2-prime submodules of modules

Fatima Dhiyaa Jasem, Alaa A.Elewi
Department of mathematics, College of Science, University of Baghdad, Baghdad, Iraq

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Abstract:
Let R be a commutative ring with unity. And let E be a unitary R-module. This paper introduces the notion of 2-prime submodules as a generalized concept of 2-prime ideal, where proper submodule H of module F over a ring R is said to be 2-prime if rx ∈ H, for r ∈ R and x ∈ F implies that x ∈ H or r² ∈ [H:F], we prove many properties for this kind of submodules, Let H is a submodule of module F over a ring R then H is a 2-prime submodule if and only if [N_f(r)] is a 2-prime submodule of E, where f ∈ R. Also, we prove that if F is a non-zero multiplication module, then [K:F] ⊆ [H:F] for every submodule k of F such that H ⊆ K. Furthermore, we will study the basic properties of this kind of submodules.

Keywords: prime ideal, prime submodules, 2-prime ideal, primary submodule.

1. Introduction
Let R be a commutative ring with unity, an ideal P of a ring R is prime if for all elements a, b ∈ R, ab ∈ P implies that either a ∈ P or b ∈ P [1, Def (2.8), p4], as a generalization of the prime ideal, [2] introduced prime submodule where a proper submodule H of module F over a ring R is said to be prime if rx ∈ H, for r ∈ R, and x ∈ F, then either x ∈ H or r ∈ [H:F]. W.Messirdi introduced in [3] 2-prime ideals where a proper ideal I of a ring R is 2-prime ideal if for all x,y ∈ R such that xy ∈ I then either x² or y² lies in I. This paper is devoted to studying a generalization of 2-prime ideals. A proper submodule of H of module F over a ring R is said

*Email: alqaisyfatima@gmail.com
to be 2-prime submodule, if \( rx \in H \), where \( r \in R \), \( x \in F \) then either \( x \in H \) or \( r^2 \in [H:F] \). This definition appeared in [4] and it is called a 2-primary submodule, however, in our work, it is convenient to call it a 2-prime submodule. We prove many properties for this kind of submodules such as if \( H \) is a submodule of module \( F \) over a ring \( R \) then \( H \) is a 2-prime submodule if and only if \( [N:F] \) is a 2-prime submodule of \( E \).

2. Basic result for a 2-prime submodule.

The concept of \( n \)-primary submodules was introduced by [4]. Let \( H \) be a proper submodule of an \( R \)-module \( F \) and \( n \in Z_+ \). \( H \) is called \( n \)-primary submodule, if whenever \( r \in R \), \( x \in F \), \( rx \in H \) implies \( x \in H \) or \( r^n \in [H:F] \) where \([H:F] = \{r \in R, rF \subseteq H\}\), we shall study the case when \( n=2 \) for this kind of submodules.

**Definition (2.1):** Let \( H \) be a proper submodule of an \( R \)-module \( F \), \( H \) is called 2-prime submodule, if whenever \( r \in R \), \( x \in F \), \( rx \in E \), implies \( x \in H \) or \( r^2 \in [H:F] \).

