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On the G-quadratic and LG-quadratic of the Exterior Algebra and Associated Algebra

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Abstract:

Given an exterior algebra E over a finite dimension vector space v, and let $R = E/I$, where I is a graded ideal in E. The relation between the algebra R and R_{mon} regarding to G-quadratic and LG- quadratic will be investigated. We show that the algebra R is G- quadratic if and only if R_{mon} is G- quadratic. Furthermore, it has been shown that the algebra R is LG- quadratic if and only if R_{mon} is LG- quadratic.

Keywords: Initial ideal, Exterior algebra, Associated algebra, G-quadratic, and LG-quadratic.

حول G التربيعيه و LG التربيعية في الجبر الخارجي والجبر المرتبط

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الخلاصة:

بالنظر إلى الجبر الخارجي E على متجه ذو ابعاد منتهية ، ونفرض $R = E/I$ حيث I مثالي متدرج في E. تم البحث في العلاقة بين الجبر R و R_{mon} فيما يتعلق بخاصية. $G-$ التربيعية و $LG-$ التربيعة. نظهر أن الجبر R هو G –التربيعي فقط إذا وفقط اذا R_{mon} هو G – تربيعي . نتيجة أخرى ايضا ، يكون الجبر R هو $LG-$ تربيعي إذا وفقط إذا كان R_{mon} هو LG –التربيعي.

1-Introduction:

Let $E = \wedge_k(e_1, \dots , e_m)$ be an exterior algebra over a field K. The objective of this paper is to study the relation between two algebras; $R = E/I$ and $R_{mon} = E/in_<(I)$ regarding to G-quadratic and LG - quadratic properties. From literature, the following relations in case of commutative are given by

$$G\text{-quadratic} \Rightarrow LG\text{-quadratic} \Rightarrow \text{Koszul} \Rightarrow \text{quadratic.}$$

The converse of the first implication holds under some conditions, where Conca [1] states that
every quadratic Artinian algebra R with $dim\, R_2 \leq 2$ is G-quadratic. Mccullough and Mere show that these implications hold in exterior algebra. However, they showed that the converse not true for implications [2]. From the algebra that given by quiver and relations, we know that every quadratic is Koszul [3].

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In this paper, G-quadratic and LG-quadratic in two algebras R and R_{mon} will be investigated. Furthermore, we will show in Theorem 3.2, that the algebra R is G-quadratic if and only if R_{mon} is G-quadratic. Moreover, we will go through the relationship between R and R_{mon} under LG-quadratic properties.

The layout of this paper will be as follows: The background of Görbner basis, properties of monomial in the exterior algebra, and the associated algebra will be explained in Section two. Finally, the relation between R and R_{mon} under G-quadratic, LG -quadratic conditions will be given in Section three.

Now, we fix some notation in this work. Let K be a field, V be a vector space over K with dimension $n, E = \wedge_k(e_1, \dots, e_n)$ be an exterior algebra of V, and $e_1, \dots, e_n$ be a basis of V.

The exterior algebra E will be considered as a skew-commutative and graded ring with $\deg(e_i) = 1$, for $i = 1, \dots, n$. Additionally, a module E will be called graded if there is K-vector space M_i; such that $M = \bigoplus_i M_i$ and $E_i\, M_j \subseteq M_{i+j}$, for all $i\ and\ j$. Let M be a finitely generated and graded E-module.

2-Gröbner basis of associated algebra

Let K be a field, and E be an exterior algebra. We call the set of monomial of E by $Mon(E)$. The ideal that generated by monomial element is called monomial ideal. Let Ɓ be the set of finite monomail in E. We write a monomail element in E as $e_{i1}e_{i2} \dots . e_{is}$ instead of $e_{i1} \wedge e_{i2} \wedge \dots \wedge e_{is}$.

