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P, P-L. Compact Topological Ring

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Abstract
In this paper, we introduced some new definitions on P-compact topological ring and PL-compact topological ring for the compactification in topological space and rings, we obtain some results related to P-compact and P-L compact topological ring.

Keywords: rings, compact topological ring , topological ring, D-cover groups, isomorpnism, direct product, D-compact group.

التراص P ، P−L. للحلقات التبولوجية

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الخلاصة
في هذا البحث قدمنا بعض التعاريف الجديدة عن التراص من نوع P ونوع P−L. للحلقات التبولوجية كنوع من التراص للفضاءات التبولوجية والحلقات. وقدمنا بعض النتائج المتعلقة بهذه الانواع اضافة الى النتائج التي تبين علاقتهم ببعض.

1. Introduction

A topological ring (R,∗, . ,τ) is a ring which is also topological space such that both the addition and the multiplication are continuous maps [1].
(R,∗, . ,τ) is said to be compact, if (R, τ) is compact as a topological space, for details see [2] . In [3] D.G. Salih gave the concept of D–compact groups. In this paper, we shall generalize this concept to the rings so we introduced P–compact and the P-L. compact topological rings, in particular case we deal with the ideals of a topological rings, so we investigated PI-compact and PI-L. compact topological rings, we obtain some good results related these concepts above. We mean throughout this paper a topological rings is just ring as a set with topology.

2. Definitions and Examples .

Definition 2.1.
Let (R,∗, .,τ) be a topological ring and I be an index set, we say that
1. The family $\{Ri \in \tau : (R_i,*,.) is\ a\ proper\ subrings\ of\ (R,*.),\ \forall i \in I\}$ is a P-cover topologica rings of (R,∗, . ,τ) if $R = \bigcup_{i \in I} R_i$

Definition 2.2.
Let $(R,*,.,\tau)$ be a topological ring we say that;
2. $(R,*,.,\tau)$ is weakly P–compact topological ring if there is a finite P– cover topological rings of $(R,*,.,\tau)$.

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3. $(R,*,.,\tau)$ is P-compact topological ring if for any P–cover topological rings of $(R,*,.,\tau)$, there is a finite sub P–cover topological rings of $(R,*,.,\tau)$.
4. $(R,*,.,\tau)$ is weakly P–L. compact topological ring if there exists a countable P–cover topological rings of $(R,*,.,\tau)$.
5. $(R,*,.,\tau)$ is P–L. compact topological ring if for any P–cover topological ring of $(R,*,.,\tau)$, there is a countable sub P–cover topological rings of $(R,*,.,\tau)$.

Definition 2.3.

Let $(R,*,.,\tau)$ be a topological ring and $(H,*,.)$ be a subring of $(R,*,.)$. The topological subring $(H,*,.,\tau_H)$ where $\tau_H = \tau \cap H$ is said to be:

P–compact topological subring (weakly P–compact topological subring, P–L. compact topological subring, weakly P.L. compact topological subring) if $(H,*,.,\tau_H)$ is P–compact (weakly P–compact, P–L. compact and weakly P–L. compact) topological ring respectively.

Definition 2.4.

1. Let $(R,*,.,\tau)$ and $(\bar{R},\bar{*},\bar{.},\bar{\tau})$ be two topological rings then,

(i) $f:(R,*,.,\tau) \to (\bar{R},\bar{*},\bar{.},\bar{\tau})$ is a homomorphism topological rings if $f:(R,\tau) \to (\bar{R},\bar{\tau})$ is continuous such that $f(x*y) = f(x) \bar{*} f(y)$ and $f(x.y) = f(x)\bar{.}f(y)$ for each pair of elements $x, y \in R$

(ii) f : $(R,*,.,\tau) \to (\bar{R},\bar{*},\bar{.},\bar{\tau})$ is a topological isomorphism if it is topological homeomorphism and ring isomorphism.

