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Properties of a General Fuzzy Normed Space

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Abstract
The aim of this paper is to introduce the definition of a general fuzzy norned space as a generalization of the notion fuzzy normed space after that some illustrative examples are given then basic properties of this space are investigated and proved.
For example when V and U are two general fuzzy normed spaces then the operator $S: V \to U$ is a general fuzzy continuous at $u \in V$ if and only if $u_n \to u$ in V implies $S(u_n) \to S(u)$ in U.

Key Words: Fuzzy Absolute value, General fuzzy normed space, General Complete fuzzy Normed space, General Fuzzy Bounded Set.

خواص فضاء القياسي الضبابي العام

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الخلاصة
الهدف من هذا البحث هو تقديم تعريف فضاء القياس الضبابي العام كتعميم لتعريف فضاء القياس الضبابي
ثم تم اعطاء بعض الامثلة التوضيحية وبعد ذلك تم بحث وبرهان الخواص الاساسية لهذا الفضاء. مثلا عندما يكون كل من V و U فضاء القياس الضبابي العام عندئذ يكون المؤثر $S: V \to U$ مستمر ضبابيا عاما عند $u \in V$ اذا وفقط اذا كانت $u_n \to u$ تؤدي الى $S(u_n) \to S(u)$ في U.

1. Introduction

Zadeh in 1965[1] was the first one who introduced the theory of fuzzy set. When Katsaras in 1984 [2] studying the notion of fuzzy topological vector spaces he was the first researcher who studied the notion of the fuzzy norm on a linear vector space. A fuzzy metric space was also studied by Kaleva and Seikkala in 1984 [3]. The fuzzy norm on a linear space has been studied by Felbin in 1992 [4] where Kaleva and Seikkala introduce this type of fuzzy metric. Another type of fuzzy metric spaces was given by Kramosil and Michalek in [5].

A certain type of fuzzy norm on a linear space was given by Cheng and Mordeson in 1994 [6] so that the corresponding fuzzy metric is of Kramosil and Michalek type. A finite dimensional fuzzy normed linear space was studied by Bag and Samanta in 2003 [7]. Saadati and Vaezpour in 2005 [8] studied complete fuzzy normed spaces and proved some results. Bag and Samanta in 2005 [9] studied fuzzy bounded linear operators on a fuzzy normed space.

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Bag and Samanta in 2006 and 2007 [10], [11] was proved the fixed point theorems on fuzzy normed linear spaces introduced by Cheng and Mordeson. The fuzzy topological structure introduced by Cheng and Mordeson of the fuzzy normed linear space was studied by Sadeqi and Kia in 2009 [12]. Kider introduced a new fuzzy normed space in 2011 [13]. Also, he proved this new fuzzy normed space has a completion in [14]. Nadaban in 2015 [15] was studied properties of fuzzy continuous mapping which defined on a fuzzy normed linear space that introduced by Cheng and Mordeson.

The definition of the fuzzy norm of a fuzzy bounded linear operator was introduced by Kider and Kadhum in 2017 [16]. Fuzzy functional analysis is developed by the concepts of fuzzy norm by different authors have been published for reference one may see [17- 25].

This paper is structured as follows:
Basic properties of fuzzy absolute value are presented in section two**.** The notion of general fuzzy normed space is introduced in section three then we proved some properties of this space. In the last section, we continue with the study of the general fuzzy normed space by proving other properties of this space.

2. Basic Properties of Fuzzy Absolute Value

Definition 2.1: [10]

A continuous **triangular norm(t-norm)** is a binary operation $\otimes : [0,1]\times[0,1]\to[0,1]$ with the following conditions hold for all m ,n ,s ,t $\in[0,1]$:

(1) $m\otimes n=n\otimes m$

(2) $m\otimes 1=m$

(3) $m\otimes(n\otimes t)=(m\otimes n)\otimes t$

(4) if $m\leq n$ and $t\leq s$ then $m\otimes t \leq n \otimes s$.

Example 2.2: [11]

(1)Let $m\otimes n=m.\,n$ for all n, m $\in [0,1]$ where m .n is multiplication in [0,1]. Then $\otimes$ is continuous t-norm.

(2)Let $m\otimes n= m \wedge n$ for all n, m$\in$[0,1] then $\otimes$ is continuous t-norm.

Remark 2.3: [24]

(1)for all n > m there is k with $n\otimes k \geq m$ where n, m, k $\in$[0,1].

(2)there is q with $q\otimes q \geq n$ where n, q $\in$[0,1].

First, we need the following definition

Definition 2.4: [26]

Let $\mathbb{R}$ be a vector space of real numbers over filed $\mathbb{R}$ and $\odot,\otimes$ be continuous t-norm. A fuzzy set $L_{\mathbb{R}}$ $:\mathbb{R}\times[0,\infty)$ is called **fuzzy absolute value on** $\mathbb{R}$ if it satisfies the following conditions for all m , n $\in\mathbb{R}$ and for all t, s $\in$[0,1]

(A1) $0\leq L_{\mathbb{R}}(n, s)<1$ for all s>0.

(A2) $L_{\mathbb{R}}(n, s) =1 \Leftrightarrow n= 0$ for all s>0.

(A3) $L_{\mathbb{R}}(n+m, s+t)\geq L_{\mathbb{R}}(n, s)\odot L_{\mathbb{R}}(m, t)$.

