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Nonoscillatory Properties of Fourth Order Nonlinear Neutral Differential equation

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Abstract
In this paper, the oscillatory and nonoscillatory qualities for every solution of fourth-order neutral delay equation are discussed. Some conditions are established to ensure that all solutions are either oscillatory or approach to zero as $\xi \to \infty$. Two examples are provided to demonstrate the obtained findings.

Keywords: Non-oscillatory, Neutral Differential Equations, Fourth Order.

الخصائص غير المتذبذبة للمعادلة التفاضلية غير الخطيه المحايده من الرتبة الرابعة

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الخلاصة
في هذا البحث تمت مناقشة الخصائص التذبذبية وغير التذبذبية لكل حل من المعادلة التباطؤية المحايدة من الرتبة الرابعة. تم وضع بعض الشروط للتأكد من أن جميع الحلول إما متذبذبة أوتقترب إلى الصفر عندما $\xi \to \infty$. يتم تقديم مثالين لتوضيح النتائج التي تم الحصول عليها.

1. Introduction

The main concept of this article is the fourth- order nonlinear neutral differential equations (NDE):

$$\left[\mathcal{P}(\xi) - \mathcal{S}(\xi)\mathcal{P}\big(u(\xi)\big)\right]^{(4)} + \mathcal{Q}(\xi)\mathcal{T}\Big(\mathcal{P}\big(v(\xi)\big)\Big) = 0, \xi \geq \xi_0 \ . \quad (1)$$

Where $\mathcal{S}(\xi) \in C([\xi_0,\infty);R^+), \mathcal{Q}(\xi) \in C([\xi_0,\infty);R), u(\xi) \in C[\xi_0,\infty);R), v(\xi) \in C[\xi_0,\infty);R), u(\xi) < \xi, v(\xi) < \xi, \lim_{t\to\infty} u(\xi) = \infty, \lim_{t\to\infty} v(\xi) = \infty,$

$$A_1: \ \mathcal{T} \in C(R;R), \vartheta\mathcal{T}(\vartheta) > 0, \ \ |\mathcal{T}(\vartheta)| \geq m|\vartheta|, \ \ m > 0.$$

By a solution eq.(1), we are currently referred to a function $\mathcal{P}(\xi)$ in the sense that the function $\mathcal{P}(\xi) - \mathcal{S}(\xi)\mathcal{P}(u(\xi))$ is four times continuously differentiable such that $\mathcal{P}(\xi)$ fulfills eq.(1) on $[\xi_0,\infty]$.

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A non-trivial solution $\mathcal{P}(\xi)$ is claimed to be oscillatory if its sign changes on $(\xi_{\mathcal{P}},\infty)$, where $\xi_{\mathcal{P}}$ is an arbitrary number. Otherwise, it is nonoscillatory. The equation (1) is called oscillatory if all its solutions oscillate [1].

Many researchers have discussed and investigated the oscillatory and asymptotic behaviour solution of NDE in their works for more details see the references listed therein [1-10].

Bazighifan [1] find novel oscillation conditions for fourth-order neutral differential equations (NDE) with a Canonical operator using a comparative approach with a first-order differential equation. Mohamad and Ketab [2] studied the oscillation behaviour of the solution of n^{th} order NDE . Moaaz and El-Nabulsi [3] studied fourth-order DE and established some conditions for asymptotic and oscillation of the solutions. Mohamad [6] considered oscillation of third-order DE. Yildiz. Karaman and Durur [7] investigated the oscillations of bounded solutions of higher-order nonlinear neutral delay differential equations. The goal of this work is to find enough novel conditions for nonoscillation to occur. The following lemma serves as the foundation for the proof of our key findings.

Lemma 1[4]: Let $f \in C^n(I;R^+)$ *and* $f^{(n)}(\xi)$ be eventually of one sign for all large ξ such that there is a $\xi_1 \geq \xi_0$. As a result $f^{(n-1)}(\xi)f^{(n)}(\xi) \leq 0$ for all $\xi \geq \xi_1$, if $\lim_{\xi\to\infty} f(\xi)$ exists, then for every $\varepsilon \in (0,1)$ there exists $\xi_\varepsilon \in [\xi_1,\infty)$ such that $f(\xi) \geq \frac{\varepsilon}{(n-1)!}\xi^{n-1}f^{(n-1)}(\xi)$, for all $\xi \in [\xi_\varepsilon,\infty)$.

