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On Closed Rickart Modules

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Abstract

In this article, we study the notion of closed Rickart modules. A right R-module M is said to be closed Rickart if, for each $\varphi \in End(M)$, $r_M(\varphi) = Ker\varphi$ is a closed submodule of M. Closed Rickart modules is a proper generalization of Rickart modules. Many properties of closed Rickart modules are investigated. Also, we provide some characterizations of closed Rickart modules. A necessary and sufficient condition is provided to ensure that this property is preserved under direct sums. Several connections between closed Rickart modules and other classes of modules are given. It is shown that every closed Rickart module is κ-nonsingular module. Examples which delineate this concept and some results are provided.

Keywords: c-Rickart Modules, Quasi-Dedekind Modules, Closed Simple Modules, Nonsingular Modules.

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حول مقاسات ريكارت المغلقة

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الخلاصة

في هذا البحث درسنا فكرة مقاسات ريكارت المغلقة. المقاس الأيمن M على الحلقة R يدعى ريكارت مغلق اذا كان لكل $\varphi \in End(M)$ فأن $r_M(\varphi) = Ker\varphi$ مقاس جزئي مغلق من M. مقاس ريكارت المغلق يمثل اعمام فعلي لمقاس ريكارت. الكثير من الخصائص لمقاسات ريكارت المغلقة تحققت. كذلك نحن برهنا بعض الشواخص لمقاسات ريكارت المغلقة. برهنا الشرط الضروري و الكافي لضمان تحقق خاصية ريكارت المغلقة تحت الجمع المباشر. أثبتنا ان كل مقاس ريكارت مغلق يكون مقاس غير منفرد من النمط K.
أمثلة لتوضيح هذا المفهوم وبعض النتائج قد بُرهنت.

1 Introduction

Throughout this article, R denotes an associative ring with identity, unless otherwise stated, and all modules will be unitary right R-modules. The concept of right Rickart modules has been extensively studied in the literature. According to [1], a module M is called Rickart if, for any $\varphi \in S = End(M)$, $r_M(\varphi) = Ker\varphi = eM$ for some $e^2 = e \in S$. A submodule N of a module M is called closed if, N has no proper essential extensions inside M [2]. Following [3], a right R-module M is said to be closed Rickart (for short c-Rickart) if, for each $\varphi \in S = End(M)$, $r_M(\varphi) = Ker\varphi$ is closed submodule of M. R is a c-Rickart ring if R_R is a c-Rickart module. It is clear that every Rickart module is c-Rickart.

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This work consists of two sections. In section 2, we establish basic properties and some characterizations of c-Rickart modules. Also, we discussed and investigated the connections between a c-Rickart modules and other types of modules. It is shown that every nonsingular module is c-Rickart. Also, we give an example to show that the convers need not be true, in general. We prove that the class of rings R for which every right R-module is c-Rickart is precisely that of the semisimple artinian. The investigations in section 3 focus on the question when the direct sum of two or more c-Rickart modules is also c-Rickart ? Further, we prove that for all $i \in I$, M_i is a c-Rickart module if and only if $\bigoplus_{i\in I} M_i$ is a c-Rickart module, where M_i is a fully invariant submodule of $\bigoplus_{i\in I} M_i$ for all $i \in I$. Some results on direct sum decompositions of c-Rickart modules are also included in this section. In what follows, we will denote the endomorphism of a module M by $End(M)$. The notations $N \subseteq M$, $N \leq M$, $N \leq_e M$, $N \leq^c M$, or $N \leq^{\oplus} M$ mean that N is a subset, a submodule, an essential submodule, a closed submodule, or a direct summand of M, respectively. $E(M)$ denotes the injective hull of a module M. For a module M, we denote $r_M(\varphi) = \{m \in M | \varphi(m) = 0\} = Ker\varphi$ for each $\varphi \in S = End(M)$.

2 Closed Rickart Modules

Let M be a right R-module with $S = End(M)$. The module M is called closed Rickart (for short c-Rickart) if, for any $\varphi \in S = End(M)$, $r_M(\varphi) = Ker\varphi$ is closed submodule of M [3]. In this section, we continue to investigate properties and characterizations of c-Rickart modules. We begin with the following remarks and examples.

Remarks and Examples (2.1)

(*i*) Every semisimple module is c-Rickart, but not conversely, for example: the Z-module Z is c-Rickart but its neither semisimple nor simple.

A right R-module M is called Baer if, the left annihilator in $S = End(M)$ of any submodule of M is generated by an idempotent of S. Equivalently, the right annihilator in M of any non-empty subset of $S = End(M)$ is generated by an idempotent of S [4]. It is clear that every Baer module is Rickart, so it is a c-Rickart module.