**Remarks and Examples (2.2):**
1. Every prime submodule is a 2-prime submodule.
   **Proof:** Let \( H \) be a prime submodule of an \( R \)-module \( F \) and let \( r \in R \), \( x \in E \) such that \( rx \in H \), since \( H \) is a prime submodule, either \( x \in H \) or \( r \in [H:F] \), hence \( r^2 \in [H:F] \) and therefore \( H \) is a 2-prime submodule.
2. The converse of (1) is not true in general for example the submodule \( 4Z \) in \( Z \)-module \( Z \) is a 2-prime submodule, however, it is not prime submodule.
3. The converse of (1) is true, if \([N:F] \) is the semi-prime ideal of \( R \) which means if a submodule \( N \) of an \( R \)-module is 2-prime and \([N:F] \) is semiprime ideal, then \( N \) is prime.
   **Proof:** Let \( rx \in N \), where \( r \in R \), \( x \in F \). Since \( N \) is a 2-prime submodule then either \( x \in N \) or \( r^2 \in [N:F] \), so \( r \in \sqrt{[N:F]} \), but \( N \) is semiprime ideal. So \( r \in [N:F] \), hence \( N \) is a prime submodule.
4. The \( Z \)-module \( Z_{p\infty} \) has no 2-prime submodules.
   **Proof:** Every proper submodule in \( Z_{p\infty} \) is of the form \((\frac{1}{p^n} + Z)\) where \( n \in Z_+ \) and by [5] if \( k \) is a proper submodule of \( Z_{p\infty} \), then \([K:Z_{p\infty}] = (0)\). Now, let \( N \subseteq Z_{p\infty} \) and thus \( N = (\frac{1}{p^n} + Z) \), where \( i \in Z_+ \). It’s clear that \( p^{i+1} + Z = \frac{1}{p^{i+1}} + Z \notin N \) and \( p^2 \notin [N:Z_{p\infty}] = (0) \). Then, \( N \) is not a 2-prime submodule.
5. Every 2-prime submodule is a primary submodule where a proper submodule \( H \) of an \( R \)-module \( F \) is primary, whenever \( rx \in H \) for \( r \in R \) and \( x \in F \) then either \( x \in H \) or \( r^2 \in [H:F] \) for some \( n \in Z_+ \) [6].
6. The converse of (4) is not true as the following example shows: Consider \( Z \) as \( Z \)-module let \( H = 8Z \) is a submodule of \( Z \), \( H \) is primary, but \( 2 \cdot 4 \in H \) and \( 2^2 \notin [8Z:Z] = 8Z \). So that \( H \) is not a 2-prime submodule.
   Note: prime submodule \( \rightarrow \) 2-prime submodule \( \rightarrow \) primary submodule.
7. Let \( F \) be a module on integral domain \( R \). Then \( \tau(F) \) is a 2-prime submodule if \( \tau(F) \neq F \) where \( \tau(F) \) is called torsion submodule defined \( \tau(F) = \{x \in F: \exists r \in R, r \neq 0 \text{ such that } rx = 0\} [7] \).
   **Proof:** By [[8], remark and examples (1.2),P6] \( \tau(F) \) is prime when\( \tau(F) \neq F \).
   Thus, by (1) \( \tau(F) \) is a 2-prime submodule.
8. Let \( F \) be a torsion-free module over an integral domain \( R \). Then every pure submodule of \( F \) is 2-prime where module \( F \) over an integral domain \( R \) is torsion-free if \( \tau(F) = 0 \). And a submodule \( K \) of \( F \) is pure if \( IF \cap K = 1K \) for every ideal \( I \) of \( R \).
   **Proof:** clear from [[8], remark, and examples (1.2),P6], and by (1)

**Proposition (2.3):** If \( N \) is a 2-prime submodule of an \( R \)-module \( F \) then \( [N:F] \) is the 2-prime ideal of \( R \).
Proof: Let \( a, b \in R \) such that \( ab \in [N:F] \), assume \( a \in [N:F] \) thus there exists \( x \in F \) and \( ax \notin N \) but \( ab \in [N:F] \), so \( abx \in N \), but \( N \) is 2-prime submodule and \( ax \notin N \) thus \( b^2 \in [N:F] \). Therefore, \([N:F] \) is the 2-prime ideal.

Remark (2.4): The converse of proposition (2.3) is not true in general for example: Let \( F = \mathbb{Z} \oplus \mathbb{Z} \) as \( Z \)-module and let \( N = (0) \oplus \mathbb{Z} \). Then \([N:F] = (0) \) which is 2-prime ideal. But, \( N \) is not a 2-prime submodule since \( 2(0,4) \in N \), but \((0,4) \notin N \) and \( 2^2 \notin [N:F] \).

Now, we give a characterization for 2-prime submodules.

