ISSN: 0067-2904

Characterization of strict local minimizers of order two for semi -infinite problems in the nonparametric constraint case

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Abstract
We extended the characterization of strict local minimizers of order two in ward's theorem for nonlinear problem to a certain class of nonsmooth semi-infinite problems with inequality constraints in the nonparametric constraint case.

Keywords: Strict local minimizer, Semi-infinite programming, Maximum function, $C^{1,1}$functions

تمييز النقطة المحلية الصغرى الصارمة ذات الرتبة الثانية لمسائل شبه اللانهائية في حالة قيد غير بارامتري

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الخلاصة
تم توسيع نظرية وورد الخاصة بتمييز النقطة المحلية الصغرى الصارمة ذات الرتبة الثاني لمسائل البرمجة اللاخطية الى فئة معينة من مسائل شبه اللانهائية غير ملساء ذات قيود على شكل متراجحة في حالة قيد غير بارامتري .

1. Introduction

The notion of a strict local minimize of order m was introduced by Cromme, under name "strongly unique minimize" , in study of iterative numerical method (see[1]) .

Strict local minimize play an important role in stability studies (see e.g [2-5]). Some results concerning characterization of such minimizers have been obtained for (i) standard nonlinear programming problems with both inequality and equality constraints, for $m = 1$ or $m = 2$ in [6-8] , (ii) non smooth static minmax problems with inequality constraints in the nonparametric and parametric constraint case . for $m = 1$ in [9-11]. These results were derived under the presence of constraint qualification leading to statements in which there is no gap between the necessary and sufficient conditions.

The aim of this paper is to extend the characterization obtain for $m = 2$ in wards theorem [6, Theorem 3.3] to a certain class of non-smooth semi-infinite problems with inequality constraints in the nonparametric constraint case. We consider problems in which the objective function $f: \mathbb{R}^n \to \mathbb{R}$ is continually differentiable on $\mathbb{R}^n$, while the inequality constraints function $g: \mathbb{R}^n \to \mathbb{R}$ are given by $g_i(x) = \max_{y_i \in Y_i} \emptyset_i(x, y_i)$, where $\emptyset_i : \mathbb{R}^n \times \mathbb{R}^{m_i} \to \mathbb{R}$ are continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^{m_i}$, and $Y_i \subset \mathbb{R}^{m_i}$, $i = 1, \dots, p$.

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2. Notation and preliminary:

We will need some notations and definitions which can be found in [12, chapter 2]. For a locally Lipschitzian function $f:\mathbb{R}^n\to\mathbb{R}$, we denote by $\partial f(x)$ the generalized gradient of f at x. We say that f is regular (or subdifferentially regular) at x if the usual one-sided directional derivative $f'(x;d)$ exists for all d and is equal to the generalized directional derivative $f^{\circ}(x;d)$.

Consider the following nonlinear programming problem:

$$(p)\qquad \min\{f(x)|x\in S\},$$

$$S:=\{x\in\mathbb{R}^n|g_i(x)\le 0, i\in I\},$$

Where $I=\{1,\dots,p\}, f:\mathbb{R}^n\to\mathbb{R}, g_i:\mathbb{R}^n\to\mathbb{R}, i\in I$.

Special problem of the form (p) is the following problem:

$$(p1)\qquad \min\{f(x)|x\in S\},$$

$$S:=\{x\in\mathbb{R}^n|g_i(x)\le 0, i\in I\},$$

Where $I=\{1,\dots,p\}, f:\mathbb{R}^n\to\mathbb{R}, f(x):=\max_{y_0\in Y_0}\emptyset_0(x,y_0), g_i:\mathbb{R}^n\to\mathbb{R},\ g_i(x):=\max_{y_i\in Y_i}\emptyset_i(x,y_i)(i\in I);\ \emptyset_i:\mathbb{R}^n\times\mathbb{R}^{m_i}\to\mathbb{R},\ Y_i\subset\mathbb{R}^{m_i}(i\in\{0\}\cup I)$.