(*ii*) The c-Rickart property does not always transfer from a module to its submodule, as example: it is well known that the Z-modules Q and Z_2 are Baer modules such that $Hom_Z(Q, Z_2) = 0$, so by [5, Prop. 3.20] $Q \oplus Z_2$ is a Baer Z-module, then it is c-Rickart. However, we can see that the submodule $Z \oplus Z_2$ is not a c-Rickart module, as follows: assume that $\varphi \in S = End(Z \oplus Z_2)$ defined by $\varphi(x, \bar{y}) = (0, \bar{x})$ for each $x \in Z$, $\bar{y} \in Z_2$. Hence $Ker\varphi = 2Z \oplus Z_2$ which is not closed in $Z \oplus Z_2$. In fact $2Z \oplus Z_2$ is an essential submodule of $Z \oplus Z_2$.

(*iii*) The c-Rickart property does not always transfer from submodules to a module, as the next example illustrate: the Z-module Z_4 is not c-Rickart, since $\varphi \in End(Z_4)$ such that $\varphi(\bar{x}) = 2\bar{x}$ for all $\bar{x} \in Z_4$, but $Ker\varphi = 2Z_4 \leq_e Z_4$, so $Ker\varphi$ is not closed in Z_4. However, the submodule $2Z_4 (\cong Z_2)$ is a c-Rickart Z-module.

(*iv*) A homomrphic image of a c-Rickart module may not be c-Rickart. Consider the natural epimorphism $\varphi: Z \longrightarrow Z_4$. The Z-module Z is c-Rickart, but $Im\varphi = Z_4$ is not c-Rickart Z-module.

(*v*) The c-Rickart property transforms under an isomorphism.

Proof. Assume that M_1 and M_2 are two R-modules such that M_1 is c-Rickart, and $\varphi: M_1 \longrightarrow M_2$ is an isomorphism. Let $\alpha \in End(M_2)$, to prove that $Ker\alpha \leq^c M_2$. Suppose that $Ker\alpha \leq_e A_2$ in M_2, hence $\varphi^{-1}(Ker\alpha) \leq_e \varphi^{-1}(A_2)$ in M_1. We claim that $\varphi^{-1}(Ker\alpha) = Ker(\varphi^{-1}\alpha\varphi)$. Let $y \in \varphi^{-1}(Ker\alpha)$, $y = \varphi^{-1}(x)$ and $x \in Ker\alpha$, hence $\varphi(y) = x$ and $\alpha(x) = 0$. Then $\varphi^{-1}\alpha\varphi(y) = \varphi^{-1}\alpha(x) = \varphi^{-1}(0) = 0$, so $y \in \mathrm{Ker}(\varphi^{-1}\alpha\varphi)$. Conversely, let $x \in Ker(\varphi^{-1}\alpha\varphi)$, $\varphi^{-1}\alpha(\varphi(x)) = 0$ and so $\varphi(x) \in \mathrm{Ker}(\varphi^{-1}\alpha)$. But $\mathrm{Ker}(\varphi^{-1}\alpha) = Ker\alpha$ (clear), hence $\varphi(x) \in \mathrm{Ker}\alpha$, $x \in \varphi^{-1}(\mathrm{Ker}\alpha)$. Thus, $\varphi^{-1}(Ker\alpha) = Ker(\varphi^{-1}\alpha\varphi)$. Since $\varphi^{-1}\alpha\varphi \in End(M_1)$ and M_1 is a c-Rickart module, then $\varphi^{-1}(Ker\alpha) = Ker(\varphi^{-1}\alpha\varphi) \leq^c M_1$, but $\varphi^{-1}(Ker\alpha) \leq_e \varphi^{-1}(A_2)$ in M_1 this implies $\varphi^{-1}(Ker\alpha) = \varphi^{-1}(A_2)$ and hence $Ker\alpha = A_2$. Therefore $Ker\alpha \leq^c M_2$ and M_2 is a c-Rickart module. ■

An R-module M is said to have the closed intersection property (for short CIP) if, the intersection of any two closed submodules of M is again closed [3].

Theorem (2.2) The following conditions are equivalent for a module M :

(*i*) M is a c-Rickart with CIP;

(*ii*) The right annihilator in M of any finitely generated left ideal $I = \langle \varphi_1, \varphi_2, \dots, \varphi_n \rangle$ of $End(M)$ is closed in M.