Theorem (2.5): Let \( N \) be a proper submodule of an \( R \)-module \( F \). The following three statements are equivalent:

1. \( N \) is a 2-prime submodule,
2. \( a^2 \in [N_R,F], a \in R \) if and only if \( a^2 \in [N_F(c)] \) for every \( c \in F \).
3. \( a^2 \in [N_R,F], a \in R \) if and only if \( a^2 \in [N:K] \) for every submodule \( K \) of \( F \) such that \( N < K \).

Proof: (1) \( \Rightarrow \) (2) let \( c \in F \setminus N \) if \( a^2 \in [N:F] \) then it is clear that \( a^2 \in [N_R(c)] \) then \( a(ac) \in N \) since \( N \) is a 2-prime submodule. Then either \( ac \in N \) or \( a^2 \in [N:F] \). If \( a^2 \in [N:F] \), then there is nothing to do. If \( ac \in N \) as \( N \) is 2-prime submodule and \( c \notin N \), so \( a^2 \in [N:F] \), hence we get the result.

(2) \( \Rightarrow \) (3) If \( a^2 \in [H:F], then \) \( a^2 \in [N:K] \), if \( a^2 \in [N:K] \) where \( N \subseteq K \). Thus \( \alpha^2 K \subseteq N \). Since \( N \subseteq K \) and \( x \notin N \) and so \( a^2x \in N \), i.e. \( a^2 \in [N:F] \). It follows \( a^2 \in [N:F] \) (by condition 2).

(3) \( \Rightarrow \) (1) Let \( x \in \mathbb{N} \) and suppose \( m \notin \mathbb{N} \), put \( K = N + <m> \), so \( k \notin \mathbb{N} \). Then \( a^2 = a^2(N + <m>) \), \( a^2N + a^2 <m> \subseteq N \) therefore \( a^2 \in [N:K] \). Hence by condition (3), \( a^2 \in [N:F] \) and thus \( N \) is a 2-prime submodule.

By using proposition (2.5), we have the following result.

Corollary (2.6): Let \( F \) be an \( R \)-module and \( N \) is a proper submodule. The following statements are equivalent:-

1. \( (0) \) is a 2-prime submodule,
2. \( a^2 \in \text{ann}(F), a \in R \) if and only if \( a^2 \in \text{ann}(c) \), where \( c \in F \),
3. \( a^2 \in \text{ann}(F), a \in R \) if and only if \( a^2 \in \text{ann}(c) \), where \( K \) is a submodule of \( F \) such that \( N \subseteq K \).

Proposition (2.7): Let \( H \) be a proper submodule of module \( F \) over a ring \( R \). Then \( H \) is a 2-prime submodule and if only if \( [H_F(r)] \) is a 2-prime submodule for every \( r \in R \).

Proof: \( (\Rightarrow) \) let \( ax \in [H_F(r)] \), so \( ax \in H \). Since \( H \) is a 2-prime submodule, Then either \( x \in H \) or \( a^2 \in [H:F] \). If \( x \notin H \), then \( x \in [H_F(r)] \) and if \( a^2 \in [H:F] \), hence \( a^2 \cap L \subseteq L \). Therefore \( a^2 \in [H:F] \). Thus, \([H_F(r)] \) is a 2-prime submodule.

(\(\Leftarrow\)) Now, let \( ax \in H, x \in F, a \in R \), so \( a \in H \) and if \( H \) is a 2-prime submodule, so \( x \in [H_F(r)] \) and \( a^2 \in [H:F] \). If \( x \in [H:F] \), take \( r = 1 \) then \( x \in H \) and if \( a^2 \in [H:F] \), \( a^2 \in [H:F] \) therefore, \( H \) is a 2-prime submodule.

Proposition (2.8): Let \( F, \hat{F} \) be \( R \)-modules and \( \theta: F \rightarrow \hat{F} \) be an epimorphism and \( L \) is a submodule of \( F \) such that \( \ker \theta \leq L \). If \( N \) is a 2-prime submodule of \( F \) such that \( L \leq N \), then \( \theta(N) \) is a 2-prime submodule of \( \hat{F} \) such that \( \theta(L) \leq \theta(N) \).