For $x_0\in\mathbb{R}^n$ and $\delta>0$, we denote $B(x_0,\delta):=\{x\in\mathbb{R}^n\,|\|x-x_0\|\le\delta\}$. We say that $x_0\in S$ is a local minimizer for problem (p) if there exists $\varepsilon>0$ such that

$$f(x)\ge f(x_0) \text{ for all } x\in S\cap B(x_0,\varepsilon).$$

Let $m\ge 1$ be an integer. We say that $x_0\in S$ is a strict local minimizer of order m for problem (p) if there exist $\varepsilon>0,\beta>0$ such that

$$f(x)\ge f(x_0)+\beta\|x-x_0\|^m \quad \text{for all } x\in S\cap B(X_0,\varepsilon).$$

Throughout the paper, we will use the following notations for a given $x\in\mathbb{R}^n$:

$$I(x):=\{i\in I|g_i(x)\}=0 \qquad (2.1)$$

We will need some additional assumptions concerning problem $(p1)$.

(A1) Assume that, for all $i\in\{0\}\cup I$,

(1) Y_i is a compact set,

(2) $\emptyset_i(x,y_i)$ is upper semi continuous in (x,y_i),

(3) $\emptyset_i(x,y_i)$ is locally Lipschitz with respect to x, uniformly for y_i in Y_i,

(4) $\emptyset_i'(x,y_i;.)=\emptyset_i^{\circ}(x,y_i;.)$, the derivatives being with respect to x,

(5) $\partial_x\emptyset_i(x,y_i)$ is upper semicontinuous in (x,y_i).

Under assumption (A1), in view of [12, Theorem 2.1],[13] the maximal-value functions $f, g_i, i\in I$, are locally Lipschitz, while their directional derivatives and generalized gradients are given as follows:

$$f'(x;d)=f^{\circ}(x;d)=\max\{\zeta_0.d\,|\zeta_0\in\partial_x\emptyset_0(x,y_0), y_0\in Y_0(x)\},$$

$$\partial f(x)=co\bigcup_{y_0\in Y_0(x)}\partial_x\emptyset_0(x,y_0),$$

$$g'_i(x;d)=g^{\circ}_i(x;d)=\max\ \{\zeta_i.d\,|\zeta_i\in\partial_x\emptyset_i(x,y_i), y_i\in Y_i(x)\}, i\in I,$$

$$\partial g_i(x)=co\bigcup_{y_i\in Y_i(x)}\partial_x\emptyset_i(x,y_i), i\in I$$

Where

$$Y_i(x):=\{y_i\mid\emptyset_i(x,y_i)=\max_{u_i\in Y_i}\emptyset_i(x,u_i)\}, i\in\{0\}\cup I \qquad (2.2)$$

Theorem 1.[14, theorem 26]. Let x_0 be a local minimizer for $(p1)$, and suppose that assumption (A1) holds. Then there exists vectors $y_{ij}\in Y_i(x_0)$ together with scalars $\lambda_{ij}\ge 0, j=1,\dots,\beta_i,\beta_i\in\mathbb{N}, i\in\{0\}\cup I$, such that

$$0\in\sum_{i\in\{0\}\cup I}\sum_{j=1}^{\beta_i}\lambda_{ij}\partial_x\emptyset_i(x_0,y_{ij}),$$

$$\lambda_{ij}\emptyset_i(x_0,y_{ij})=0, j=1,\dots,\beta_i, i\in I$$

Furthermore, if α is the number of nonzero $\lambda_{ij}, j=1,\dots,\beta_i, i\in\{0\}\cup I$ then $1\le\alpha\le n+1$