Lemma 2. [5]:
i-If $\mathcal{Q},\upsilon \in C(R^+;R^+), \upsilon(\xi) < \xi$ and

$$\liminf_{\xi\to\infty} \int_{\upsilon(\xi)}^{\xi} \mathcal{Q}(\omega)\,d\omega > \frac{1}{e}.$$

Then, $\mathfrak{D}'(\xi) + \mathcal{Q}(\xi)\mathfrak{D}'\big(\upsilon(\xi)\big) \leq 0$ has no eventually positive solution.
ii-If $\mathcal{Q},\upsilon \in C(R^+;R^+), \upsilon(\xi) > \xi$,and

$$\liminf_{\xi\to\infty} \int_{\xi}^{\upsilon(\xi)} \mathcal{Q}(\omega)\,d\omega > \frac{1}{e}.$$

Then, $\mathfrak{D}'^{(\xi)} - \mathcal{Q}(\xi)\mathfrak{D}'\big(\upsilon(\xi)\big) \geq 0$ has no eventually positive solution.

2-Mains Results:

The following outcomes give some necessary requirements for nonoscillation of all solutions of eq.(1). For simplicity, we defined the function:

$$\mathfrak{D}(\xi) = \mathcal{P}(\xi) - \mathcal{S}(\xi)\mathcal{P}\big(u(\xi)\big). \tag{2}$$

So eq.(1) can be written as

$$\mathfrak{D}^{(4)}(\xi) + \mathcal{Q}(\xi)\mathcal{T}\left(\mathcal{P}\big(\upsilon(\xi)\big)\right) = 0 \tag{3}$$

Lemma 3: Suppose that $0 \leq \mathcal{S}(\xi) \leq \mathcal{S}_1 < 1, \mathcal{Q}(\xi) \geq 0$, A_1 hold and eq.(1) has a solution that is nonoscillatory. Then there are only the following cases to consider:

1. $\mathfrak{D}(\xi) > 0, \mathfrak{D}'(\xi) > 0,\ \mathfrak{D}''(\xi) > 0, \mathfrak{D}'''(\xi) > 0,\ \mathfrak{D}^{(4)}(\xi) \leq 0, \lim_{\xi\to\infty}\mathfrak{D}(\xi) = \infty$,
2. $\mathfrak{D}(\xi) > 0, \mathfrak{D}'(\xi) > 0,\ \mathfrak{D}''(\xi) < 0, \mathfrak{D}'''(\xi) > 0,\ \mathfrak{D}^{(4)}(\xi) \leq 0,$
3. $\mathfrak{D}(\xi) < 0, \mathfrak{D}'(\xi) > 0,\ \mathfrak{D}''(\xi) < 0, \mathfrak{D}'''(\xi) > 0,\ \mathfrak{D}^{(4)}(\xi) \leq 0, \lim_{\xi\to\infty}\mathfrak{D}(\xi) = \lim_{\xi\to\infty}\mathcal{P}(\xi) = 0.$

Proof. Suppose that eq.(1) has a nonoscillatory solution $\mathcal{P}(\xi)$, so $\mathcal{P}(\xi)$ is eventually positive (eventually negative). Let $\mathcal{P}(\xi) > 0,\ \mathcal{P}\big(u(\xi)\big) > 0,\ and\ \mathcal{P}\big(\upsilon(\xi)\big) > 0$, it follows from eq.(3) that $\mathfrak{D}^{(4)}(\xi) \leq 0$, hence $\mathfrak{D}'''(\xi),\ \mathfrak{D}''(\xi),\ \mathfrak{D}'(\xi)\ and\ \mathfrak{D}(\xi)$ are monotone, that is there is $\xi_1 \geq \xi_0$ such that,, $\mathfrak{D}'''(\xi)$, $\mathfrak{D}''(\xi)$, $\mathfrak{D}'(\xi)$, $\mathfrak{D}(\xi)$ of constant sign for $\xi \geq \xi_1$. We assert: $\mathfrak{D}'''(\xi) > 0, \xi \geq \xi_1$. Otherwise, $\mathfrak{D}'''(\xi) < 0$, then consequently,

$$\mathfrak{D}''(\xi) < 0, \mathfrak{D}'(\xi) < 0, \mathfrak{D}(\xi) < 0, \text{ and } \lim_{\xi\to\infty} \mathfrak{D}(\xi) = -\infty,$$