Proof: First we claim that \( \theta(N) \) is a proper submodule of \( \hat{F} \). If not i.e. \( \theta(N) = \hat{F} \), thus for every \( a \in F \) there exists \( n \in N \) such that \( \theta(n) = \theta(a) \) so \( a - n \in \ker \theta \leq L \subseteq N \), hence \( a \in N \) therefore, \( N = F \), but this is a contradiction. Thus, \( \theta(N) \leq \hat{F} \).

Now, let \( r \in R \) and \( x \in \hat{F} \) such that \( rx \in \theta(N) \), but \( \theta \) is an epimorphism and thus there exists \( x \in F \) such that \( \theta(x) = x \). So \( rx = r\theta(x) = \theta(rx) \in \theta(N) \) then there exists \( y \in N \) such that \( \theta(y) = \theta(rx) \). i.e. \( rx - y \in \ker \theta \leq L \leq N \), so \( rx \notin N \) but \( N \) is a 2-prime submodule of \( F \) and thus \( x \in N \) or \( r^2 \in [\theta(N):\hat{F}] \), hence \( \theta(N) \) is the 2-prime submodule of \( \hat{F} \). It’s clear that \( \theta(L) \leq \theta(N) \).
Corollary (2.9): Let $N$, $H$ be submodules of an $R$-module $F$ such that $H \leq N$ and $N$ is 2-prime of $F$, $\frac{N}{H}$ is a 2-prime submodule of $\frac{F}{H}$.

Proof: Let $\pi : F \rightarrow \frac{F}{H}$ be an $R$-homomorphism since $N$ is 2-prime of $F$ then by proposition (2.8), $\pi(N)$ is a 2-prime submodule of $\frac{F}{H}$.

Corollary (2.10): Let $F$ be an $R$-module and $H \leq N \leq L \leq F$ such that $\frac{L}{H}$ is a 2-prime submodule of $\frac{F}{H}$, then $\frac{L}{N}$ is a 2-prime submodule of $\frac{F}{N}$.

Proof: $\theta : \frac{F}{H} \rightarrow \frac{F}{N}$ be the map defined by $\theta(x + H) = x + N$, $\forall x \in F$. Clear that $\theta$ is an epimorphism, since $\frac{L}{H}$ is a 2-prime submodule of $\frac{F}{H}$ then $\theta(\frac{L}{H})$ is 2-prime in $\frac{F}{H}$. That means $\frac{L}{N}$ is a 2-prime submodule of $\frac{F}{N}$.

Corollary (2.11): If $T$, $Y$ are two submodules of module $F$ such that $T$ is 2-prime of $F$ then $T$ is 2-prime in $Y$.

Proof: Let $\tau \in R$, $a \in Y$ such that $ra \in T$ since $T$ is 2-prime in $F$. Then either $a \in X$ or $r^2 \in [T : F]$ i.e. $r^2F \subseteq T$ since $Y \subseteq F$. Then $r^2 \in [T : Y]$ therefore, $T$ is 2-prime in $Y$.

Corollary (2.12): Let $F$ be an $R$-module and $H < N < F$. If $N$ is a direct summand of $F$ and $H$ is a 2-prime submodule of $F$. Then $H$ is a 2-prime submodule of $N$.

Proof: since $N$ is the direct summand of $F$, $N \oplus L = F$ for $L < F$ but $H$ is a 2-prime submodule of $F$, then $H \oplus (0)$ is 2-prime of $N \oplus L = F$ by (proposition 3.12).

Thus, $H$ is a 2-prime submodule of $N$.

Remark (2.13): If $T$ and $Y$ are two submodules such that $T \leq Y$ and $Y$ is a 2-prime submodule of module $F$, then $T$ is not 2-prime of $F$, for example: Consider $Z_{12}$ as a $Z$-module and $T= \{0,6\}$, $Y= \{0,3,6,9\}$. $Y$ is a 2-prime submodule of $Z_{12}$ since $2^2 \cdot 3 = 6 \in Y$ but $2^2 \notin [Y : Z_{12}] = 3Z$. but $T$ is not a 2-prime of $Z_{12}$ since $2^2 \cdot 3 \in T$ and $2^2 \cdot 3 \notin [T : Z_{12}]$.

