a 2-prime submodule of F such that $L \leq N$ then $\theta(N)$ is a 2-prime submodule of $\acute{F}$ such that$\theta(L) \leq \theta(N)$.
5. If T, Y two submodules such that $T \leq Y$ and Y is a 2-prime submodule of a module F. Then T not necessary to be 2-prime of F, for example: Consider Z_{12} as a Z-module and T= $\{\bar{0}, \bar{6}\}, Y = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$.Y is a 2-prime submodule of Z_{12} since $\bar{2} \cdot \bar{3} = \bar{6} \in Y, \bar{3} \in$

Y but $2^2 \notin [Y: Z_{12}] = 3Z$. but T is not a 2-prime of Z_{12}, since $\bar{2} \cdot \bar{3} \in T, \bar{2}, \bar{3} \notin T$, and $2^2, 3^2 \notin [T: Z_{12}]$.

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