From (2) note that $\mathfrak{D}(\xi) \geq -\mathcal{S}(\xi)\mathcal{P}(u(\xi)) \geq -\mathcal{S}_1\mathcal{P}(u(\xi))$

$$\mathcal{P}(u(\xi)) \geq \frac{-1}{\mathcal{S}_1}\mathfrak{D}(\xi)$$

Hence, $\mathcal{P}(\xi) \to \infty$, as $\xi \to \infty$. On the other hand, since $\mathfrak{D}(\xi) < 0$ then from (2)

$$\mathcal{P}(\xi) < \mathcal{S}(\xi)\mathcal{P}(u(\xi)) \leq \mathcal{S}_1\mathcal{P}(u(\xi)) < \mathcal{P}(u(\xi)).$$

It implies that $\mathcal{P}(\xi)$ is decreasing function, this is a contradiction with $\mathcal{P}(\xi) \to \infty$.
Thus, $\mathfrak{D}'''(\xi) > 0$, $for\ \xi \geq \xi_1$. If $\mathfrak{D}''(\xi) > 0, for\ \xi \geq \xi_2 \geq \xi_1$, consequently, $\mathfrak{D}'(\xi) > 0, \mathfrak{D}(\xi) > 0$ and $\lim_{\xi\to\infty} \mathfrak{D}(\xi) = \infty$, this suggests $\mathcal{P}(\xi) \to \infty$.
If $\mathfrak{D}''(\xi) < 0, \xi \geq \xi_2 \geq \xi_1$, then by the same way to $\mathfrak{D}'''(\xi)$,we can show that $\mathfrak{D}'(\xi) > 0, \xi \geq \xi_2 \geq \xi_1$, so either $\mathfrak{D}(\xi) > 0$, or $\mathfrak{D}(\xi) < 0$. When $\mathfrak{D}(\xi) < 0$. It remains to show that $\lim_{\xi\to\infty} \mathfrak{D}(\xi) = 0$. Now if we assume that it is not true, then

$$\lim_{\xi\to\infty} \mathfrak{D}(\xi) = l < 0.$$

From (2) we get

$$\mathcal{S}_1\mathcal{P}(u(\xi)) \geq \mathcal{S}(\xi)\mathcal{P}(u(\xi)) = \mathcal{P}(\xi) - \mathfrak{D}(\xi)$$

$$\mathcal{P}(u(\xi)) \geq \frac{\mathcal{P}(\xi) - \mathfrak{D}(\xi)}{\mathcal{S}_1} \tag{4}$$

Since $\mathfrak{D}(\xi) < 0$, then $\mathcal{P}(\xi) < \mathcal{P}(u(\xi))$ and $\mathcal{P}(\xi)$ is decreasing, so let $\lim_{\xi\to\infty} \mathcal{P}(\xi) = k \geq 0$, letting $\xi \to \infty$, it follows from (4) that $k \geq \frac{k-l}{\mathcal{S}_1}$, this is a contradiction. Therefore, we have $\lim_{\xi\to\infty} \mathfrak{D}(\xi) = 0, \lim_{\xi\to\infty} \mathcal{P}(\xi) = 0$,
The proof is complete.

Lemma 4: Suppose that $Q(\xi) \leq 0$, $0 \leq \mathcal{S}(\xi) \leq \mathcal{S}_1 < 1$, A_1 hold and eq.(1) has a solution that is nonoscillatory. Then, there are only the following cases to consider:

1. $\mathfrak{D}(\xi) > 0, \mathfrak{D}'(\xi) > 0, \mathfrak{D}''(\xi) > 0, \mathfrak{D}'''(\xi) > 0, \mathfrak{D}^{(4)}(\xi) \geq 0, \lim_{\xi\to\infty} \mathfrak{D}(\xi) = \infty, \lim_{\xi\to\infty} \mathcal{P}(\xi) = \infty,$
2. $\mathfrak{D}(\xi) > 0, \mathfrak{D}'(\xi) > 0, \mathfrak{D}''(\xi) > 0, \mathfrak{D}'''(\xi) < 0, \mathfrak{D}^{(4)}(\xi) \geq 0, \lim_{\xi\to\infty} \mathfrak{D}(\xi) = \infty, \lim_{\xi\to\infty} \mathcal{P}(\xi) = \infty,$
3. $\mathfrak{D}(\xi) < 0, \mathfrak{D}'(\xi) < 0, \mathfrak{D}''(\xi) > 0, \mathfrak{D}'''(\xi) < 0, \mathfrak{D}^{(4)}(\xi) \geq 0.$
4. $\mathfrak{D}(\xi) > 0, \mathfrak{D}'(\xi) < 0, \mathfrak{D}''(\xi) > 0, \mathfrak{D}'''(\xi) < 0, \mathfrak{D}^{(4)}(\xi) \geq 0.$