A monomial order [4] on E is a total order $\leq$ on $Mon(E)$ satisfying the following conditions:

1. $if\ u \in Mon(E)\ and\ u \neq 1, then\ 1 < u$.
2. $Let\ u, v\ and\ w\ in\ Mon(E).\ If\ u < v, then\ uw < vw$.

Now we fix a monomial order on Ɓ. We use the monomail order to define the leading term of element in E.

Definition 2.1: Let $0 \neq f \in E$ and if $f = \sum_{\{g \backslash in\, Ɓ\}} c_g g$, where $c_g \in K$ with some $c_g \neq 0$, then the leading term of f is defined by $LT(f) = g$ where $c_g \neq 0$ and $h \leq g$ for all h with $c_h \neq 0$.

Definition 2.2:[2] Let I be a graded ideal in E. The initial ideal I is defined as $in_<(I) = (LT(f): f \in I)$, with leading coefficients equal 1.

Example 2.3: Let $f = 3x^3 + 2x^2y^2 - 4xy^2z^3$, where $f \in C[x, y, z]$ and let $<$ be a monomail order with $z < y < x$. Then $LT(f) = 3x^3$ and $in_<(f) = x^3$.

We state here some properties of monomial ideal.

Proposition 2.4: (Properties of monomial in the exterior algebra see [2])

1. Let $g_1, \dots, g_t \in I$ and let $T = \{g_1, \dots, g_t\}$ be the minimal set of monomials which generate $in_<(I)$. In other words $in_<(I) = \big(LT(g_1), LT(g_2), \dots, LT(g_t)\big)$.
2. $I\backslash$ has unique reduced Gröbner basis T.

Let $N =$ Ɓ $\setminus \{LT(\{f\}): \{f\} \in I\}$ and let $T = \{g_1, \dots, g_t\}$ be the minimal set of monomails which generate $in_<(I)$.

The result that shows the exterior algebra $\wedge(V)$ can be written as a direct sum of K vector space will be stated now.

Lemma 2.5: Let K be a field and let $E = \wedge(V) = \wedge_k(e_1, \dots, e_n)$ be an exterior algebra of V. Then

$$\wedge(V) = in_<(I) \oplus_k Span_k N$$

as the K vector space.

Proof: Let $0 \neq x \in in_<(I) \cap Span_K N$. Then $x \in LT(f)$ where $f \in I$ which is contradiction, since $\in N$.

Assuming that$\wedge(V) \neq in_{<(I)} + Span_K N$. So there is $x \in \wedge(V)$ such that $LT(x)$ is minimal, and $x \notin in_<(I) + Span_K N$. Let $f = LT(x)$ and so that $f \in in_<(I)$ or $f \in N$. We let c_f be the coefficient of f in x, where $x = \sum_{h \in \mathrm{B}} c_h h$, for all $c_h \in K. If\ f \in in_< (I)$, then there exists $g \in I$ such that $LT(g) = f$. Since $f \in in_< (I)$, then $c_{g=} = 1$. So $LT(x - c_f g)$ is either 0 or has minimal leading term than$f$. With the minimalty of the $LT(x)$, we have $x - c_f g \in in_<(I) + Span_K N$. By hypothesis $g \in in_<(I)$, and so $x \in in_< (I) + Span_K N$. Therefore, we get a contradictory.

On the other side, let $f \in N$, then $x - c_f f = 0$ or has minimal than x. Using the minimally fact of $f, x - c_f f \in in_<(I) + Span_K N$ and so $x \in in_<(I) + Span_K N$ which contradict with hypothesis. Therefore, we can write an element in E uniquely. If x in E, then $x = g_x + n_x$ where g_x unique in $in_<(I)$ and n_x unique in $Span\_K N$.

The basis of $\wedge(V)/in_<(I)$ will be characterized in the following proposition.

Proposition 2.6: Let K be a field and let $E = \wedge(V) = \wedge_k(e_1, \cdots, e_n)$ be an exterior algebra of V. Then $Span_K N \simeq \wedge(V)/ i\, n_<(I)$ as K vector space.