Proof. $(i) \Rightarrow (ii)$ Suppose that M is a c-Rickart module, let $I \leq S = End(M)$ be a nonzero left ideal with a finite number of generators $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$. Since $r_M(I) = \bigcap_{i=1}^{n} r_M(\varphi_i) \leq^c M$ for $1 \leq i \leq n$, as M has the CIP. Hence $r_M(I) \leq^c M$.
$(ii) \Leftarrow (i)$ Let $\psi \in S = End(M)$, then $\langle \psi \rangle$ is a left ideal of S with one generator, hence $r_M(\psi) \leq^c M$. Therefore M is c-Rickart. ■

Proposition (2.3) Every direct summand of a c-Rickart module is c-Rickart.

Proof. Let M be a c-Rickart module, and let $N \leq^{\oplus} M$. Then $M = N \oplus K$ for some $K \leq M$. Let $\varphi \in S = End(N)$, to prove $Ker\varphi \leq^c N$. Consider the sequence $M \overset{\rho}{\rightarrow} N \overset{\varphi}{\rightarrow} N \overset{i}{\rightarrow} M$ where ρ is the natural projection, i is the inclusion mapping. Hence $i \circ \varphi \circ \rho \in End(M)$, and so $Ker(i \circ \varphi \circ \rho)$ is closed in M. It is easy to see that $Ker(i \circ \varphi \circ \rho) = Ker\varphi \oplus K$, implies $Ker\varphi \oplus K \leq^c M$, but $Ker\varphi \leq^c Ker\varphi \oplus K$, hence $Ker\varphi \leq^c M$ by [6, Prop. 6.24 (2)], but $Ker\varphi \subseteq N$, so $Ker\varphi \leq^c N$ by [6, Prop. 6.24 (1)]. Thus N is c-Rickart. ■

The converse of Proposition 2.3 need not be true, in general, as the following example shows.

Example (2.4) Consider the Z-module Z_{12}. Let $\varphi \in S = End(Z_{12})$ such that $\varphi(\overline{x}) = 6\,\overline{x}$ for all $\overline{x} \in Z_{12}$, then $Ker\varphi = \langle \overline{2} \rangle \leq_e Z_{12}$ this implies $Ker\varphi$ is not closed in Z_{12}, and hence Z_{12} is not c-Rickart. On the other hand, $N = \langle \overline{4} \rangle \leq^{\oplus} Z_{12}$, then $N(\cong Z_3)$. But Z_3 is a simple Z-module, so it is c-Rickart. Thus N is a c-Rickart Z-module.

Now, we need to recall of definitions for a module M such as: (C_1) every submodule of M is essential in a direct summand of M; (C_2) if a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M; and (C_3) if A and B are direct summands of M such that $A \cap B = 0$, then $A \oplus B$ is a direct summand of M. Modules with the C_1 property are called extending (or CS)-modules. A module M is an extending module if and only if every closed submodule of M is a direct summand [7]. A module M is called continuous if M has C_1 and C_2, and quasi-continuous if M has C_1 and C_3 [7]. An R-module M is said to be a quasi-injective module if, for any submodule L of M, any $\varphi \in Hom_R(L, M)$ can be extended to an endomorphism of M [6].

The next result give a condition under which the concepts of Rickart and c-Rickart modules are equivalent.

Proposition (2.5) Let M be an extending R-module. Then M is c-Rickart if and only if M is Rickart.

Proof. It is easy to check. ■

Corollary (2.6) Let M be an injective or (quasi-injective, continuous, quasi-continuous) R-module. Then M is c-Rickart if and only if M is Rickart.

Corollary (2.7) If M is a c-Rickart extending R-module, then so are $Ker\varphi$ and $Im\varphi$ for every $\varphi \in End(M)$.

Proof. By Proposition 2.5, M is a Rickart module. Hence, for any $\varphi \in End(M)$, $Ker\varphi \leq^{\oplus} M$ this implies $Ker\varphi \oplus K = M$ for some $K \leq M$. But $Im\varphi \cong \frac{M}{Ker\varphi} \cong K$. Thus $Ker\varphi$ and $Im\varphi$ are Rickart modules, by [1, Th. 2.7], and so they are c-Rickart. ■

Our next aim is to find some conditions under which every submodule of a c-Rickart module is also c-Rickart. First, we give the following Lemma.

Lemma (2.8) Let M be a c-Rickart R-module, and $N \leq M$. If every $\varphi \in End(N)$ can be extended to an $\overline{\varphi} \in End(M)$, then N is a c-Rickart module.

Proof. Let $\varphi \in End(N)$, so by assumption, there is an $\overline{\varphi} \in End(M)$ such that $\overline{\varphi}|_N = \varphi$. Since M is c-Rickart, then $Ker\overline{\varphi} \leq^c M$ and so $Ker\varphi \leq^c M$, but $Ker\varphi \subseteq N$ implies $Ker\varphi \leq^c N$ by [6, Prop. 6.24 (1)]. Hence N is a c-Rickart module. ■

The following Corollary is an immediate consequence of Lemma 2.8.