Proof. Suppose that eq.(1) has a solution that is nonoscillatory. $\mathcal{P}(\xi)$, we can assume that $\mathcal{P}(\xi) > 0, \mathcal{P}(u(\xi)) > 0$ $and\ \mathcal{P}(v(\xi)) > 0$, it follows from eq.(3) that $\mathfrak{D}^{(4)}(\xi) \geq 0$, hence $\mathfrak{D}'''(\xi)$, $\mathfrak{D}''(\xi)$, $\mathfrak{D}'(\xi)$, $\mathfrak{D}(\xi)$ are monotone and they have a constant sign for $\xi \geq \xi_1 \geq \xi_0$. If $\mathfrak{D}'''(\xi) > 0, \xi \geq \xi_1$,then consequently,
$\mathfrak{D}''(\xi) > 0, \mathfrak{D}'(\xi) > 0, \mathfrak{D}(\xi) > 0$, and $\lim_{\xi\to\infty} \mathfrak{D}(\xi) = \infty$, this implies $\lim_{\xi\to\infty} \mathcal{P}(\xi) = \infty$,
If $\mathfrak{D}'''(\xi) < 0, for\ \xi \geq \xi_1$, we claim that $\mathfrak{D}''(\xi) > 0, \xi \geq \xi_2 \geq \xi_1$. Otherwise, $\mathfrak{D}''(\xi) < 0, \xi \geq \xi_2 \geq \xi_1$, hence $\mathfrak{D}'(\xi) < 0, \mathfrak{D}(\xi) < 0$, and $\lim_{\xi\to\infty} \mathfrak{D}(\xi) = -\infty$. From (2) it should be noted that

$$\mathfrak{D}(\xi) \geq -\mathcal{S}(\xi)\mathcal{P}(u(\xi)) \geq -\mathcal{S}_1\mathcal{P}(u(\xi))$$

$$\mathcal{P}(u(\xi)) \geq -\frac{1}{\mathcal{S}_1}\mathfrak{D}(\xi)$$

Hence, $\mathcal{P}(\xi) \to \infty$, as $\xi \to \infty$. On the other hand, since $\mathfrak{D}(\xi) < 0$ then from (2)

$$\mathcal{P}(\xi) < \mathcal{S}(\xi)\mathcal{P}\big(u(\xi)\big) \le \mathcal{S}_1\mathcal{P}\big(u(\xi)\big) < \mathcal{P}\big(u(\xi)\big)$$

It implies that $\mathcal{P}(\xi)$ is decreasing function, this is a contradiction with $\mathcal{P}(\xi) \to \infty$.
Thus, $\mathfrak{D}''(\xi) > 0,\ \xi \ge \xi_2$. If $\mathfrak{D}'(\xi) > 0,\ \xi \ge \xi_2 \ge \xi_1$, Consequently,
$\mathfrak{D}(\xi) > 0$ and $\lim_{\xi\to\infty} \mathfrak{D}(\xi) = \infty$, which implies that $\lim_{\xi\to\infty} \mathcal{P}(\xi) = \infty$.
If $\mathfrak{D}'(\xi) < 0, \xi \ge \xi_2 \ge \xi_1$, so either $\mathfrak{D}(\xi) < 0$, or $\mathfrak{D}(\xi) > 0$. Therefore, the proof is finished.

Theorem 4: Suppose that A_1 holds $\mathcal{Q}(\xi) \ge 0$, $0 \le \mathcal{S}(\xi) \le \mathcal{S}_1 < 1$,

$$\liminf_{\xi\to\infty} \int_{v(\xi)}^{\xi} \mathcal{Q}(\omega)\, v^3(\omega) d\omega > \frac{6}{m\varepsilon e}. \tag{5}$$

Then, each solution of eq.(1) oscillates or approaches to 0 as $\xi \to \infty$.
Proof. Suppose that eq.(1) has a nonoscillatory solution $\mathcal{P}(\xi)$, let $\mathcal{P}(\xi) > 0$, $\mathcal{P}\big(u(\xi)\big) > 0$, $\mathcal{P}\big(v(\xi)\big) > 0$, According to Lemma 3, there are only the following cases for discussion:

1. $\mathfrak{D}(\xi) > 0, \mathfrak{D}'(\xi) > 0,\ \mathfrak{D}''(\xi) > 0, \mathfrak{D}'''(\xi) > 0,\ \mathfrak{D}^{(4)}(\xi) \le 0, \lim_{\xi\to\infty} \mathfrak{D}(\xi) = \infty$
2. $\mathfrak{D}(\xi) > 0, \mathfrak{D}'(\xi) > 0,\ \mathfrak{D}''(\xi) < 0, \mathfrak{D}'''(\xi) > 0,\ \mathfrak{D}^{(4)}(\xi) \le 0,$
3. $\mathfrak{D}(\xi) < 0, \mathfrak{D}'(\xi) > 0,\ \mathfrak{D}''(\xi) < 0, \mathfrak{D}'''(\xi) > 0,\ \mathfrak{D}^{(4)}(\xi) \le 0, \lim_{\xi\to\infty} \mathfrak{D}(\xi) = \lim_{\xi\to\infty} \mathcal{P}(\xi) = 0.$

Since cases 1 and 2 are similar for $\mathfrak{D}(\xi) > 0,\ and\ \mathfrak{D}'(\xi) > 0$, then we will discuss them together.
Cases 1 and 2. Since $\mathfrak{D}^{(4)}(\xi)\mathfrak{D}'''(\xi) < 0$, and $\mathfrak{D}(\xi)$ is positive increasing so $\lim_{\xi\to\infty} \mathfrak{D}(\xi) > 0$, then by Lemma 1, we have $\mathfrak{D}(\xi) \ge \frac{\varepsilon}{3!}\xi^3\mathfrak{D}'''(\xi)$, for $\xi \in [\xi_\varepsilon, \infty)$. So

$$\mathfrak{D}\big(v(\xi)\big) \ge \frac{\varepsilon}{3!} v^3(\xi)\mathfrak{D}'''\big(v(\xi)\big). \tag{6}$$

By A_1, the eq.(3) leads to:

$$\mathfrak{D}^{(4)}(\xi) + m\mathcal{Q}(\xi)\mathcal{P}\big(v(\xi)\big) \le 0$$

By (2), it follows that $\mathcal{P}\big(v(\xi)\big) \ge \mathfrak{D}\big(v(\xi)\big)$ so

$$\mathfrak{D}^{(4)}(\xi) + m\mathcal{Q}(\xi)\mathfrak{D}\big(v(\xi)\big) \le 0$$

By substituting (6) in the last inequality, we get

$$\mathfrak{D}^{(4)}(\xi) + \frac{m\varepsilon}{3!}\mathcal{Q}(\xi)v^3(\xi)\mathfrak{D}'''\big(v(\xi)\big) \le 0$$

Let $\varphi(\xi) = \mathfrak{D}'''(\xi)$, then

$$\varphi'(\xi) + \frac{m\varepsilon}{3!}\mathcal{Q}(\xi)v^3(\xi)\varphi\big(v(\xi)\big) \le 0,$$

In terms of condition (5) and Lemma 2, it follows that the last inequality is impossible to have eventually positive solution, this is incongruent.

Example 1. Take into account fourth order nonlinear neutral equation

$$\left[\mathcal{P}(\xi) - \left(\tfrac{1}{5} + e^{-1}\right)\mathcal{P}(\xi - 1)\right]^{(4)} + \tfrac{1}{10}e^{-1}\mathcal{T}\big(\mathcal{P}(\xi - 2)\big) = 0,\ \xi \ge 0. \tag{7}$$

$\mathcal{S}(\xi) = \frac{1}{5} + e^{-1}$, $\mathcal{Q}(\xi) = \frac{1}{10}e^{-1}$, $u(\xi) = \xi - 1, v(\xi) = \xi - 2, \mathcal{T}(\mathcal{P}) = 2\mathcal{P},\ m = 2$. To check the condition (5)

$$\liminf_{\xi\to\infty} \int_{v(\xi)}^{\xi} \mathcal{Q}(\omega)\, v^3(\omega) d\omega = \frac{e^{-1}}{10} \lim_{\xi\to\infty} \int_{\xi-2}^{\xi} (\omega - 2)^3\, d\omega = \infty.$$