(A4) $L_{\mathbb{R}}(nm, st)\geq L_{\mathbb{R}}(n, s) \otimes L_{\mathbb{R}}(m, t)$.

(A5) $L_{\mathbb{R}}(n, .):[0,\infty)\to[0,1]$ is continuous function of t.

(A6) $\lim_{s\to\infty} L_{\mathbb{R}}(n, s) =1$.

Then $(\mathbb{R}, L_{\mathbb{R}}, \odot,\otimes)$ is called **a fuzzy absolute value space**.

Example 2.5: [26]

Define $L_{\mathbb{R}}(a, t) = \frac{t}{t+|a|}$ for all a $\in \mathbb{R}$ then $L_{\mathbb{R}}$ is a fuzzy absolute value on $\mathbb{R}$ where $a \odot b = a \otimes b = a \cdot b$ for all a, b $\in \mathbb{R}$ where $a \cdot b$ is the ordinary multiplication of a and b.

The proof of the following example is clear hence is omitted.

Example 2. 6:

Define L_d: $\mathbb{R}\times [0 ,\infty) \to[0 ,1]$ by

$$L_d(u, t)=\begin{cases}1 & \text{if } t > |u| \\ 0 & \text{if } t \leq |u|\end{cases}$$

then L_d is a fuzzy absolute value on $\mathbb{R}$. L_d is called the **discrete fuzzy absolute value on $\mathbb{R}$.**

3. General Fuzzy Normed Space

The following definition is the key of all results in this section

Definition 3.1:

Let V be a vector space over the filed ℝ and ⊙,⊗ be continuous t-norms. A fuzzy set G_V:V×[0,∞) is called a **general fuzzy norm on V** if it satisfies the following conditions for all u, v∈V and for all α ∈𝔽, s, t ∈ [0,∞):

(G1) 0≤ G_V(u, s)<1 for all s>0.

(G2) G_V(u, s)=1 ⟺ u= 0 for all s>0.

(G3) G_V(αu, st) ≥ $L_{\mathbb{R}}(\alpha$, s) ⊗ G_V(u, t) for all α ≠0 ∈ℝ.

(G4) G_V(u+ v, s +t) ≥ G_V(u, s) ⊙ G_V(v, t).

(G5) G_V(u, .) : [0,∞) → [0,1] is continuous function of t.

(G6) $\lim_{t\to\infty} G_V$(u ,t) = 1

Then (V, G_V,⊙,⊗) is called **a general fuzzy normed space.**

The following example is an application to definition 3.1

Example 3.8:

Define $G_{|.|}$(u,s)=$\frac{t}{t+|u|}$ for all u∈ℝ .Then (ℝ ,$G_{|.|}$,⊙,⊗) is a general fuzzy normed space with a⊙ b=a⊗ b=a . b for all a ,b ∈ [0,1]. Then $G_{|.|}$ is called the **standard general fuzzy norm induced by the absolute value |.|**.

Proof:

(G1) since |u|≥0 for all u∈ V and 0 ≤ t so 0≤$G_{|.|}$(u ,t) <1.

(G2)$G_{|.|}$(u ,t) =1 if and only if $\frac{t}{t+|u|}$ = 1 if and only if |u| = 0 if and only if u= 0.

(G3) To verify $G_{|.|}$(αu, st) ≥ $L_{|.|}(\alpha$,s) . $G_{|.|}$(u, t) for all u∊V 0 ≠ α ∈𝔽 and s, t ∈[0,∞) first for t =0 or s =0 the inequality is clear . When t≠0 and s≠ 0 the inequality becomes $\frac{st}{st+|\alpha u|} \geq \frac{s}{s+|\alpha|} \otimes \frac{t}{t+|u|}$ that is $G_{|.|}$(αu, st) ≥ $L_{|.|}(\alpha$, s) ⊙ $G_{|.|}$(u, t).

(G4) For s ,t >0 we will show that

$G_{|.|}$(u +v, s +t) ≥ $G_{|.|}$ (u,s) ⊙ $G_{|.|}(v,s)$. For t =0 or s =0 the inequality is clear. But for t≠ 0, $s \neq 0$ this inequality is becomes $\frac{s+t}{s+t+|u+v|} \geq \frac{s}{s+|u|} \cdot \frac{t}{t+|v|}$ which is true for every u, v ∈ V and s, t > 0

$(G5)$ Let (t_n) be sequence in [0,∞) such that $t_n \to t$. Now for every u ∈ V, $\lim_{n\to\infty} G_{|.|}(u,t_n) = \lim_{n\to\infty} \frac{t_n}{t_n+|u|} = \frac{\lim_{n\to\infty} t_n}{\lim_{n\to\infty} t_n + \lim_{n\to\infty} |u|}$

$\frac{t}{t+|u|}$ = $G_{|.|}$ (u,t). So $G_{|.|}$(u,t_n) → $G_{|.|}$(u,t).

Hence G_V: [0,∞) → [0,1] is continuous of t

(G6) $\lim_{t\to\infty} G_{|.|}$(u,t) = $\lim_{t\to\infty} \frac{t}{t+|u|} = \lim_{t\to\infty} \frac{\frac{t}{t}}{\frac{t}{t}+\frac{|u|}{t}}$ = 1.

Then (V, $G_{|.|}$, . , .) is a general fuzzy normed space.