Corollary (2.9) Let M be a quasi-injective R-module. If $E(M)$ is a c-Rickart module, then so is M.

Corollary (2.10) Let M be a quasi-injective and c-Rickart R-module, then every submodule of M is c-Rickart.

Proof. Assume M is a quasi-injective and c-Rickart R-module. Let $N \leq M$ and $\varphi \in End(N)$, so φ extends to an $\overline{\varphi} \in End(M)$, as M is quasi-injective. Since M is c-Rickart, then by Lemma 2.8, N is a c-Rickart module. ■

Consider the following condition $(\boldsymbol{I_c})$ for an R-module M :

- For any submodule A of M for which $\frac{M}{A} \cong B \leq^{\oplus} M$, then A is a closed submodule of M … $(\boldsymbol{I_c})$

Proposition (2.11) Every c-Rickart module satisfy the condition $\boldsymbol{I_c}$.

Proof. Assume that M is a c-Rickart module. Let N be any submodule of M with $\frac{M}{N} \cong K \leq^{\oplus} M$, then there is an isomorphism $\varphi: \frac{M}{N} \to K$. Consider the sequence $M \overset{\pi}{\to} M/N \overset{\varphi}{\to} K \overset{i}{\to} M$ where π is the natural projection map, and i is the inclusion map. Thus $i \circ \varphi \circ \pi \in End(M)$, and so $Ker(i \circ \varphi \circ \pi) = \pi^{-1}\big(Ker(i \circ \varphi)\big) = \pi^{-1}(0) = N$. Since M is c-Rickart, $N = Ker(i \circ \varphi \circ \pi)$ is closed in M. Therefore M satisfies $\boldsymbol{I_c}$. ■

Corollary (2.12) Let M be a c-Rickart such that $\frac{M}{N} \cong L \leq^{\oplus} M$ for all $N \leq M$, then M is semisimple.

Proof. Since M is a c-Rickart module and $\frac{M}{N} \cong L \leq^{\oplus} M$ for all $N \leq M$, thus by previous proposition, every submodule of M is closed. Hence M is semisimple. ■

In the next Proposition, we put certain condition under which the converse of Proposition 2.11 is true.

Proposition (2.13) If M is a module satisfies $\boldsymbol{I_c}$, and $Im\varphi$ is isomorphic to a direct summand of M, for each $\varphi \in End(M)$, then M is a c-Rickart module .

Proof. Let $\varphi \in End(M)$, so by assumption, $Im\varphi \cong A \leq^{\oplus} M$. But $\frac{M}{Ker\varphi} \cong Im\varphi$, hence $\frac{M}{Ker\varphi} \cong A \leq^{\oplus} M$, then $Ker\varphi$ is closed in M, by the condition $\boldsymbol{I_c}$. Therefore M is a c-Rickart module. ■

Notice that the factor module of a c-Rickart module may not be c-Rickart, for example; the Z-module Z is c-Rickart, but we know $\frac{Z}{4Z} \cong Z_4$ is not a c-Rickart Z-module.

However, we have the following Remarks:

Remarks (2.14)

(*i*) If M is a c-Rickart module, then it is clear that $\frac{M}{N}$ is c-Rickart for each direct summand N of M. In particular, if M is a semisimple module, then $\frac{M}{N}$ is c-Rickart for every submodule N of M .

(*ii*) Let M be a module, and $N \leq M$. If N and $\frac{M}{N}$ are both c-Rickart modules, then M may not be c-Rickart, for example; let $M = Z \oplus Z_2$ and $N = (0) \oplus Z_2 \leq M$. It is clear that N and $\frac{M}{N} \cong Z$ are c-Rickart Z-modules, but M is not c-Rickart, see [Rem. and Ex. 2.1(*ii*)].

Now, we consider the following definitions:

An R-module M is called nonsingular if, for all $m \in M$ with $r_R(m) \leq_e R$ implies $m = 0$ [2]. Recall that an R-module M is said to be κ-nonsingular if, for each $\varphi \in End(M)$ and $Ker\varphi \leq_e M$ implies $\varphi = 0$ [8]. It is clear that every nonsingular module is κ-nonsingular. Following [9], an R-module M is called monoform (polyform) if, for any submodule K of M and for all $\varphi \in Hom(K, M)$, $Ker\varphi = 0$ (resp. $Ker\varphi$ is closed in K). Let M and N be R-modules. Then M is called N-c-Rickart (or relatively c-Rickart to N) if, for each $\varphi \in Hom(M, N)$, $Ker\varphi \leq^c M$ [3]. Clearly, a module M is c-Rickart if and only if M is M-c-Rickart.