All conditions of theorem 4 are satisfied, To clarify, note that $\mathcal{P}(\xi) = e^{-\xi}$ is a solution to the eq.(7) and approaches zero as $\xi \to \infty$.
Theorem 5: Suppose that A_1 holds, $\mathcal{Q}(\xi) \le 0, v(\xi) \ge \xi$ for some $\eta(\xi) > \xi$ and

$$\liminf_{\xi\to\infty} \int_{\xi}^{v(\xi)} |\mathcal{Q}(\varrho)| v^3(\varrho) d\varrho > \frac{6}{\varepsilon m e}, \tag{8}$$

$$\limsup_{\xi\to\infty}\int_{\xi_2}^{\xi}\int_{\omega}^{\eta(\omega)}\int_{\sigma}^{\eta(\sigma)}\int_{\tau}^{\eta(\tau)}|\mathcal{Q}(\varrho)|d\varrho\, d\tau\, d\sigma d\omega=\infty\,. \quad (9)$$

Then, every bounded solution of eq.(1) either oscillates or approaches to 0 as $\xi\to\infty$,
Proof. Suppose eq.(1) has a nonoscillatory solution $\mathcal{P}(\xi)$, so $\mathcal{P}(\xi)$ is eventually positive (eventually negative). Let $\mathcal{P}(\xi)>0,\ \mathcal{P}(u(\xi))>0,\ \mathcal{P}(v(\xi))>0$. According to lemma 4, there are only the following cases for discussion:
1. $\mathfrak{D}(\xi)<0, \mathfrak{D}'(\xi)<0,\ \mathfrak{D}''(\xi)>0, \mathfrak{D}'''(\xi)<0,\ \mathfrak{D}^{(4)}(\xi)\geq 0$.
2. $\mathfrak{D}(\xi)>0, \mathfrak{D}'(\xi)<0,\ \mathfrak{D}''(\xi)>0, \mathfrak{D}'''(\xi)<0,\ \mathfrak{D}^{(4)}(\xi)\geq 0$.

Cases 1: Since $\mathfrak{D}'''(\xi)\mathfrak{D}''''(\xi)<0$, and $\mathfrak{D}(\xi)$ is negative decreasing , $\lim_{\xi\to\infty}\mathfrak{D}(\xi)<0$, then by lemma 1 we have $\mathfrak{D}(\xi)\ \geq\frac{\varepsilon}{6}\xi^3\mathfrak{D}'''(\xi)$, for $\xi\in[\xi_\varepsilon,\infty)$. So

$$\mathfrak{D}(v(\xi))\ \geq\frac{\varepsilon}{6}v^3(\xi)\mathfrak{D}'''(v(\xi)) \quad (10)$$

Eq.(3) may be written as well as the

$$\mathfrak{D}^{(4)}(\xi)\geq m|\mathcal{Q}(\xi)|\mathcal{P}(v(\xi))\geq m|\mathcal{Q}(\xi)|\mathfrak{D}(v(\xi)) \quad (11)$$

By substituting (10) in (11) we get

$$\mathfrak{D}^{(4)}(\xi)\geq\frac{\varepsilon m}{6}|\mathcal{Q}(\xi)|v^3(\xi)\mathfrak{D}'''(v(\xi)). \quad (12)$$

Let $\varphi(\xi)=\mathfrak{D}'''(\xi)$ then

$$\varphi'(\xi)-\frac{\varepsilon m}{6}|\mathcal{Q}(\xi)|v^3(\xi)\varphi(v(\xi))\geq 0.$$

In terms of condition (8) and Lemma 2, it follows that the last inequality is impossible to have eventually positive solution, this is incongruous.
Case 2: We claim that $\liminf_{\xi\to\infty}\mathcal{P}(\xi)=0$, otherwise $\liminf_{\xi\to\infty}\mathcal{P}(\xi)=l_1>0$. So there is $\xi_2\geq\xi_1$ such that $\mathcal{P}(\xi)\geq l_1,\ \xi\geq\xi_2$.
Hence, the inequality (11) leads to

$$\mathfrak{D}^{(4)}(\xi)\geq m|\mathcal{Q}(\xi)|\mathcal{P}(v(\xi))\geq ml_1|\mathcal{Q}(\xi)|,\ \ \xi\geq\xi_2$$