2. Suppose that $\wedge$ is a non-empty set and $(R_\lambda,*_\lambda,._\lambda,\tau_\lambda)$ is a topological rings for each $\lambda\epsilon\wedge$, their product is $\pi_{\lambda\epsilon\wedge}R_\lambda$ equipped with the usual product topology $\tau_{\pi_{\lambda\in\Lambda}}R_\lambda$ and with multiplication given by $(x \otimes y) = x_\lambda \otimes y_\lambda$ for each $x_\lambda, y_\lambda \epsilon R_\lambda$ and $\lambda \in \wedge$.

3. If $R_\lambda = R$ and $\tau_\lambda = \tau, \forall\lambda\epsilon\wedge$, then we denoted that
$R^\wedge = \pi_{\lambda\epsilon\wedge}R_\lambda$ and $\tau^\wedge = \pi_{\lambda\in\Lambda}\,\tau_\lambda$

Example 2.5.

Let $X = \{a, b, c, d\}$ and P(X) the power set of X i.e.
$P(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, \{a,c,d\}\}$
One can show easily that $R = (P(X), \Delta, \cap)$ where $\Delta\ B = A \cup B - A \cap B$, is a ring
Let τ be the discrete topology defined on $P(X)$ and let P_i , $1 \le i \le 12$ be the following sets :
$P_1 = \{\emptyset, X\}$
$P_2 = \{\emptyset, \{a\}\}$
$p_3 = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$
$P_4 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$
$P_5 = \{\emptyset, \{d\}\}$
$P_6 = \{\emptyset, \{d\}, \{a\}, \{a,d\}\}$
$P_7 = \{\emptyset, \{d\}, \{a\}, \{c\}, \{a,d\}, \{a,c\}, \{d,c\}, \{a,d,c\}\}$
$P_8 = \{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}\}$
$P_9 = \{\emptyset, \{b\}, \{a,d\}, \{a,b,d\}\}$
$P_{10} = \{\emptyset, \{a\}, \{c,d\}, \{a,c,d\}\}$
$P_{11} = \{\emptyset, \{d\}, \{b,c\}, \{b,c,d\}\}$
$P_{12} = \{\emptyset, X, \{a,c\}, \{b,d\}\}$
It is clear that the family $\{(P_i, \Delta, \cap), 1 \le i \le 12\}$ is a P–cover topological rings of $(P(X), \Delta, \cap)$, which has a finite sub P-cover $\{p_8, p_9, p_{10}, p_{11}, p_{12}\}$ so $(P(X), \Delta, \cap)$ is weakly P–compact in fact it is P–compact .
Recall that see[4 , P. 190] , a sub ring I of the ring R is said to be a two side ideal of R if and only if r ∈ R and $a \in I$ imply both $ra \in I$ and $ar \in I$.
For rings with identity 1, it is clear that if $1 \in I$ then $\equiv R$, so we give the following modification for the preceding definitions.

Definition 2.6.
Let $(R,*,.,\tau)$ be a topological rings and let Λ be an index we say that the family $\{I_i \in \tau : (I_i,*,.)$ is a proper ideal of $(R,*,.), \forall i \in \Lambda\} \cup \{1\}$ is a PI-cover topological ideals of $(R,*,.,\tau)$, if $R = \cup_{i\in\Lambda} I_i \cup \{1\}$

Definition 2.7.
Let $(R,*,.,\tau)$ be a topological ring, we say that

1. $(R,*,.,\tau)$ is weakly PI – compact topological ring, if there exists a finite PI – cover topological ideals of $(R,*,.,\tau)$
2. $(R,*,.,\tau)$ is PI–compact topological ring, if for any PI–cover topological ideals of $(R,*,.,\tau)$, there is a finite sub PI-cover topological ideals of $(R,*,.,\tau)$
3. $(R,*,.,\tau)$ is weakly PI–L. compact topological ring, if there exists a countable PI – cover topological of $(R,*,.,\tau)$
4. $(R,*,.,\tau)$ is PI–L. compact topological ring, if for any PI- cover topological ideals of $(R,*,.,\tau)$, there is a countable sub PI- cover topological rings of $(R,*,.,\tau)$.