Proposition (2.15) The following conditions hold for a module M :

(*i*) If M is a polyform module then M is c-Rickart.

(*ii*) For any $N \leq M$, N is M-c-Rickart module if and only if M is a polyform module.

Proof. It is easy to check. ■

It is known that every nonsingular module is polyform, by [7, 4.10(1)], also every prime (or monoform) module is polyform. So we have:

Corollary (2.16) Every nonsingular (prime, or monoform) module is c-Rickart.

The converse need not be true, in general, as example shows: it is clear that the Z-module $Z_{p\times q}$ is semisimple, where p and q are prime, so it is c-Rickart. But, it is not nonsingular (not prime, not monoform) as Z-module.

By using Corollary 2.16, we shall give another short proof of the following Proposition which appeared in [4].

Proposition (2.17) If M is a nonsingular and extending module, then M is Rickart.

Proof. By Corollary 2.16, M is c-Rickart. But M is c-Rickart and extending implies M is Rickart, by Proposition 2.5. ■

The following result is appeared in [1, Prop. 2.12].

Proposition (2.18) Every Rickart module is κ-nonsingular.

However, we presented the next strong Proposition.
Proposition (2.19) Every c-Rickart module is κ-nonsingular.
Proof. Suppose that M is a c-Rickart module. Let $\varphi \in End(M)$ such that $Ker\varphi \leq_e M$. Since M is c-Rickart, $Ker\varphi \leq^c M$ which implies that $Ker\varphi = M$, and so $\varphi = 0$. Thus M is κ-nonsingular. ■
Proposition (2.20) Let M be an R-module. If $S = End(M)$ is a regular ring, then M is c-Rickart.
Proof. Let $\varphi \in End(M)$. Since $S = End(M)$ is a regular ring, so by [10, Th.4] $Ker\varphi$ is a direct summand of M, thus $Ker\varphi \leq^c M$. Hence M is c-Rickart. ■
The converse is false, in general, for example: it is well known that Z as Z-module is c-Rickart, but $End_Z(Z) \cong Z$ is not a regular ring.

Recall that an R-module M is called quasi-Dedekind if, for each nonzero endomorphism φ of M, φ is a monomorphism [11]. A module M is said to be closed simple if, the trivial submodules are the only closed submodules of M [3]. Each of the Z-module Z, Q, Z_4 is closed simple .

In the next result, we give a condition under which the concepts Baer, Rickart, c-Rickart, quasi-Dedekind and κ-nonsingular modules are coincide. Before, we need the following Lemma which appeared in [4, Lemma 2.2.4].
Lemma (2.21) Every κ-nonsingular extending module is a Baer module.
Proposition (2.22) Consider the following conditions for an R-module M :
(*i*) M is a Baer module;
(*ii*) M is a Rickart module;
(*iii*) M is a c-Rickart module;
(*iv*) M is a quasi-Dedekind module;
(*v*) M is a κ-nonsingular module.
Then $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (v)$. If M is extending, then $(v) \Longrightarrow (i)$. (i) through (v), whenever M is a closed simple module.
Proof. $(i) \Longrightarrow (ii) \Longrightarrow (iii)$, obvious.
$(iii) \Longrightarrow (v)$ It follows by Proposition 2.19.
$(v) \Longrightarrow (i)$ Since M is extending, then the result follow by Lemma 2.21.
$(iii) \Longrightarrow (iv)$ Let φ be any nonzero endomorphism of M, $Ker\varphi \leq^c M$ (since M is c-Rickart). But M is closed simple and $\varphi \neq 0$, this implies $Ker\varphi = 0$, thus φ is a monomorphism and hence M is a quasi-Dedekind module.
$(iv) \Longrightarrow (iii)$ Obvious. ■
The condition " closed simple " in above Proposition, is necessary as the following example shows: let $M = Z_2 \oplus Z_2$ be a Z-module. It is clear that M is semisimple, then M is c-Rickart. Consider the short exact sequence $M \overset{\rho}{\to} Z_2 \oplus (0) \overset{i}{\to} M$, where ρ is a projection map, and i is the inclusion map. Then $h = i \circ \rho \in End(M)$ and $h \neq 0$, but $Kerh = (0) \oplus Z_2$, so h is not a monomorphism. Hence M is not quasi-Dedekind Z-module. Notice that M is not closed simple.