Example 3. 9:

If (V, ||.||) is normed space and $G_{\|.\|}$: V × [0,∞) → [0,1] is defined by :

$G_{\|.\|}$(u,t) = $\frac{t}{t+\|u\|}$ then (V,$G_{\|.\|}$,⊙ ,⊗) is general fuzzy normed space where u⊙ v= u ∧ v and u ⊗ v= u. v for all u ,v ∈ [0,1]. Then $G_{\|.\|}$ is called the **standard general fuzzy norm induced by the norm ||.||**.

Proof :

According to example 3.8 conditions (G1), (G2) ,(G3) , (G5) and (G6) are holds . it remains to prove (G4) that is for s ,t > 0

$$G_{\|.\|}(u+v, t+s) \geq G_{\|.\|}(u,t) \wedge G_{\|.\|}(v,s)$$

For t ≠0 and s≠ 0 this inequality is equivalent to

$$\frac{t+s}{t+s+\|u+v\|} \geq \frac{t}{t+\|u\|} \wedge \frac{s}{s+\|v\|}$$

Then (V, $G_{\|.\|}$,∧ , .) is general fuzzy normed space .

Example 3.10 :

Let (V,|| .||) be vector space over ℝ, define

$$G_d(u,t)= \begin{cases} 1 & if \quad \|u\| < t \\ 0 & if \quad \|u\| \geq t \end{cases}$$

Where u ⨀ v = u ⊗ v = u∧ v for all u, v∈ [0, 1] and u⊗ v= u . v for all u, v∈ [0,1]. Then G_d is called the **discrete general fuzzy norm on V.**

Proof:

It is easy to check (G1), (G2), (G5) and (G6) we verify the condition (G3) for any u ∈ V, $0 \neq \alpha \in \mathbb{R}$ and t >0

$$G_d(\alpha u, t^2) = \begin{cases} 1 & if\ \|\alpha u\| < t^2 \\ 0 & if\ \|\alpha u\| \geq t^2 \end{cases}$$

$$G_d(\alpha u, t^2) = \begin{cases} 1 & if\ |\alpha|\|u\| < t^2 \\ 0 & if\ |\alpha|\|u\| \geq t^2 \end{cases}$$

$$L_d(\alpha, t) \otimes G_d(u, t) = \begin{cases} 1 & if\ |\alpha|\|u\| < t^2 \\ 0 & if\ |\alpha|\|u\| \geq t^2 \end{cases}$$

Hence $G_d(\alpha u, t^2) \geq L_d(\alpha, t) \otimes G_d(u, t)$

Now to check the condition (G4) let u, v∈ V, s, t ∈ [0 , ∞) if ||u + v|| ≥ t + s then t ≤||u|| or s ≤||v|| . If t >||u|| or s >||v|| then t + s >||u||+||v|| ≥|| u+ v|| which is contradiction . So when t ≤||u|| then G_d(u ,t) = 0 also when s ≤ ||v|| implies G_d(v ,s) = 0. Thus G_d(u ,t) ⨀ G_d(v ,s) = 0 . Therefore the inequality G_d(u + v , t + s) ≥G_d(u ,t) ⨀ G_d(v ,s) holds . If ||u + v || < t + s then G_d(u + v , t + s) =1 and G_d(u + v , t + s)≥ G_d(u ,t) ⨀ G_d(v ,s) holds.

The proof of the following example is clear and hence is deleted.

Proposition 3.11:

Suppose that (V, ‖.‖) is a normed space define $G_V(u, s) = \frac{s}{s+\|u\|}$ for all u ∈ V and 0< s. Then (V, G_V,⨀,⊗) is generala fuzzy normed space where a ⨀ b = a ⊗ b = a . b for all a, b ∈ [0,1] .

Lemma 3.12:

Let (V, G_V,⨀,⊗) be a general fuzzy normed space then G_V(u, .) is a nondecreasing function of t for all u ∈V that is if 0< t <s then G_V(u, t) < G_V(u, s) .

Proof:

Suppose that 0<t < s and G_V(u, s) < G_V(u, t). Then
G_V(u, t)⨀ G_V(0, s−t) ≤ G_V(u, s)< G_V (u, t). Thus G_V(u, t) ⨀1<G_V(u ,s) that is
G_V(u ,t) < G_V(u, s) .This is contradiction.

The proof of the following result is clear, hence is omitted.

Remark 3.13:

Let (V,G_V,⨀,⊗) be a general fuzzy normed space and let u ∈ V, s>0 ,0<n<1 .

1-If G_V(u, s)≥(1−n) we can find 0< t< s with G_V(u ,t)> (1−n).

2- If G_V(u, s)≥(1−n) we can find 0< s< t with G_V(u ,t)> (1−n).

Definition 3.14:

Let (V,G_V,⨀,⊗) be a general fuzzy normed space. Then GFB(u ,n ,s) = { m ∈V: G_V(u−m, s)>(1−n) } is called a **general fuzzy open ball** with center u∈V radius n and s>0 and GFB[u, n ,s] = {m ∈V : G_V(u−m ,s) ≥ (1−n)} is called a **general fuzzy closed ball** with center u∈V radius n and s>0**.**

Lemma 3.15:

Assume that (V, G_V, ⨀, ⊗) is a general fuzzy normed space then
G_V(u−m, t^2) ≥ G_V(m−u, t) for all u, m ∈V and t>0 with fuzzy absolute value of this type L_d(a, t) for all a ∈ ℝ and t>1.