Remark (2.14): The 2-prime submodule is not transitive which means if $H \leq N \leq F$ and $H$ is a 2-prime submodule of $N$ and $N$ is a 2-prime submodule, then $H$ does not a 2-prime submodule of $F$, as the example in Remark (2.12) in the $Z$-module $Z_{12}$ and $\{0,3,6,9\}$ is a 2-prime submodule of $Z_{12}$, however, $\{0,6\}$ is not 2-prime submodule of $Z_{12}$.

3 The other result about 2-prime submodules.

In this section, we give the relation between 2-prime submodules and maximal submodule, primary. Also, we study the 2-prime submodules in the module of fractions.

Definition (3.1) [9]: Let $F$ be an $R$-module and $H$ be a proper submodule of $F$. $H$ is called maximal if for every submodule $L$ of $F$ such that $L \subseteq H$ then either $H=L$ or $F=L$.

Lemma (3.2) [10]: If H is a maximal submodule of F, then [H : R] is maximal ideal in $R$.

Proposition (3.3) [10]: Let $H$ a submodule of $F$, if $[H : F]$ maximal ideal in $R$, then $H$ is prime submodule in $F$.

Corollary (3.4): Let $H$ be a submodule of $F$. If $[H : F]$ maximal ideal in $R$, then $H$ is a 2-prime submodule of $F$.

Proof: By Proposition 3.3 and by Remarks and Examples (2.2) (1).

Corollary (3.5): If $K$ is a proper submodule of an $R$-module $F$ such that $H \subseteq K$ and the ideal $[H : F]$ is maximal in $R$ then $K$ is a 2-prime submodule of $F$.

Proof: It’s clear that $[H : F] \subseteq [K : F]$ since $H \subseteq K$. But $[K : F]$ is proper in $R$ and $K$ is proper submodule in $F$. Also, $[H : F]$ is maximal in $R$ then $[K : F] = [H : F]$ and by corollary (3.4) then $K$ is 2-prime submodule of $F$.

Corollary (3.6): Every maximal submodule is a 2-prime submodule.

Proof: By [8, corollary (2.5), P14] and by remarks and examples (2.2) (1).
Corollary (3.7): Let F is an R-module and I is maximal ideal if F ≠ IF, then IF is a 2-prime submodule of F.

Proof: By [8, cor. (2.6), p14], IF is prime and by Remarks and Examples (2.2) (1), IF is a 2-prime submodule.

Proposition (3.8): Let H be a proper submodule of module F over Ring R such that [K: F] ⊆ [H: F] for every submodule K of F with H ⊆ K. Then H is 2-prime of F if and only if the ideal [H: F] is 2-prime ideal in R.

Proof: (⇐) suppose [H: F] is 2-prime ideal in R and let rx ∈ H where r ∈ R, x ∈ F and x ∉ H. Hence, tx ∈ [H: F] for every submodule K of F with H ⊆ K. Thus, by Proposition (3.10), H is 2-prime of F if and only if the ideal [H: F] is 2-prime in R.

(⇒) (By proposition 2.3).

Remark (3.9): [8, Remark (2.15), P18] If E ≠ 0 is multiplication module and N is a proper submodule of E, then [K: E] ⊆ [H: E] for every submodule K of E such that H ⊆ K.

Corollary (3.10): Let H be a proper submodule of a multiplication module E, then H is a 2-prime in E if and only if the ideal [H: E] is 2-prime in R.

Proof: It’s obvious from Proposition (3.9) and Remark (3.10).

Proposition (3.11): Let E1 and E2 be two R-modules. Then N1 and N2 are 2-prime submodules of E1 and E2, respectively if and only if N1 ⊕ N2 is 2-prime submodule of E1 ⊕ E2.