An R-module M is said to be Hopfian if, every epimorphism $\varphi \in End(M)$, is an isomorphism [12].
Corollary (2.23) Every closed simple c-Rickart module is Hopfian.
Proof. Let $\varphi: M \to M$ be a non-zero epimorphism. Since M is c-Rickart, $Ker\varphi \leq^c M$ but M is closed simple and $\varphi \neq 0$, this implies $Ker\varphi = 0$, hence φ is a monomorphism. ■
The following result is appeared in [8, Th. 2.20].
Proposition (2.24) Let R be a ring. Then the following conditions are equivalent :
(*i*) Every injective R-module is a Baer module;
(*ii*) Every R-module is a Baer module;
(*iii*) R is a semisimple artinian ring.
However, we can prove the following Proposition.

Proposition (2.25) The following conditions are equivalent for a ring R :
(*i*) Every R-module is a c-Rickart module;
(*ii*) Every extending R-module is a c-Rickart module;
(*iii*) Every injective R-module is a c-Rickart module;
(*iv*) Every injective R-module is a Baer module;
(*v*) Every R-module is a Baer module;
(*vi*) R is a semisimple artinian ring.

Proof. $(i) \Longrightarrow (ii) \Longrightarrow (iii)$, obvious. Assume (iii), let M be an injective module, then M is c-Rickart, and hence by Proposition 2.19, M is κ-nonsingular. But M is extending, hence by Lemma 2.21, M is a Baer module, so (iv) holds.
$(iv) \Longrightarrow (v) \Longrightarrow (vi)$ It follows by Proposition 2.24.
$(vi) \Longrightarrow (i)$ Since R is semisimple, every R-module is semisimple, and so every R-module is a c-Rickart module. ■

Recall that an R-module M is said to be finitely related if there exists an exact sequence $0 \to P \to F \to M \to 0$ where F is free (of arbitrary rank) and P is finitely generated [6].
Proposition (2.26) Consider the following conditions for a ring R :
(i) Every flat R-module is c-Rickart;
(ii) Every projective R-module is c-Rickart;
(iii) Every free R-module is c-Rickart.
Then $(i) \Longrightarrow (ii) \Longrightarrow (iii)$. However, (i) through (iii) for finitely related R-modules.
Proof. $(i) \Longrightarrow (ii) \Longrightarrow (iii)$, obvious. Assume (iii), let M be a projective module, so by [13, Th. 5.4.1] $M \cong A \leq^{\oplus}$ (F free R-module). By (iii), F is c-Rickart, then by Proposition 2.3, A is a c-Rickart module, and hence M is c-Rickart, so (ii) holds.
$(ii) \Longrightarrow (i)$ By [6, Th. 4.30], every finitely related flat module is projective, so the result holds. ■
Proposition (2.27) Let R be a ring, and $S = End(R)$ be a commutative ring. If R is a c-Rickart R-module, then S is nonsingular.
Proof. Assume R is a c-Rickart R-module. Let $\varphi \in S = End(R)$ such that $r_R(\varphi) = Ker\varphi \leq_e R$. Since R is c-Rickart, $Ker\varphi \leq^c R$ this implies $Ker\varphi = R$ and so $\varphi = 0$. Therefore $S = End(R)$ is a nonsingular R-module. ■
Corollary (2.28) Let R be a commutative ring. Then R is c-Rickart if and only if R is nonsingular.
Proof. Since R is a c-Rickart R-module, so by previous Proposition, $S = End(R)$ is nonsingular, but $End(R) \cong R$, hence R is a nonsingular ring. The converse, follows from Corollary 2.16. ■

Recall that an R-module M is called scalar if, for all $\varphi \in End(M)$, there exists an $x \in R$ such that $\varphi(m) = mx$ for all $m \in M$ [14].
Proposition (2.29) Let M be a scalar R-module. If M is a c-Rickart R-module, then for any $(0 \neq)\varphi \in End(M)$, there exists $(0 \neq)a \in R$ such that $r_M(a) \leq^c M$.
Proof. For any $\varphi \in End(M)$ and $\varphi \neq 0$, then there is $(0 \neq)a \in R$ such that $\varphi(x) = ax$ for all $x \in M$, as M is scalar. Since M is c-Rickart, thus $Ker\varphi = r_M(a) \leq^c M$. ■
Corollary (2.30) Let R be a commutative ring with identity. If R is a c-Rickart R-module, then for any $(0 \neq)\varphi \in End(R)$, there exists $(0 \neq)a \in R$ such that $r_R(a) \leq^c R$
Proof. It is clear, since every commutative ring R is a scalar R-module . ■
Corollary (2.31) Let M be a scalar faithful R-module, and $S = End(M)$. Then R is c-Rickart if only if S is c-Rickart.
Proof. Since M is a scalar R-module, so by [15, Lemma 6.2] $S = End(M) \cong R/r_R(M)$. But M is faithful, so $S = End(M) \cong R$. Thus the result is obtained. ■
Proposition (2.32) Let M be an R-module, and $\overline{R} = R/r_R(M)$. Then M is c-Rickart as R-module if only if M is c-Rickart as $\overline{R}$-module.
Proof. It is easy to check. ■