Proof:

G_V(u – m, t^2) = $G_V[(-1)($ m – u, $t^2)]$ ≥ L_d(−1 ,t)⊗ G_V(m – u, t)
=1 ⊗ G_V(m − u, t) = G_V(m – u , t).

Lemma 3.16:

Let (V,G_V,⨀,⊗) be fuzzy normed space then :

(1) the function (a, b)→ a+b is continuous .

(2) the function (α, a) → αa is continuous .

For all a, b ∈ V and $0 \neq \alpha \in \mathbb{F}$.

Proof:

1-if a_n → a and b_n → b then as n→ ∞ then $\lim_{n\to\infty} G_V(a_n - a, \frac{t}{2})$=1 and

$\lim_{n\to\infty} G_V(b_n -$ b, $\frac{t}{2}$)=1. Now
$G_V[(a_n + b_n) -$ (a+ b) ,t)] ≥ $G_V(a_n -$ a, $\frac{t}{2}$) ⊗ G_V ($b_n -$ b,$\frac{t}{2}$)
$\lim_{n\to\infty} G_V[(a_n + b_n) -$ (a+ b) ,t)] ≥
$\lim_{n\to\infty} G_V(a_n -$ a, $\frac{t}{2}$) ⊗ $\lim_{n\to\infty} G_V(b_n$ −b, $\frac{t}{2}$) ≥ 1 ⊗ 1 = 1.
Hence the addition is a fuzzy continuous function.
2-Suppose that $a_n \to$ a and $\alpha_n \to \alpha$ as n→ ∞ that is for any given 0 <r<1 there is N_1 such that $G_V(a_n$ − a,t) > (1 −r) for all n≥ N_1 Also for any given p>0 there is N_2 such that $L_{\mathbb{R}}(\alpha_n - \alpha$,t) > (1 −p) for all n≥ N_2. Now take N= min{ N_1, N_2} we have
$G_V[\alpha_n a_n - \alpha$ a, $2t^2$]= $G_V[\alpha_n(a_n -$ a)+ a$(\alpha_n - \alpha)$, $2t^2$]
≥ $G_V[\alpha_n(a_n -$ a), t^2] ⊙ G_V[a $(\alpha_n - \alpha)$, t^2]
≥ $L_{\mathbb{R}}(\alpha_n$,t) ⊗ $G_V[a_n -$ a),t] ⊙ $L_{\mathbb{R}}(\alpha_n - \alpha$,t) ⊗ G_V(a,t)
≥ $L_{\mathbb{R}}(\alpha_n$,t) ⊗ (1 −r) ⊙ (1 −p) ⊗ G_V(a,t)
Now choose q>0 such that
$L_{\mathbb{R}}(\alpha_n$,t) ⊗ (1 −r) ⊙ (1 −p) ⊗ G_V(a,t) > (1 −q). Thus
$G_V[\alpha_n a_n - \alpha$ a, $2t^2$] > (1 −q) for all n> N.

Definition 3.17:

A subset M of general fuzzy normed space (V, G_V ,⊙,⊗) is called a **general fuzzy open** for any u ∈ M we can find 0< n<1, s>0 with FB(u, n, t) ⊆ M . A subset W ⊆ V is called a **general fuzzy closed** set if W^C is a general fuzzy open.

Definition 3.18:

Let (V,G_V,⊙,⊗) be a general fuzzy normed space. A sequence (u_n) in V is said to be **general fuzzy approaches** to u if every 0 < ε < 1 and 0< s there is N ∈ ℤ such that G_V[u_n − u , s] > (1 −ε) for every n≥ N. If (u_n) is general fuzzy approaches to the fuzzy limit u we write $\lim_{n\to\infty} u_n$= u or u_n → u. Also $\lim_{n\to\infty} G_V(u_n -$ u, s) = 1 if and only if (u_n) is general fuzzy approaches to u.

Definition 3.19:

A sequence (v_n) in a general fuzzy normed space $(V, G_V,⊙,⊗)$ is said to be a **general Cauchy sequence** if for each $0 < r < 1$, $t > 0$ there exists a positive number N ∈ ℤ such that $G_V[v_m - v_n, t] > (1 - r)$ for all m , n ≥ N.

Definition 3.20:

Let (V,G_V,⊙,⊗) be a general fuzzy normed space and let M⊆ V. Then the **general closure** of M is denote by $\overline{M^G}$ or GCL(M) is smallest general fuzzy closed set contains M .

Definition 3.21:

A subset M of general fuzzy normed space (V, G_V,⊙,⊗) is called **general fuzzy dense** in M if $\overline{M^G}$ = V

Definition 3.22:

Let (V,G_V,⊙,⊗) be a general fuzzy normed space. A sequence (u_n) is said to be **general fuzzy bounded** if there exists 0< q <1 such that G_V(u_n , s) >(1−q) for all s>0

4.Other Properties Of a General Fuzzy Normed Space

Definition 4. 1:

Let (V,G_V,⊙ ,⊗)and (U,G_U,⊙ ,⊗) be two general fuzzy normed spaces the operator S: V→ U is called **general fuzzy continuous at** v_0 ∈V for every s>0 and every 0 < γ < 1 there exist t(depends on s and γ and v_0) and there exists δ (depends on s and γ and v) such that for all v∈V with $G_V[v - v_0, s] > (1 - \delta)$ we have $G_U[S(v) - S(v_0), t] > (1 - \gamma)$ if S is fuzzy continuous at each point v∈V then S be **general fuzzy continuous .**

Theorem 4.2:

Suppose that $(V, G_V,⊙ .⊗)$ and $(U, G_U,⊙,⊗)$ are two general fuzzy normed spaces. The operator S: V → U is a general fuzzy continuous at u ∈ V if and only if u_n →u in V implies $S(u_n)$ →S(u) in U.