Proposition 2. [6, Corollary 2.3 (d)]. Let x_0 be a strict local minimizer of order $m \geq 1$ for (p). Then

$$d^m f^{IK}(x_0; d) > 0, \ \ \forall d \in k(S, x_0) \setminus \{0\},$$

Where $d^m f^{IK}(x_0; d) := inf_{\varepsilon>0} \lim_{t\to 0+} inf \ sup_{d'\in B(d,\varepsilon)} \big(f(x_0 + td') - f(x_0)\big)/t^m$,
and $k(S, x_0)$ denotes the Ursescu tangent cone, defined by

$$k(S, x_0) := \{d | \forall t_n \to 0+, \exists d_n \to d \ \text{ with } \ x_0 + t_n d_n \in S, \forall n\}.$$

Definition 3. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be $C^{1,1}$ at x if f is continuously Fr'echet differentiable at x and $\nabla f(.)$ is locally Lipschitz near x .

Lemma 4. [6, Lemma 3.1 (i)]. If f is $C^{1,1}$ at x $\nabla f(x) = 0$, then

$$d^2 f^k(x; .) = d^2 f^{IK}(x, .),$$

Where $d^2 f^K(x; d) := \lim_{(t,d')\to(0+,d)} (f(x + td') - f(x))/t^2.$

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Now, we set the following notation. For $\lambda_i \geq 0, i = 1, \dots, p$, we define the Lagrangian function

$$L(x) := f(x) + \sum_{i=1}^{p} \lambda_i g_i(x)$$

And partition $I(x)$, where $I(x)$ is defined by (2.2), into the sets

$$J(x) := \{i \in I(x) | \lambda_i > 0\},$$
$$Q(x) := \{i \in I(x) \ | \lambda_i = 0\},$$

We also define the set of directions

$$D(x) := \{d \in \mathbb{R}^n \mid \nabla g_i(x). d \leq 0, \forall i \in Q(x), \nabla g_i(x) . \ d = 0, \forall i \in J(x)\}.$$

Definition 5. Let g_i , $i = 1, \dots, p$, be C^1 at x_0 . We say that the Strict Mangasarian-Fromowitz Constraint Qualification (SMFCQ) holds at x_0 if

(i) $\nabla g_i(x_0), i \in J(x_0)$, are linearly independent;

(ii) there exist $d' \in \mathbb{R}^n$ such that

$$\nabla g_i(x_0). \ d' < 0, \forall i \in Q(x_0),$$
$$\nabla g_i(x_0). \ d' = 0, \ \ \forall i \in J(x_0).$$

Theorem 6. [6, theorem 3.3]. Let $f, g_i, i = 1, \dots, p,$ be $C^{1,1}$ at x_0 . Suppose that SMFCQ is satisfied at x_0 for some $\lambda_i \geq 0$ such that $\nabla L(x_0) = 0$, $\lambda_i g_i(x_0) = 0$, $\forall i = 1, \dots, p$. Then x_0 is a strict local minimizer of order two for (p) if and only if

$$d^2 L^K(x_0; d) > 0, \forall d \in D(x_0) \setminus \{0\}.$$

3. Necessary optimality conditions:

In this section , we present necessary optimality conditions which are satisfied by all local minimizers (not necessary strict) for the given problem . These conditions include a restriction on the nonzero multipliers .

In this paper, we consider the following semi-infinite problem:

$$\text{(P2)} \qquad \min \ \{f(x) | \ x \in S\} ,$$
$$S := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in I\} ,$$

Where $I = \{1, \dots, p\}, f : \mathbb{R}^n \to \mathbb{R}$ is C^1 on $\mathbb{R}^n$, $g_i : \mathbb{R}^n \to \mathbb{R}, g_i(x) := \max_{y_i \in Y_i} \emptyset_i(x, y_i)$; $\emptyset_i : \mathbb{R}^n \times \mathbb{R}^{m_i} \to \mathbb{R}$ are C^1 on $\mathbb{R}^n \times \mathbb{R}^{m_i}$, $Y_i \subset \mathbb{R}^{m_i}$ is compact set $(i \in I)$.