Consequently, using Proposition 2.6, we can determine the elements in $\wedge(V)/in_<(I)$ with the elements in $Span_K N$. So for all $x, y \in Span_K N$, the products of x and y in $\wedge(V)/in_<(I)$ which equals to $N_{x \wedge y}$, where $x \wedge y$ is the wedge product in $\wedge(V)$.(see[2]).

The associated algebra R_{mon} will be introduced now.

Definition 2.7: Let E be an exterior algebra. Let $in_<(I)$ be the ideal generated by $(LT(g_1), \dots, LT(g_t))$ in E. Then $in_<(I)$ is a monomial ideal and set $R_{mon} = E/ in_<(I)$.

The following proposition shows that some facts of reduced Gröbner bases and monomial algebra R_{mon}.

Proposition 2.8: Let I be a graded ideal and let E be an exterior algebra. Assuming that T is the unique minimal set of monomial generating $in_<(I),\ and\ g_1, \dots, g_t$ is the unique reduced Gröobner basis generating for I. Then

1. T is the reduced Gröbner basis for $in_<(I)$.
2. Using the description of E with $Span\, N$, The $dim_k(R_{mon}) = |N|$, where $|N|$ is the cardinality of N.
3. The dimension of R is equal to the dimension of R_{mon}
4. The exterior algebra E is finite dimensional if and only if the set N is finite

3− The algebra R and the associative algebra R_{mon}

We state here the relationship between R and R_{mon} with regard to G-quadratic algebra, and LG-quadratic algebra. We keep previous notation and start with definition of the G-quadratic algebra.

Definition 3.1: [2] Let $R = E/I$ be a quotient of an exterior algebra E, and I be a graded ideal. Let $<$ be a monomial order on E. Then R is G-quadratic algebra if I has Gröbner basis consisting of homogeneous elements of degree two (quadric) with respect to some coordinate on E_1 .

From the definition, we can see that quadratic Gröbner basis is precisely the G-quadratic.
We stated here one of the main result of this paper.

Theorem 3.2: Let $R = E/I$ be a quotient of an exterior algebra E and let I be a graded ideal. Then $R = E/I$ is G-quadratic if and only if $R_{mon} = E/\, in_{<}(I)$ is G-quadratic.
Proof: We suppose that R is G-quadratic, we need to prove R_{mon} is G-quadratic algebra. Since $in_{<}(I)$ is an ideal generated by $\big(LT(g_1), \dots, LT(g_t)\big)$, where $g_s \in I$, and $s = 1, \dots, t$. By hypothesis I has reduced Gröbner basis consisting of homogenous elements of degree 2 (quadric). So $LT(g_s)$ is homogenous element of degree 2 for all $g_s \in I$ and $s = 1, \dots, t$. Therefore, $in_{<}(I)$ is generated by homogenous elements of degree 2 and then R_{mon} is G-quadratic algebra.

Conversely, we have to prove R is G-quadratic algebra. Since I has reduced Gröbner basis $g_1, \dots, g_t$, Moreover, $in\_<\ (I) = \big(LT(g_1\,), \dots, LT(g_t)\big)$. Then $g_1, \dots, g_s$ are homogenous elements of degree 2, since R_{mon} is G-quadratic. Therefore, R is G-quadratic algebra.

Fröberg [5] proved that R is Koszul if I is a monomail ideal and he showed by using a stranded deformation argument R is Kozul when R is G- quadratic.