Proof:

Let the operator S be a general fuzzy continuous at u ∈ V and assume that

$(u_n) \in V$ and $u_n \to u$. Suppose that $\varepsilon \in (0, 1)$ and $t > 0$ so we can find $0 < \delta < 1$ and $s > 0$ for every v∈V whenever $G_V[v - u, s] > (1 - \delta)$ implies $G_U[S(v) - S(u), t] > (1 - \varepsilon)$ but $u_n \to u$ then there is $N \in \mathbb{Z}$ with for every $n \geq N$ we have $G_V[u_n - u, s] > (1 - \delta)$. Therefore for $n \geq N$ we have $G_U[S(u_n) - S(u), t] > (1 - \varepsilon)$. Thus $S(u_n) \to S(u)$
Conversely, assume that for every sequence (u_n) in V with $u_n \to u$ implies $(S(u_n)) \to S(u)$ suppose that S is not fuzzy continuous at u. This means that for every $\gamma, 0 < \gamma < 1$ and for every $s > 0$ there exists $0 < \varepsilon < 1$ and u∈V with $G_V[v - u, s] > (1 - \varepsilon)$ but $L_U[G(v) - G(u), t] \leq (1 - \gamma)$ Now , for any $n \in N$ we can find $u_n \in X$ with $G_V[u_n - u, s] > \left(1 - \frac{1}{n}\right)$ but $G_U[S(u_n) - S(u), t] \leq (1 - \gamma)$. Then $(S(u_n))$ does not general fuzzy approaches to S(u). Therefore S is general fuzzy continuous at u.

Definition 4.3:

Let $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ be two general fuzzy normed spaces. Let $T: V \to U$ be an operator T is called **uniformly general fuzzy continuous** if for $t > 0$ and for every $0 < \alpha < 1$ there exists β [depends on t and α] and there exists $s > 0$ [depends on t and α] such that $G_U[T(v) - T(u), t] > (1 - \alpha)$ whenever $G_V[v - u, s] > (1 - \beta)$ for all $v, u \in V$

Theorem 4.4:

Suppose that $(V, G_V, \odot, \otimes)$ and $(U, G_U, \odot, \otimes)$ are two general fuzzy normed spaces. Let $T: V \to U$ be uniformly general fuzzy continuous operator. If (u_n) is a general Cauchy sequence in V then $(T(u_n))$ is a general Cauchy sequence in U

Proof:

Because T is uniformly general fuzzy continuous then for any $t > 0$ and for every $0 < \varepsilon < 1$ there is $s > 0$ and there is $0 < \delta < 1$ with $G_U[T(v) - T(u), t]\,] > (1 - \varepsilon)$ whenever $G_V[v - u, t] > (1 - \delta)$ for all $u, v \in V$. But (v_n) is Cauchy sequence in V so corresponding to $0 < \delta < 1$ and $s > 0$ there is $N \in \mathbb{Z}$ with $G_V[v_n - v_m, s] > (1 - \delta)$ for any $m, n \geq N$, We now conclude that $G_U[T(v_n) - T(v_m), t]\,] > (1 - \varepsilon)$ for all $m, n \geq N$ this implies that $(T(v_n))$ is general Cauchy.

Definition 4.5

Let $(V, G_1, \odot, \otimes)$ and $(V, G_2, \odot, \otimes)$ be general fuzzy normed spaces and for all $(u_n) \in V$, u in V then $\lim_{n\to\infty} G_1[u_n - u, s] = 1$ if and only if $\lim_{n\to\infty} G_2[u_n - u, t] = 1$ for all $t > 0, s > 0$. Then G_1 and G_2 are said to be **equivalent general fuzzy norms** on V. Also $(V, G_1, \odot, \otimes)$ and $(V, G_2, \odot, \otimes)$ are equivalent general fuzzy normed spaces.

Theorem 4.6

If we find $k \in \mathbb{R}$ with $\frac{1}{k} G_2(v, t) \leq G_1(v, s) \leq k\, G_2(v, t)$ for all $v \in V$ and t>0, s>0 then the two general fuzzy normed spaces $(V, G_1, \odot, \otimes)$ and $(V, G_2, \odot, \otimes)$ are equivalent.

Proof

Let $v_n \to v$ in $(V, G_1, \odot, \otimes)$ so for all $0 < \varepsilon < 1, s > 0$ so we can find $N \in \mathbb{Z}$ with $G_1[v_n - v, s] > (1 - \varepsilon)$. Hence $G_2[v_n - v, t] \geq \frac{1}{k} G_1[v_n - v, s] > \frac{(1-\varepsilon)}{k} = (1 - r)$ for some $o < r < 1$.Therefore $v_n \to v$ in $(V, G_2, \odot, \otimes)$. Now assume that $v_n \to v$ in $(V, G_2, \odot, \otimes)$ so for all $0 < r < 1$, $t > 0$ so we can find $N \in \mathbb{Z}$ with $G_2[v_n - v, s] > (1 - r)$ for all $n \geq N$. Now $G_1[v_n - v, s] \geq \frac{1}{k} G_2[v_n - v, t] > \frac{(1-r)}{k} = (1 - q)$ for some $0 < q < 1$. Therefore $v_n \to v$ in $(V, G_1, \odot, \otimes)$.