Example 2.8.
Let R be the ring $(Z,+,.,\tau)$ where τ is the discrete topology defined on Z. Note that the following:

$2Z = \{0, \mp 2, \mp 4, \ldots\}$
$3Z = \{0, \mp 3, \mp 6, \ldots\}$
$4Z = \{0, \mp 4, \mp 6, \ldots\}$
$5Z = \{0, \mp 5, \mp 10, \ldots\}$
$6Z = \{0, \mp 6, \mp 12, \ldots\}$ …….etc.

are all proper ideals of $(Z,+,.,\tau)$. Now it is easy to show that $I_k = \{kZ \mid k \in Z^+\} \cup \{1\}$ is a countable PI – cover toplological ideals of $(Z,+,.,\tau)$, that's mean $Z = \cup_{k\in Z^+} I_k \cup \{1\}$
Which has been a countable sub PI-cover since for example $4Z \subseteq 2Z$ and $6Z \subseteq 3Z$, …. etc. Hence $(Z,+,.,\tau)$ is PI–L. compact which is not PI-compact because the prime numbers are infinite see [5].

3. Main results .
It is easy to prove direct from definitions the following Lemmas

Lemma 3.1.

1. Any P –compact topological ring is weakly P–compact.
2. Any P – compact topological ring is P–L. compact .

Lemma 3.2.

1. Any PI – compact topological ring is weakly PI–compact .
2. Any PI – compact topological ring is PI–L. compact .

Also we can prove directly by Lemma (1) , the following theorem

Theorem 3.3.
Let (R , $*$, . , τ) be a topological ring such that R is finite set, then the following are equivalents :

1. (R , $*$, . , τ) is P–compact topological ring .
2. (R , $*$, . , τ) is P–L. compact topological ring .

If we replace P–(P–L.) compact topological with PI–(PI–L.) compact topological ring respectively , the result is true .

Example 3.4.
Let R = $Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ and
$\tau = \{\{\bar{0}\}, \{\bar{0}, \bar{3}\}, \{\bar{0}, \bar{2}, \bar{4}\}, \{\bar{1}, \bar{5}\}, \{\bar{2}, \bar{3}, \bar{4}\}, \{\bar{0}, \bar{1}, \bar{3}, \bar{5}\}, \emptyset, R\}$
Although (R , + , . , τ) is finite topological ring but it is not P-compact and not PI-compact since $\{\bar{0}\}, \{\bar{0}, \bar{3}\}, \{\bar{0}, \bar{2}, \bar{4}\}$ are the only proper sub rings which are not cover (R , + , . , $\bar{\tau}$).
Recall that a ring (R , $*$, .) is said to be a field provided that the set R/{0} is commutative group under the multiplication of R. It is known that if R is a field, then R has no non trivial ideals see [4], so we have the following theorem .

Theorem 3.5.
Any infinite ring (not field) can be PI–compact.

Proof.

Suppose first (R,*,.,τ) is a ring without identity . Let I be a set (finite or infinite) defined = $\{A_i \subseteq R :\ A_i^C \text{ is finite set },\ (A_i ,*,.) \text{ proper ideal } \forall_i \in I \text{ and } A_{i_1} \subseteq A_{i_2} \text{ for } i_1 \leq i_2\} \cup \{\emptyset\}$.

Now since any arbitrary set { na : a ∈ $R , n \in Z^+$} is ideal see [4], hence Z ∈ $\emptyset$. Clear that (R , ∗ , . , τ) is topological ring since .

(1) $\emptyset \in \tau$ and $R^c = \emptyset$ is finite i.e. $R \in \tau$.

(2) Let $A_1 , A_2 \in \tau$, so A_1^c , A_2^c are finite but $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$ implies $(A_1 \cap A_2)^c$ is finite and since $(A_1 \cap A_2 ,*,.)$ is ideal see[3] hence $A_1 \cap A_2 \in \tau$.