Proof: (⇒) Let r∈R and x=(x1, x2)∈E1 ⊕ E2 where x1 ∈ E1 and x2 ∈ E2 such that rx ∈ N1 ⊕ N2. Thus (rx1, rx2) = (n1, n2) for some n1 ∈ N1 and n2 ∈ N2, this implies that rx1 = n1 ∈ N1 and rx2 = n2 ∈ N2. But each of N1 and N2 is 2-prime submodules of E1 and E2 respectively. And therefore either x1 ∈ N1 or r2 ∈ [N1: E1] and either x2 ∈ N2 or r2 ∈ [N2: E1]. Hence either x=(x1, x2) ∈ N1 ⊕ N2 or r2 ∈ [N1 + N2: E1 + E2]

(⇐) Let ρ1:E1 ⊕ E2 → E1 be the natural projection and suppose that H is the 2-prime submodule of E. By proposition (2.8) ρ1(H) is the 2-prime submodule of E1, i.e. H1 is a 2-prime submodule of E1. Similarly, H2 is a 2-prime submodule of E2.

Proposition (3.12): Let N be a 2-prime submodule of an R-module E such that S⁻¹(N) ≠ S⁻¹(E). Then S⁻¹N is a 2-prime submodule of, S⁻¹E [where S⁻¹E = {s⁻¹ : s ∈ S} (S is a multiplicative subset of R)].

Proof: First, notice that S⁻¹(N) is proper of S⁻¹(E). Now, let r ∈ R and r.s ∈ S⁻¹(R) and r₂.s ∈ S⁻¹(E) such that r.s,r₂.s ∈ S⁻¹(N) and thus ∃ n ∈ N and t ∈ S such that r.s.n = r₂.n = t this means that ∃ w ∈ S such thatwtm ∈ N. So r(wtm) ∈ N. But N is a 2-prime submodule. Then, either wt ∈ N or r² ∈ [N: E]. Thus, S⁻¹N is a 2-prime submodule of, S⁻¹E [where S⁻¹E = {s⁻¹ : s ∈ S} (S is a multiplicative subset of R)] then r².s = r².s ∈ S⁻¹(N) then r².s ∈ S⁻¹(E) then 1.s.r².s = r².s ∈ S⁻¹(N) then r².s ∈ S⁻¹(E).

Proposition (3.13): Let E be an R-module, S multiplicative subset of R, if W is a 2-prime submodule in S⁻¹(E), then φ⁻¹(W) is a 2-prime submodule of E.

Proof: since W ≠ S⁻¹(E) so S ∩ φ⁻¹(W) = φ ∩ φ⁻¹(W) = E. Let r ∈ R, m ∈ E such that rm ∈ φ⁻¹(W) ⇒ φ(rm) = rm ∈ W but W is 2-prime in S⁻¹(E) then either m ∈ W or r² ∈ [W: S⁻¹(E)] then φ⁻¹(m) = m ∈ W or r² ∈ [φ⁻¹(W): E] = φ⁻¹(W) is 2-prime ideal in E.

Definition (3.14): An R-module M is called Noetherian if the set of all sub-modules of M is Noetherian. [7].

Definition (3.15): An R-module E is faithful if its annihilator is zero [7].

Proposition (3.17): Let F be a faithful multiplication Noetherian R-module. Then, F satisfies d.c.c on 2-prime submodules.

Proof: Let $H_1 \supseteq H_2 \supseteq \cdots$ be descending chain of 2-prime submodules of F. Then, $[H_1 : F] \supseteq [H_2 : F] \supseteq \cdots \ (1)$.

On the other hand, by proposition (2.3), for each $i \in Z_+[H_i : F]$ is the 2-prime ideal of R, hence (1) is a descending of 2-prime ideals of R. But F is a faithful multiplication Noetherian R-module, so R is a Noetherian [6, 5.3, P.767], hence R satisfies d.c.c on 2-prime, it follows that $\exists k \in Z_+$ such that $[H_k : F] = [H_{k+1} : F] = \cdots$. Therefore, $[H_k : F]F = [H_{k+1} : F]F = \cdots$. Thus, $H_k = H_{k+1} = \cdots$.

We will give the difference between our work and the work in [4] and we give a conclusion for each of them.