Definition 4.7:

A general fuzzy normed space $(V, G_V, \odot, \otimes)$ is said to be **a general complete** if every general Cauchy sequence in V is general fuzzy approaches to a vector in V.

Definition 4.8:

The continuous t-norms $\otimes$ is said to be distributive on the continuous t-norms $\odot$ if
$a \otimes [b \odot c] = [a \otimes b] \odot [a \otimes c]$ for all a, b. c ∈ [0, 1].

Theorem 4. 9

Let $(V_1, G_1, \odot, \otimes)$, $(V_2, G_2, \odot, \otimes)$,… , $(V_n, G_n, \odot, \otimes)$ be general fuzzy normed spaces, then $(V, G_V, \odot, \otimes)$ is a general fuzzy normed space.Where $V = V_1 \times V_2 \times \ldots \times V_n$ with $G_V[(v_1, v_2, \ldots, v_n, t) = G_V[(v_1, t) \odot G_V[(v_2, t) \odot \ldots \odot G_V[(v_n, t)$ and $\otimes$ is distributive on $\odot$.

Proof:

(G1) Since $0 \leq G_1(v_1,t) < 1$, $0 \leq G_2(v_2,t) < 1$,… , $0 \leq G_n(v_n,t) < 1$.

Then $0 \leq G_v(v,t) < 1$.

(G2) $0 \leq G_v[(v_1,v_2,\dots,v_n,t) < 1$, if and only if $G_1(v_1,t) \odot G_2(v_2,t) \odot \dots \odot G_n(v_n,t) = 1$, if and only if $G_1(v_1,t)$ =1, $G_2(v_2,t) = 1$, … , $G_n(v_n,t)$=1, if and only if $v_1 = 0, v_2 = 0, \dots, v_n = 0$ if and only if v =$(v_1,v_2,\dots,v_n)$ = (0,0,…,0) = 0 .

(G3) For any $0 \neq \alpha \epsilon F$

$G_v[\alpha(v_1,v_2,\dots,v_n),ts] = G_v[(\alpha v_1,\alpha v_2,\dots,\alpha v_n),ts]$

$= G_1(\alpha v_1,ts) \odot G_2(\alpha v_2,ts) \odot \dots \odot G_n(\alpha v_n,ts)$

$\geq L_{\mathbb{F}}(\alpha,t) \otimes G_1(v_1,s) \odot L_{\mathbb{F}}(\alpha,t) \otimes G_2(v_2,s) \odot \dots \odot L_{\mathbb{F}}(\alpha,t) \otimes G_n(v_n,s)$

By definition 4.8 then

$G_v[\alpha(v_1,v_2,\dots,v_n),ts] \geq L_{\mathbb{F}}(\alpha,t) \otimes [G_1(v_1,s) \odot G_2(v_2,s) \odot \dots \odot G_n(v_n,s)]$

then $G_v[\alpha(v_1,v_2,\dots,v_n),ts] \geq L_{\mathbb{F}}(\alpha,t) \otimes G_v[(v_1,v_2,\dots,v_n),s]$

(G4) $G_v[(v_1,v_2,\dots,v_n,)+(u_1,u_2,\dots,u_n),t+s] =$

$G_v[(v_1+u_1,v_2+u_2,\dots,v_n+u_n),t+s]$

$= G_1(v_1+u_1,t+s) \odot G_2(v_2+u_2,t+s) \odot \dots \odot G_n(v_n+u_n,t+s)$

$\geq G_1(v_1,t) \odot G_1(u_1,s) \odot G_2(v_2,t) \odot G_2(u_2,s) \odot \dots \odot G_n(v_n,t) \odot G_n(u_n,s)$

$\geq G_v[(v_1,v_2,\dots,v_n),t] + G_v[(u_1,u_2,\dots,u_n),s]$

(G5) Since $G_1(v_1,.)$, $G_2(v_2,.)$, …, $G_n(v_n,.)$ are continuous function of t then $G_v[(v_1,v_2,\dots,v_n,.)$ is continuous function of t .

(G6)$\lim_{n\to\infty} G_V[(v_1,v_2,\dots,v_n,t) =$

$\lim_{n\to\infty}[\, G_1(v_1,t) \odot G_2(v_2,t) \odot \dots \odot G_n(v_n,t)] =$

$\lim_{n\to\infty}[\, G_1(v_1,t) \odot \lim_{n\to\infty}[\, G_2(v_2,t) \odot \dots \odot \lim_{n\to\infty}[\, G_n(v_n,t)$

$=1 \odot 1 \odot \dots \odot 1 = 1$

Thus $(V,G_V,\odot,\otimes)$ is a general fuzzy normed space.