(1) Let Λ be any index and let $A_s \in \tau$, $\forall_s \in \Lambda$, hence A_s^c is finite for each s ∈ Λ which leads to $\bigcap_{s\in\Lambda} A_s^c$ is finite . Now since $\left(\bigcup_{s\in\Lambda} A_s\right)^c = \bigcap_{s\in\Lambda} A_s^c$, but $\bigcup_{s\in\Lambda} A_s = A_t$, for each s ∈ Λ . So $(\bigcup_{s\in\Lambda} , A_i ,*,.)$ is an ideal , hence (R , ∗ , . , τ) is topological ring.

Now let $\{A_\lambda \epsilon \tau , \lambda \in \Lambda\}$ be any PI–cover topological rings of (R,∗,. , τ), that is R= $\bigcup_{\lambda\in\Lambda} A_\lambda$.

Let $A_0 \epsilon \{A_\lambda\}_{\lambda\epsilon\hat{\Lambda}}$ implies $(A_0 ,*,.)$ is an ideal and A_0^c is finite set. Suppose that A_0^c = $\{a_1 , a_2 , \dots\dots , a_n\}$ where a_j ∈ R for each 1 ≤ j ≤ n, but $\{A_\lambda \in \tau, \lambda \in \Lambda\}$ is PI–cover of (R,∗,.,τ) so there is $A_{\lambda_j} \in \{A_\lambda \in \tau , \lambda \in \Lambda\}$ such that $a_j \in A_{\lambda j}$ for each j implies $A_0^c \subseteq \bigcup_{j=1}^n A_{\lambda_j}$. Thus $R \subseteq \bigcup_{j=1}^n A_{\lambda_j} \cup A_0$ (of course R = $A_0 \cup A_0^c$) that means there is a finite sub PI – cover topological rings {A_0 , $A_{\lambda 1}$, … , $A_{\lambda n}$} each of which is ideal , therefore (R , ∗ , . , τ) is PI–compact topological ring . If (R , ∗ , . , τ) is a ring with identity we take $\acute{\tau} = \tau \cup \{1\}$ and the prove is similar

Corollary 3.6.

Any infinite ring (not field) can be a weakly PI–compact .

For product P – compact rings we have the following two theorems .

Theorem 3.7.

Let (R , ∗ , . , τ) and $(\bar{R} ,\bar{*} ,\bar{.} ,\bar{\tau})$ be two topological rings if $(\bar{R} ,\bar{*} ,\bar{.} ,\bar{\tau})$ is a P–compact topological ring . Then (R × $\bar{R}$, ⊗ , ⨀, τ × $\bar{\tau}$) is P–compact topological ring.

Proof.

Let $\{(R \times \bar{R}_i , \otimes , \odot) :\ \bar{R}_i \in \bar{\tau} \text{ and } (\bar{R}_i ,\bar{*} ,\bar{.}) \text{ rings } \forall i \in I\}$ be any P–cover topological rings of R × $\bar{R}$, i.e. R × $\bar{R}$ = $\bigcup_{i\in I} (R \times \bar{R}_i) = R \times (\bigcup_{i\in I} \bar{R}_i)$ implies $\bar{R} = \bigcup_{i\in I} \bar{R}_i$, but $(\bar{R} ,\bar{*} ,\bar{.} ,\bar{\tau})$ is P–compact topological ring so there is finite subset J ⊆ I such that $\bar{R} = \bigcup_{j\in J} \bar{R}_j$. Thus R × $\bar{R}$ = R × $\left(\bigcup_{j\in J} \bar{R}_j\right) = \bigcup_{j\in J} \left(R \times \bar{R}_j\right)$, where $(R \times \bar{R}_j , \otimes , \odot)$ is a ring for each j ∈ J. Therefore (R × $\bar{R}$, ⊗ , ⨀, τ × $\bar{\tau}$) is P-compact topological ring.

Theorem 3.8.

Let (R , ∗ , . , τ) and $(\bar{R} ,\bar{*} ,\bar{.} ,\bar{\tau})$ be two P–compact topological rings then (R × $\bar{R}$, ⊗ , ⨀, τ × $\bar{\tau}$) is P–compact topological ring.

Proof.