3 Direct Sums Of c-Rickart Modules

In this section, we will discuss the direct sum of c-Rickart modules. Before that, we give the following example which illustrates that the direct sum of c-Rickart modules is not c-Rickart.
Example (3.1) It is easy to see that the Z-modules Z, Z_p are both c-Rickart, where p is prime. But $M = Z \oplus Z_p$ is not c-Rickart. Consider the endomorphism $\varphi \in End(M)$ such that $\varphi(x, \overline{y}) = (0, \overline{x})$ for all $x \in Z$ and $\overline{y} \in Z_p$, then $Ker\varphi = pZ \oplus Z_p$ is an essential in $Z \oplus Z_p$. Thus $Ker\varphi$ is not closed in M, and hence $M = Z \oplus Z_p$ is not a c-Rickart Z-module.
Our next results generalize Example 3.1 to arbitrary modules.
Proposition (3.2) Let M be a closed simple and c-Rickart module which has a nonzero maximal submodule N, then $M \oplus (M/N)$ is not a c-Rickart module.
Proof. Assume that $M \oplus (M/N)$ is a c-Rickart module. Consider an $\varphi \in End(M \oplus (M/N))$ defined by $\varphi(m, \overline{n}) = (0, \overline{m})$ for all $m \in M$, $\overline{n} \in M/N$. So $= \{(m, \overline{n}) \in M \oplus (M/N) \,|\, \varphi(m, \overline{n}) = (0, \overline{0})\} =$

$\{(m,\overline{n})|m+N=N\}=\{(m,\overline{n})|m\in N\}=N\oplus(M/N)$. Since $M\oplus(M/N)$ is a c-Rickart module, $Ker\varphi=N\oplus(M/N)\leq^{c}M\oplus(M/N)$, but $N\leq^{c}Ker\varphi$, so by [6, Prop. 6.24 (2)] $N\leq^{c}M\oplus(M/N)$ this implies $N\leq^{c}M$ by [6, Prop. 6.24 (1)], which contradicts that M is closed simple. Therefore $M\oplus(M/N)$ is not c-Rickart. ■

Recall that a submodule *N* of a module *M* is called a fully invariant submodule if, $\varphi(N)\subseteq N$ for each $\varphi\in End(M)$ [16].

Now, we give a condition under which the *c*-Rickart property is closed under direct sums.

Proposition (3.3) Let $M=\oplus_{i\in I}M_i$ be an *R*-module such that M_i is fully invariant for each $i\in I$. Then *M* is *c*-Rickart if and only if M_i is *c*-Rickart, for each $i\in I$.

Proof. Assume that M_i is a *c*-Rickart module, for each $i\in I$. Let $M=\oplus_{i\in I}M_i$ and $\varphi=(\varphi_{ij})\in End(M)$ where $\varphi_{ij}\in Hom(M_i,M_j)$. Since M_i is fully invariant, then by [17, Lemma 1.9] $Hom(M_i,M_j)=0$ for $i\neq j$, also $\varphi(M_i)\subseteq M_i$ for each $i\in I$. So $Ker\varphi=\oplus_{i\in I}Ker\varphi_{ii}$. However, $Ker\varphi_{ii}\leq^{c}M_i$ for each $i\in I$, as M_i is a *c*-Rickart module, then $\oplus_{i\in I}Ker\varphi_{ii}\leq^{c}\oplus_{i\in I}M_i$ [2], this means that $Ker\varphi\leq^{c}M$ and hence M is a *c*-Rickart module. The converse, follows directly from Proposition 2.3. ■

Proposition (3.4) Let *M* be an *R*-module. Then the following conditions are equivalent.

(*i*) *M* is a *c*-Rickart module;

(*ii*) For $N\leq M$, every direct summand *K* of *M* is *N*-*c*-Rickart;

(*iii*) For any direct summand *K* of *M*, and for any $L\leq^{c}M$ such that $\varphi\in Hom(M,L)$, $Ker(\varphi|_K)$ is a closed submodule of *K*.