Theorem 4.10 :

Let $(V_1,G_1,\odot,\otimes)$, $(V_2,G_2,\odot,\otimes)$, … , $(V_n,G_n,\odot,\otimes)$ be general fuzzy normed spaces, then $(V,G_V,\odot,\otimes)$ is general complete general fuzzy space if and only if $(V_1,G_1,\odot,\otimes)$, $(V_2,G_2,\odot,\otimes)$, … , $(V_n,G_n,\odot,\otimes)$ are general complete general fuzzy normed space. Where $V=V_1 \times V_2 \times \dots \times V_n$ and $G_V[(v_1 \times v_2 \times \dots \times v_n,t) = G_1[(v_1,t) \odot G_2[(v_2,t) \odot \dots \odot G_n[(v_n,t)$ also $\otimes$ is distributive on $\odot$.

Proof:

Assume that$(V_1,G_1,\odot,\otimes)$, $(V_2,G_2,\odot,\otimes)$, …, $(V_n,G_n,\odot,\otimes)$, are general complete fuzzy normed spaces.

Let (v_m) be a general Cauchy sequence in V then $v_m=(v_1^{(m)}, v_2^{(m)}, \dots, v_n^{\{m)})$ and $v_k=(v_1^{(k)}, v_2^{(k)}, \dots, v_n^{(k)})$. Now since (v_m) is a general Cauchy sequence in V this means that $G_V(v_m - v_k,t)$ converges to 1 but this implies that $G_V[(v_1^{(m)}, v_2^{(m)}, \dots, v_n^{\{m)}) - (v_1^{(k)}, v_2^{(k)}, \dots, v_n^{(k)}), t] = G_1(v_1^{(m)} - v_1^{(k)},t) \odot G_2(v_2^{(m)} - v_2^{(k)},t) \odot, \dots, \odot G_n(v_n^{(m)} - v_n^{(k)},t)$ converges to 1.

It follows that $G_1(v_1^{(m)} - v_1^{(k)},t)$ converges to 1, $G_2(v_2^{(m)} - v_2^{(k)},t)$ converges to 1, …, $G_n(v_n^{(m)} - v_n^{(k)},t)$ converges to 1. Hence $(v_1^{(j)})$ is a general Cauchy sequence in $(V_1,G_1,\odot,\otimes)$,$(v_2^{(j)})$ is a Cauchy sequence in $(V_2,G_2,\odot,\otimes)$, …, $(v_n^{(j)})$ is a Cauchy sequence in $(V_n,G_n,\odot,\otimes)$.

But $(V_1,G_1,\odot,\otimes)$, $(V_2,G_2,\odot,\otimes)$… , $(V_n,G_n,\odot,\otimes)$are general complete so there is v_1 in V_1, v_2 in V_2, … , v_n in V_n such that $\lim_{j\to\infty} G_1(v_1^{(j)} - v_1,t)$=1, $\lim_{j\to\infty} G_2(v_2^{(j)} - v_2,t)$ =1, … , $\lim_{j\to\infty} G_n(v_n^{(j)} - v_n,t)$= 1. Now put $v=(v_1, v_2, \dots, v_n)$ then $v \in V_1 \times V_2 \times \dots \times V_n$ and $\lim_{j\to\infty} G_v(v_j - v,t) = \lim_{j\to\infty} G_1(v_1^{(j)} - v_1,t) \odot \lim_{j\to\infty} G_2(v_2^{(j)} - v_2,t) \odot \dots \odot \lim_{j\to\infty} G_n(v_n^{(j)} - v_n,t) = 1 \odot 1 \odot \dots \odot 1 = 1$.

Conversely assume that $(V,G_v,\odot,\otimes)$ is general complete let $(v_1^{(j)})$ be a general

Cauchy sequence in $(V_1, G_1, \odot, \otimes)$, $(v_2^{(j)})$ is a general Cauchy sequence in $(V_2, G_2, \odot, \otimes)$…, $(v_n^{(j)})$ is a general Cauchy sequence in $(V_n, G_n, \odot, \otimes)$. It follows that
$G_1\ (v_1^{(m)} - v_1^{(k)}, t)$ converges to 1, $G_2\ (v_2^{(m)} - v_2^{(k)}, t)$ converges to 1, …,
$G_n\ (v_n^{(m)} - v_n^{(k)}, t)$ converges to 1. Put $v_m = (v_1^{(m)}, v_2^{(m)}, \ldots, v_n^{\{m\}})$ and
$v_k = (v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)})$. Now
$G_v\ (v_m - v_k, t) = G_v\ [(v_1^{(m)}, v_2^{(m)}, \ldots, v_n^{\{m\}}) - (v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)}), t]$
$= G_1\ (v_1^{(m)} - v_1^{(k)}, t) ,\odot\ G_2\ (v_2^{(m)} - v_2^{(k)}, t) ,\odot , \ldots, \odot\ G_n\ (v_n^{(m)} - v_n^{(k)}, t)$
Converges to 1 $,\odot$ 1 $,\odot \ldots ,\odot$ 1 = 1. Hence $G_v(v_m - v_k, t)$ converges to 1.
Therefore (v_m) is a general Cauchy sequence in V so there is $v = (v_1, v_2, \ldots, v_n)$
$v \in V = V_1 \times V_2 \times \ldots . \times V_n$ such that
$1 = \lim_{j\to\infty} G_v(v_j - v, t) = \lim_{j\to\infty} G_1(v_1^{(j)} - v_1, t) \circledast \lim_{j\to\infty} G_2(v_2^{(j)} - v_2, t) \circledast \ldots \circledast \lim_{j\to\infty} G_n(v_n^{(j)} - v_n, t)$. This implies that
$\lim_{j\to\infty} G_1(v_1^{(j)} - v_1, t) = 1$, $\lim_{j\to\infty} G_2(v_2^{(j)} - v_2, t) = 1, \ldots , \lim_{j\to\infty} G_n(v_n^{(j)} - v_n, t) = 1$. Hence $(V_1, G_1, \odot, \otimes)$, $(V_2, G_2, \odot, \otimes)$, …, $(V_n, G_n, \odot, \otimes)$ are general complete general fuzzy normed spaces.