Let (R,∗,., τ) and $(\bar{R} ,\bar{\otimes} ,\bar{\odot} ,\bar{\tau})$ be any two P–compact topological rings . Then there exists a P–cover topological rings $\{R_a\}_{a\in A}$ and $\{R_b\}_{b\in B}$ of R and $\bar{R}$ respectively (A , B any index), that's mean R × $\bar{R}$ = $(\bigcup_{a\in A} R_a) \times (\bigcup_{b\in B} \bar{R}_b) = \bigcup_{a\in A, b\in B} (R_a \times \bar{R}_b)$ implies $\{R_a \times \bar{R}_b\}_{a\in A, b\in B}$ is a P–cover topological rings of (R × $\bar{R}$, ⊗ , ⨀, τ × $\bar{\tau}$).

Let $\{W_i\}_{i\in\Lambda}$ be any P – cover topological rings of (R × $\bar{R}$, ⊗ , ⨀, τ × $\bar{\tau}$) then R × $\bar{R}$ = $\bigcup_{i\in\Lambda} W_i$ such that $W_i = U_i \times V_i$, where $U_i \in \tau$ and $V_i \in \bar{\tau}$ for each $i \in \Lambda$. But (R , ∗ , . , τ) is P–compact ring , so there is a finite sub set $J \subseteq \Lambda$ such that R = $\bigcup_{j\in J} U_j$ and $(U_j ,*,.)$ is a ring for each $j \in J$. Let $U_{j_1} \in \{U_j\}_{j\in J}$ implies $\{U_{j_1} \times V_i\}_{i\in\Lambda}$ is a P–cover topological rings of $\left(U_{j_1} \times \bar{R} , \otimes , \odot\right)$

hence $U_{j_1} \times \bar{R} = \bigcup_{i\in\Lambda} (U_{j_1} \times V_i)$, but $(U_{j_1} \times \bar{R} , \otimes , \odot)$ is P–compact topological ring since $(U_{j_1} ,*,.)$ is a ring and $(\bar{R} ,\bar{*} ,\bar{.})$ is P-compact topological ring (theorem 3) so there is a finite set S ⊂ Λ such that $\{U_{j_1} \times V_s\}_{s\in S}$ is a ring , $\forall s \in S$. Now $U_{j_1} \times \bar{R}$ = $\bigcup_{s\in S} (U_{j_1} \times V_s)$ hence $U_{j_1} \times \bar{R} = U_{j_1} \times (U_{s\in S} V_s)$ see [6] and hence

R × $\bar{R}$ = $\left(\bigcup_{j\in J} U_j\right) \times (U_{s\in S} V_s) = \bigcup_{j\in J, s\in S} \left(U_j \times V_s\right)$, where

$(U_j \times V_s , \otimes , \odot)$ are rings for each j ∈ J , $s \in S$. Therefore

$(R \times \bar{R}, \otimes, \odot, \tau \times \bar{\tau})$ is P–compact topological rings.
If we replace P–compact topological ring with PI–compact in Teorems (3.7 , 3.8) the result is true since the product of ideals is also ideal (for instance see [4]) .

Theorem 3.9. [1]

Let $\{R_i : i \in I\}$ be a family of topological rings . Then the direct product $= \prod_{i \in I} R_i$, equipped with the product topology is topological rings.
From Theorem (3.8) and Theorem (3.9) , respectively , and by induction we can prove the following theorem

Theorem 3.10.

The product of any finite collection of P–compact topological rings is P–compact topological ring.
If we replace P–compact topological ring with P–L. compact topological ring , the result is true .

Corollary 3.11.

If $(R, *, ., \tau)$ is a P – compact topological ring . Then $(R^n, \otimes, \odot, \tau^n)$ is P – compact topological ring , where
$R^n = \frac{R \times R \times .....\times R}{n-time}$ and $\tau^n = \frac{\tau \times \tau \times .....\times \tau}{n-time}$

Theorem 3.12.

Let $(R, *, ., \tau)$ and $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ be two topological rings , and let
$f : (R, *, ., \tau) \rightarrow (\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ be a homomorphism. Then

(1) If S is a P–compact topological subring in $(R, *, ., \tau)$, then f(S) is P–compact topological subring in $(\bar{R}, \bar{*}, ., \bar{\tau})$.
(2) If T is a P – compact topological subring in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ and f is an isomorphism, then $f^{-1}(T)$ is P–compact topological subring in $(R, *, ., \tau)$.