Proof. $(i)\Longrightarrow(ii)$ Assume $N\leq M$. Let *K* be a direct summand of *M*, $K=eM$ for some $e^2=e\in End(M)$. Let $\varphi\in Hom(K,N)$, so $\alpha=\varphi e\in End(M)$. On the other hand, $Ker\alpha=Ker\varphi\oplus(1-e)M$, then $Ker\varphi\leq^{\oplus}Ker\alpha$, and hence $Ker\varphi\leq^{c}Ker\alpha$. Since M is *c*-Rickart and $\alpha\in End(M)$, $Ker\alpha\leq^{c}M$ so by [6, Prop. 6.24 (2)] $Ker\varphi\leq^{c}M$, but $Ker\varphi\subseteq K$, hence by [6, Prop. 6.24 (1)] $Ker\varphi\leq^{c}K$. Therefore *K* is *N*-*c*-Rickart.

$(ii)\Longrightarrow(iii)$ Let *K* be a direct summand of *M* and so $L\leq^{c}M$. If $\varphi\in Hom(M,L)$, $\varphi|_K\in Hom(K,L)$. By (ii), *K* is *L*-*c*-Rickart, and hence $Ker(\varphi|_K)$ is a closed submodule of *K*.

$(iii)\Longrightarrow(i)$ By taking $L=K=M$, we get M is *M*-*c*-Rickart this mean *M* is a *c*-Rickart module. ■

Proposition (3.5) The following conditions are equivalent for a ring *R* and a fixed $n\in\mathbb{N}$.

(i) Every *n*-generated projective *R*-module is a *c*-Rickart module;

(ii) The free *R*-module $R^{(n)}$ is a *c*-Rickart module.

Proof. $(i)\Longrightarrow(ii)$ Since every free *R*-module is projective, so it is clear that *R*-module $R^{(n)}$ is *n*-generated projective, so by (i), $R^{(n)}$ is a *c*-Rickart *R*-module.

$(ii)\Longrightarrow(i)$ Assume *M* is an *n*-generated projective *R*-module. So there exists a free *R*-module *F*, and an epimorphism $\varphi:F\to M$. Since *F* is free and *M* is *n*-generated projective *R*-module, then $\alpha:R^{(n)}\to M$ splits, where $F\cong R^{(n)}$. On the other hand, *M* is isomorphic to a direct summand of $R^{(n)}$, but $R^{(n)}$ is *c*-Rickart, hence *M* is a *c*-Rickart *R*-module. ■

Proposition (3.6) The following conditions are equivalent for a ring *R*.

(i) Every finitely generated free (projective) *R*-module is *c*-Rickart;

(ii) For each finite index set *I*, $\oplus_{i\in I}R^{(n)}$ is a *c*-Rickart *R*-module for some $n\in\mathbb{N}$.

Proof. $(i)\Longrightarrow(ii)$ Since $R^{(n)}$ is a free *R*-module for some $n\in\mathbb{N}$, then $\oplus_{i\in I}R^{(n)}$ is a finitely generated free *R*-module, for each finite index set *I* , so by (i), $\oplus_{i\in I}R^{(n)}$ is a *c*-Rickart *R*-module.

$(ii)\Longrightarrow(i)$ Let $R^{(t)}$ be a finitely generated free *R*-module, for some $t\in\mathbb{N}$. Then $R^{(t)}\leq^{\oplus}R^{(nt)}=\oplus_{i=1}^{t}R^{(n)}$ which is *c*-Rickart by (ii). Hence $R^{(t)}$ is a *c*-Rickart *R*-module, by Proposition 2.3. ■

Corollary (3.7) The following conditions are equivalent for a ring *R*.

(i) Every free (projective) *R*-module is *c*-Rickart;

(ii) For any index set *I*, $\oplus_{i\in I}R^{(n)}$ is a *c*-Rickart *R*-module for some $n\in\mathbb{N}$.

Proof. $(i)\Longrightarrow(ii)$ Since $R^{(n)}$ is a free *R*-module for some $n\in\mathbb{N}$, so $\oplus_{i\in I}R^{(n)}$ is a free *R*-module, for any arbitrary index set *I* , so by (i), $\oplus_{i\in I}R^{(n)}$ is a *c*-Rickart *R*-module.

$(ii)\Longrightarrow(i)$ Let $R^{(I)}$ be a free *R*-module, where I is an arbitrary index set, so we have two cases: if $|I|$ (i.e. length of *I*) is finite, then by a similar to the proof $(ii)\Longrightarrow(i)$ of Proposition 3.6, $R^{(I)}$ is a *c*-

Rickart R-module. Now, if $|I|$ is infinite, then $R^{(I)}$ is a direct sum of copies of $R^{(n)}$. Hence $R^{(I)}$ is a c-Rickart R-module. ■

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