5.Conclusion

The main goal of this paper is to define a general fuzzy norm and started to prove its basic corresponding theory as a new approach in the study of fuzzy functional analysis but before that, we have to introduce definitions, properties that related with a general fuzzy normed space. Then try to introduce other notions such as general fuzzy continuous operator, uniform general fuzzy continuous operator, general complete fuzzy normed space. At this end, we prove that the general fuzzy normed spaces $V_1, V_2, \ldots, V_n$ are general complete if and only if $V_1 \times V_2 \times \ldots . \times V_n$ is a general complete general fuzzy normed space.

References

1. Zadeh, L**. 1965.** Fuzzy sets, Inf. Control. 8: 338-353.
2. Katsaras, A. **1984.** Fuzzy topological vector spaces II, *Fuzzy sets and Systems*, **12**: 143-154.
3. Kaleva, O. and Seikkala, S.1984. On fuzzy metric spaces, Fuzzy sets and systems, 121: 215-229.
4. Felbin, C. **1992.** Finite dimensional fuzzy normed linear spaces, Fuzzy sets and Systems, 48: 239-248.
5. Kramosil O. and Michalek, J. **1975**. Fuzzy metrics and statistical metric spaces, Kybernetika, 11: 326-334
6. Cheng, S. and Mordeson, J. **1994.** Fuzzy linear operators and fuzzy normed linear spaces, Ball. Cal. Math. Soc. 86: 429 - 436 .
7. Bag, T. and Samanta, S. **2003**. Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11(3): 687-705.
8. Saadati, R. and Vaezpour, M. **2005.** Some results on fuzzy Banach spaces, J. Appl. Math. and Computing. 171: 475-484.
9. Bag, T. and Samanta, S. **2005**. Fuzzy bounded linear operators, Fuzzy sets and Systems, 151(3): 513-547.
10. Bag, T. and Samanta, S. **2006.** Fixed point theorems on Fuzzy normed spaces, *Inf. sci.***176**: 2910-2931.
11. Bag, T. and Samanta, S. **2007.** Some fixed point theorems in fuzzy normed linear spaces, *Information sciences*, **177**: 3271-3289.
12. Sadeqi, I. and Kia, F. **2009.** Fuzzy normed linear space and its topological structure, *Chaos Solitions and Fractals*, **40**(5): 2576-2589.
13. Kider, J. **2011.** On fuzzy normed spaces, *Eng. & Tech. Journal*, **29**(9):1790-1795.
14. Kider, J. **2011**. Completion of fuzzy normed spaces, *Eng. & Tech. Journal*, **29**(10): 2004-2012.

15. Nadaban, S**. 2015.** Fuzzy continuous mapping in fuzzy normed linear spaces, *I. J. Computer Communications and Control*, **10**(6): 834-842.
16. Kider, J.R. and Kadhum, N.A. **2017** Properties of fuzzy norm of fuzzy bounded operators, *Iraqi Journal of Science*, **58**(3A) : 1237-1281.
17. Si, H., Cao, H. and Yang, P. **2010.** Continuity in an intuitionistic fuzzy normed Space, Seventh I Conference on fuzzy systems and knowledge Discovery, 144-148.
18. Congxin, W. and MMing, M. **1993.** Continuity and bounded mappings between fuzzy normed spaces, *Fuzzy Math*, **1**: 13-24.
19. Goudarzi, M. and Vaezpour, S. **2010**. Best simultaneous approximation in fuzzy normed spaces, *Iranian J. of Fuzzy Systems*, **7**: 87-69
20. Jameel, R. **2014.** On some results of analysis in a standard fuzzy normed spaces, M.Sc. Thesis, University of Technology, Iraq.
21. Kider, J.R. **2012.** New fuzzy normed spaces, *J. Baghdad Sci*. , **9**: 559-564.
22. Oregan, D. and Saadati, R. **2010.** L-Random and fuzzy normed spaces and classical theory, *CUBO Mathematical J*. **2**: 71-81.
23. Xiao, J. and Zhu, X. **2004**. Fuzzy normed spaces of operators and its completeness, *Fuzzy sets and Systems*. **133**: 437-452.
24. George, A. and Veeramani, P. **1994.** On some results in fuzzy metric spaces, *Fuzzy sets and Systems*, **64**: 395-399.
25. Golet, I. **2010.** On generalized fuzzy normed spaces and coincidence theorems, *Fuzzy sets and Systems*, **161**: 1138-1144.
26. Jehad R. Kider and Ahmed H. Ali. **2018.** Properties of fuzzy absolute value on and properties finite dimensional fuzzy normed space. *Iraqi Journal of Science*, **59**(2B): 909-916