We view here the depth and regular sequence. Let M be a graded module over the exterior algebra E. We call $l \in E_1$ regular on M if $lm = 0$, for all $m \in M$. Otherwise, l is M singular. Let $l_1, \dots, l_s$ be a sequence of elements on E_1 , if l_i is regular element on $M/l_1, \dots, l_{i-1}$, we say $l_1, \dots, l_s$ is regular sequence on M , for all $i = 1, \dots, s$. The

$$depth_E(M) = Max\,\{\ell\,(l_1, \dots, l_s) | l_1, \dots, l_s\ is\ regular\ sequence\}$$

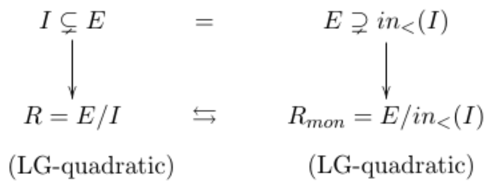
where $\ell(l_1, \dots, l_s)$ is the length of regular sequence $l_1, \dots, l_s$.

We call the algebra R, LG- quadratic, if there exists G- quadratic algebra S and r regular sequence elements $t_1, \dots, t_r$, with $deg(t_i) = 1$ in S, and $i = 1, \dots, r$ such that $R \simeq S/(t_1, \dots, t_r)$.
Caviglia [6], Avramov, Conca, and Iyenger [7] studied the LG- quadratic algebra in the commutative case.

Mccullough and Merf [2, Proposition 2.3] showed that the $depth_E(E/\, in_<(I)) \leq depth\_E(E/\,I)$.

The following theorem proves that the algebras $R$and R_{mon} are corresponding under LG-quadratic.

Theorem 3.3: Let $R = E/I$ be a quotient of an exterior algebra E and let I be a graded ideal. Then $R = E/I$ is LG-quadratic if and only if $R_{mon} = E/in_<(I)$ is LG-quadratic.



Proof: We have R is LG- quadratic. So there is a G- quadratic algebra A and linear regular sequence elements $t_1, \ldots, t_r$ on A_1 such that $R \simeq A/(t_1, \ldots, t_r)$. In other words, there exists a surjective homorphism $\varphi : A \to R$ defined as $\varphi(a) = x + I$, for all $x \in E$. Using Theorem 3.2, we have the G-quadratic algebra, A_{mon} . From [2, Proposition 2.3] we get $depth_A(A_{mon}) \leq depth_A(A)$. So there exist m linear regular sequence elements $t_1, \ldots, t_m$ such that $m \leq r$. Our aim is to show that $R_{mon} \simeq A_{mon}/(t_1, \ldots, t_m)$. Let $\varphi: A \to R$ be epimorphism algebra. By using the first isomorphism theorem we have $A/Ker\varphi \simeq R$. It can be seen that $ker\varphi = (t_1, \ldots, t_r)$. Define $\varphi^*: A_{mon} \to R_{mon}$ to be the K- algebra homomorphism which is given by $\varphi^*(a) = \varphi(a)$, for all $a \in A_{mon}$. By first isomorphism theorem, we have $A_{mon}/Ker\varphi^* \simeq R_{mon}$. It can be seen that $ker\varphi^* = (t_1, \ldots, t_m)$, for all $m \leq r$ and hence the direction is proved.

Conversely, we suppose that R_{mon} is LG- quadratic. We have $R_{mon} \simeq A_{mon}/(t_1, \ldots, t_m)$, such that A_{mon} is G- quadratic algebra. Hence A is G- quadratic by Theorem 3.2. From [2, Proposition 2.3] and [8, Theorem 3.2] we get a finite linear regular sequence $t_1, \ldots, t_p$ on A such that $m \leq p$. Now we define the algebra homorphism $\varphi: A \to R$ via $\varphi(a) = \varphi^*(a)$ for all $a \in A$. By hypothesis, we have $A/Ker\varphi \simeq R$. Thus, $A/(t_1, \ldots, t_p) \simeq R$.

Conclusions:

In this article, the relation between two algebras R and R_{mon} has been investigated. Also, the results regarding to the algebras R and R_{mon} have been proved. Furthermore, thecoincide of algebras R and R_{mon} regarding to G-quadrattic has been shown. Therefore, we proved that the algebra R is LG-quadratic if and only if the algebra R_{mon} is LG-quadratic.

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