Proof .

Let $\{\bar{R}_i\}_{i \in I}$ be any P– cover topological rings of f(S) in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ that is $f(S) = \bigcup_{i \in I} \bar{R}_i$. Now since $S \subseteq f^{-1}(f(S))$ see [7] , implies $S \subseteq f^{-1}(\bigcup_{i \in I} \bar{R}_i)$ but $f^{-1}(\bigcup_{i \in I} \bar{R}_i) = \bigcup_{i \in I} f^{-1}(\bar{R}_i)$, see also [7], hence $S \subseteq \bigcup_{i \in I} f^{-1}(\bar{R}_i)$ on the other hand $f^{-1}(R_i)$ for each $i \in I$ is a sub ring in R for enstance see [4, p. 186], and since S is P-compact and f continuous hence there exists a finite set $J \subset I$, such that
$S = \bigcup_{j \in J} f^{-1}(R_j) = f^{-1}(\bigcup_{j \in J} R_j)$ implies $f(S) = f\left(f^{-1}(\bigcup_{j \in J} R_j)\right)$. But $f\left(f^{-1}(\bigcup_{j \in J} R_j)\right) \subseteq \bigcup_{j \in J} R_j$ see [7] , i.e. $f(S) \subseteq U_{j \in J} R_j$.Thus $f(S)$ is P-compact topological sub ring in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$

2- Let $\{R_i\}_{i \in I}$ be any P-cover topological rings of $f^{-1}(T)$ in $(R, *, ., \tau)$
that is $f^{-1}(T) = \bigcup_{j \in I} R_i$, $R_i \in \tau$, $\forall i \in I$ implies $T = f(\bigcup_{i \in I} R_i) = f(\bigcup_{i \in I} R_i)$. It is clear that $f(R_i) \in \bar{\tau}$, $\forall i \in I$ since f is isomorphism (definition 2.4), but T is a P-compact topological subring of $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$, so there is a finite subset $J \subseteq I$ such that $T = \bigcup_{j \in J} f(R_j)$ where $(f(R_j), \bar{*}, \bar{.})$ is a ring, $\forall j \in J$ see [4 , p. 186]. Thus $T = f(\bigcup_{j \in J} R_j)$ hence
$f^{-1}(T) = U_{j \in J} R_j$ and hence $f^{-1}(T)$ is P-compact topological subring of $(R, *, ., \tau)$.

For PI-compact ring we have the following theorem.

Theorem 3.13.

Let $(R, *, ., \tau) \rightarrow (\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$ be an isomorphism, then

(1) If S is a PI-compact topological ideal in $(R, *, ., \tau)$, then f (S) is PI-compact topological ideal in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$.
(2) If T is a PI-compact topological ideal in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$, then $f^{-1}(T)$ is PI-compact topological ideal in $(R, *, ., \tau)$.

Proof.

Let $\{I_i\}_{i \in \wedge}$ be any PI-cover topological ideal of f(S) in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$, that is $f(S) = \bigcup_{i \in \wedge} I_i \cup \{1\}$, *hence* $S = f^{-1}(\bigcup_{i \in \wedge} I_i) \cup \{1\}$. It is known that $f^{-1}(I_i)$, $\forall i$ are ideals see [4 , P.198].
Also $f^{-1}(I_i) \in \tau$, $\forall i$ since f is isomorphism. Now S is PI compact ideal in $(R, *, ., \tau)$, hence there exists a finite set $J \subseteq \wedge$ such that
$S = \bigcup_{j \in J} f^{-1}(I_j) \cup \{1\}$ implies $(S) = f\left(f^{-1} \bigcup_{j \in J} I_j\right) \cup f\{1\}$, hence
$f(S) = \bigcup_{j \in J} I_j \cup \{1\}$, that means $f(S)$ is PI- compact ideal in $(\bar{R}, \bar{*}, \bar{.}, \bar{\tau})$.