Integrating the last inequality from ξ to $\eta(\xi)$ three times for some $\eta(\xi)>\xi$, this yields

$$-\mathfrak{D}'(\xi)\geq ml_1\int_{\xi}^{\eta(\xi)}\int_{\sigma}^{\eta(\sigma)}\int_{\tau}^{\eta(\tau)}|\mathcal{Q}(\varrho)|d\varrho\, d\tau\, d\sigma$$

$$\mathfrak{D}'(\xi)\leq -ml_1\int_{\xi}^{\eta(\xi)}\int_{\sigma}^{\eta(\sigma)}\int_{\tau}^{\eta(\tau)}|\mathcal{Q}(\varrho)|d\varrho\, d\tau\, d\sigma$$

Integrating the last inequality from ξ_2 to ξ yields

$$\mathfrak{D}(\xi)-\mathfrak{D}(\xi_2)\leq -ml_1\int_{\xi_2}^{\xi}\int_{\omega}^{\eta(\omega)}\int_{\sigma}^{\eta(\sigma)}\int_{\tau}^{\eta(\tau)}|\mathcal{Q}(\varrho)|d\varrho\, d\tau\, d\sigma$$

As $\xi\to\infty$, taking into account (9), the last inequality implies $\lim_{\xi\to\infty}\mathfrak{D}(\xi)=-\infty$, a contradiction. Thus $\liminf_{\xi\to\infty}\mathcal{P}(\xi)=0$, implies that $\lim_{\xi\to\infty}\mathfrak{D}(\xi)=0$. We claim that $\limsup_{\xi\to\infty}\mathcal{P}(\xi)=0$, otherwise $\limsup_{\xi\to\infty}\mathcal{P}(\xi)=l_2>0$, From (2) we get

$$\mathcal{P}(\xi)-\mathfrak{D}(\xi)=\mathcal{S}(\xi)\mathcal{P}(u(\xi))\leq\mathcal{S}_1\mathcal{P}(u(\xi))<\mathcal{P}(u(\xi))$$

As $\xi\to\infty$, the last inequality implies $l_2<l_2$, this is incongruous. The proof is finished.

Example 2: Consider the fourth-order NDE

$$\left[\mathcal{P}(\xi)-\frac{e^{-1}}{\xi-1}\mathcal{P}(\xi-1)\right]^{(4)}-\frac{\xi-5}{\xi-2}e^{-2}\mathcal{T}(\mathcal{P}(\xi-2))=0,\ \xi>5. \quad (13)$$

$\mathcal{S}(\xi)=\dfrac{e^{-1}}{\xi-1}$, $u(\xi)=\xi-1, \mathcal{Q}(\xi)=-\dfrac{\xi-5}{\xi-2}e^{-2}$, $\mathcal{T}(\mathcal{P})=\mathcal{P}$,
$v(\xi)=\xi-2, m\in(0,1]$, $\eta(\xi)=\xi+1, v(\eta(\xi))=\xi-1$.

$$\liminf_{\xi\to\infty}\int_{\xi-1}^{\xi}\int_{\sigma}^{\sigma+1}\left((\varrho+4)e^{2}-e^{1}\right)(\varrho-2)d\varrho\, d\sigma=\infty$$

$$\limsup_{\xi\to\infty}\int_{\xi_2}^{\xi}\int_{\omega}^{\eta(\omega)}\int_{\sigma}^{\eta(\sigma)}\int_{\tau}^{\eta(\tau)}|\mathcal{Q}(\varrho)|d\varrho\, d\tau\, d\sigma d\omega$$

$$=\liminf_{\xi\to\infty}\int_{5}^{\xi}\int_{\omega}^{\omega+1}\int_{\sigma}^{\sigma+1}\int_{\tau}^{\tau+1}\frac{(\varrho-5)e^{-4}}{\varrho-4}d\varrho\, d\tau\, d\sigma d\omega=\infty.$$

All of the requirements of theorem 5 are met, hence every solution of eq.(13) either oscillates or tends to zero as $\xi\to\infty$. To clarify, note that $\mathcal{P}(\xi)=\xi e^{-\xi}$ is a solution to eq.(13) and approaches zero as $\xi\to\infty$.

3. Conclusions:

In this paper, the oscillatory and non-oscillatory traits are proposed and studied to solve the fourth-order neutral delay equation. Some important conditions that guarantee the oscillation of the solutions to these equations are deduced, while others are close to zero $\xi\to\infty$.The results that illustrate the accuracy and efficiency of these conditions are obtained by displaying illustrative examples.

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