The difference between the work in [4] and our work is that every prime submodule is a 1-primary, hence it is an n-primary submodule, [for each $n \in Z_+$]. It makes a difference when $n = 1$, $n = 2$.

Now, we will summarize the main result of [4] as follows:

1. Every n-primary submodule is a primary submodule.
2. If N is an n-primary submodule of an R-module F, then N is an n+1-primary submodule.
3. It is clear every prime submodule is a 1-primary, and hence it is an n-primary submodule, for each $n \in Z_+$.
4. If N is a 2-primary submodule of an R-module F, then it is not necessary that N is an n-primary submodule as the following example shows; Let F be the Z module Z. Let $N = 125\mathbb{Z}$. It is easy to show that N is a 3-primary submodule of F. However N is not 2-primary since $125 \cdot 5 = 25 \in \mathbb{N}$ and $25^2 = 625 \in N : (N : Z)$.
5. If N is an n-primary submodule of an R-module F, then $(N : F)$ is an n-primary ideal of R.
6. The converse of (5) is not true as the following example; Let F be the Z-module $\mathbb{Z} \oplus \mathbb{Z}$, let $N = 0 \oplus 8\mathbb{Z}$ then $[N : M] = (0)$, which is an n-primary ideal, for each $n \in Z_+$. However, N is not an n-primary submodule, for each $n \in Z_+$, since $2(0, 4) \in N$, but $(0, 4) \in N$ and $2^n \in (N : F) = (0)$, for any $n \in Z_+$.
7. Let N submodule of an R-module F, such that for each submodule K of F, $K \supseteq N$ and $[K : F] \not\in [N : F]$. Then N is n-primary submodule if and only if $[N : F]$ is an n-primary ideal of R.
8. Let F be a multiplication R-module, N a submodule of F and $n \in Z_+$. Then N is an n-primary submodule of F if and only if $N \subseteq N : (N : F)$ is a 2-primary submodule, for every $r \in R$.
9. Let N be a primary submodule of an R-module F, let $n \in Z_+$. Then, N is the n-primary submodule of F if and only if $N : F$ is an n-primary ideal of R.
10. Let F be a multiplication R-module, and let I be an n-primary ideal of R. Then, IF is an n-primary submodule of F.

Conclusions

1. If N is a 2-prime submodule of an R-module F, Then $[N : F]$ is the 2-prime ideal of R.
2. The converse of (1) is not true in general for example: Let $F = \mathbb{Z} \oplus \mathbb{Z}$ as Z-module and let $N = (0) \oplus 8\mathbb{Z}$. Then $[N : F] = (0)$ which is 2-prime ideal. But N is not a 2-prime submodule since $2(0, 4) \in N$, but $(0, 4) \in N$ and $2^2 \notin (N : F) = (0)$.
3. Let H be a proper submodule of module F over a ring R. Then H is a 2-prime submodule if and only if $H : (H : \mathbb{R})$ is a 2-prime submodule, for every $r \in R$.
4. Let $F, \hat{F}$ be an R-modules and $\theta : F \to \hat{F}$ be an epimorphism and L is a submodule of F such that $\ker \theta \leq L$. If N is a 2-prime submodule of F such that $L \leq N$ then $\theta(N)$ is a 2-prime submodule of $\hat{F}$ such that $\theta(L) \leq \theta(N)$.
5. If T, Y two submodules such that $T \leq Y$ and Y is a 2-prime submodule of a module F. Then T not necessary to be 2-prime of F, for example: Consider $Z_{12}$ as a Z-module and $T = \{0, 6\}, Y = \{0, 3, 6, 9\}$, Y is a 2-prime submodule of $Z_{12}$ since $2 \cdot 3 = 6 \in Y, 3 \in$
Y but $2^2 \not\in [Y: \mathbb{Z}_{12}] = 3\mathbb{Z}$. but T is not a 2-prime of $\mathbb{Z}_{12}$, since $\overline{2} \cdot \overline{3} \in T$, $\overline{2}, \overline{3} \not\in T$, and $2^2, 3^2 \not\in [T: \mathbb{Z}_{12}]$.

References