2. Let $\{I_i\}_{i\in\wedge} \cup \{1\}$ be any PI-cover topological ideal of $f^{-1}(T)$ that is
$f^{-1}(T) = (\cup_{i\in\wedge} I_i) \cup \{1\}$, $\{I_i \in \tau, \forall i \in \wedge\}$ implies
$T = f(\cup_{i\in\wedge} I_i) \cup \{1\}$
$= \cup_{i\in\wedge} \left(f(I_i)\right) \cup \{1\}$
It is clear that $f(I_i) \in \bar{\tau}$, $\forall i \in \wedge$ since f is an isomorphism (definition 2.4), but T is PI-compact topological ideal in ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$), so there is a finite subset J $\subseteq \wedge$ such that $T = \cup_{j\in J} f(I_j) \cup \{1\}$ where $(f(I_j), \bar{*}, \bar{.})$ are ideals for each j∈J see [4, p.198] . Now $T = f(\cup_{j\in J} I_j) \cup \{1\}$, hence $f^{-1}(T) = U_{j\in J} I_j \, U \{1\}$ means $f^{-1}(T)$ is PI-compact topological ideal and we have done.
The following theorem show that the P-compact is topological property.

Theorem 3.14.
Let (R , ∗ , . , τ) and ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) be two topological rings and
f: (R , ∗ , . , τ) → ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) be an isomorphism, then the following are equivalents:
1- (R , ∗ , . , τ) is P-compact topological ring.
2- ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is P-compact topological ring.

Proof.
suppose that (R , ∗ , . , τ) is P-compact topological ring, let $\{R_i\}_{i\in I}$ be any P-cover topological rings of ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) , that is $\bar{R} = \cup_{i\in\wedge} \bar{R}_i$ gives
$R = f^{-1}(\bar{R}) = f^{-1}(\cup_{i\in\wedge} \bar{R}_i) = \cup_{i\in\wedge} f^{-1}(\bar{R}_i)$. But (R , ∗ , . , τ) is P-compact topological ring, so there is a finite subset J∈Λ , such that $R = \cup_{j\in J} f^{-1}(\bar{R}_j)$. Clear that $\left(f^{-1}(\bar{R}_j), *, .\right)$ is subring $\forall j \in J$, hence $R = f^{-1}(\cup_{j\in J} \bar{R}_j)$ implies $\bar{R} = f(R) = f\left(f^{-1}\left(\cup_{j\in J} \bar{R}_j\right)\right) = \cup_{j\in J} \bar{R}_j$
therefore ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is P-compact topological ring.
suppose that ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is a P-compact topological ring, let $\{R_i\}_{i\in\wedge}$ be any P-cover topological rings of (R , ∗ , . , τ), i.e. $R = \cup_{i\in\wedge} R_i$.
Clear that $\bar{R} = f(R) = f(\cup_{i\in\Lambda} R_i) = \cup_{i\in\Lambda} f(R_i)$ where $(f(R_i), \bar{*}, \bar{.})$ Is a ring $\forall i \in \Lambda$ see [4], and since f is isomorphism (definition 2.4) implies $f(R_i) \in \bar{\tau}, \forall i \in \Lambda$. But ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is P-compact so there is a finite subset $J \subseteq \Lambda$ such that $\bar{R} = \cup_{j\in J} f(R_j)$.
Now
$R = f^{-1}(\bar{R}) = f^{-1}\left(\cup_{j\in J} f(R_j)\right) = f^{-1}\left(f\left(\cup_{j\in J} R_j\right)\right) = \cup_{j\in J} R_j$. Thus (R , ∗, . , τ) is P-compact which complete the proof.
We can prove by the similar way the following theorem.

Theorem 3.15.
Let (R , ∗, . , τ) and ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) be two topological rings and
f : (R , ∗ , . , τ) → ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) be an isomorphism. Then the following are equivalents
1. (R , ∗ , . , τ) is PI-compact topological ring.
2. ($\bar{R}$, $\bar{*}$, $\bar{.}$, $\bar{\tau}$) is PI-compact topological ring.

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