Looking at problem (P2) from a different perspective, we can write the inequality constraints $g_i(x) \leq 0, i \in I$, alternatively as follows:

$$g_i(x) := \max_{y_i \in Y_i} \emptyset_i(x, y_i) \leq 0 \Leftrightarrow \emptyset_i (x, y_i) \leq 0, \forall y_i \in Y_i, \qquad (3.1)$$

Which implies that problem (P2) can be rewritten as follows:

$$\text{(P3)} \qquad \min \ \{f(x) | \ x \in S\}$$
$$S := \{x \in \mathbb{R}^n \mid \emptyset_i(x, y_i) \leq 0 , \ \forall y_i \in Y_i , i \in I\} .$$

Analogously to (2.1), we define the active index set for $x \in \mathbb{R}^n$ as follows:

$$I(x) := \{i \in I \mid \emptyset_i(x, y_i) = 0 , \forall y_i \ \in Y_i(x) \} .$$

Where $Y_i(x)$ is defined by (2.2).

For a given $x_0 \in S$, let us assume that:

(A2) For any number $\lambda_{ij} \geq 0$ and vectors $z_{ij}^* = \nabla_x \emptyset_i(x_0, y_{ij}), y_{ij} \in Y_i(x_0)$, $(j = 1, \ldots, \beta_1, i \in I(x_0))$, (where $\beta_i \geq 1$), the following implication holds:

$$\sum_{i \in I(x_0)} \sum_{j=1}^{\beta_i} \lambda_{ij} z_{ij}^* = 0 \Longrightarrow \lambda_{ij} = 0, \qquad (j = 1, \ldots, \beta_i, i \in I(x_0))$$

The next result is an easy consequence of Theorem 1 .

Corollary 7. Let x_0 be a local minimize for (P2), and suppose that assumption (A1) holds. Then there exists vectors $y_{ij} \in Y_i(x_0)$ together with scalars $\lambda_{ij} \geq 0, j = 1, \ldots, \beta_i, \beta_i \in \mathbb{N}, i \in I$, such that

$$\nabla f(x_0) + \sum_{i \in I} \sum_{j=1}^{\beta_i} \lambda_{ij} \nabla_x \emptyset_i(x_0, y_{ij}) = 0 \qquad (3.2)$$

$$\lambda_{ij} \emptyset_i(x_0, y_{ij}) = 0, j = 1, \ldots, \beta_i, i \in I \qquad (3.3)$$

Furthermore, if α is the number of nonzero $\lambda_{ij}, j = 1, \ldots, \beta_i, i \in I$, then

$$\alpha \leq n \qquad (3.4)$$

Proof. Suppose that x_0 is a local minimize for problem (P3). Since $\emptyset_i$ are C^1 on $\mathbb{R}^n \times \mathbb{R}^{m_i}$ and Y_i is compact set $(i \in I)$, then assumption (A1) holds similarly as in the case of [11, Theorem 6.7.2]. Then, by Theorem 1 implies that there exists a scalar $\lambda_0 \geq 0$, vectors $y_{ij} \in Y_i(x_0)$ together with scalars $\lambda_{ij} \geq 0, j = 1, \ldots, \beta_i, \beta_i \in \mathbb{N}, i \in I$, such that

$$\lambda_0 \nabla f(x_0) + \sum_{i \in I} \sum_{j=1}^{\beta_i} \lambda_{ij} \nabla_x \emptyset_i(x_0, y_{ij}) = 0, \qquad (3.5)$$

$$\lambda_{ij} \emptyset_i(x_0, y_{ij}) = 0, \qquad j = 1, \ldots, \beta_i, i \in I, \qquad (3.6)$$

Furthermore, if α_1 is the number of nonzero λ_0 and $\lambda_{ij}, j = 1, \ldots, \beta_1, i \in I$, then

$$1 \leq \alpha_1 \leq n + 1 \qquad (3.7)$$

Now , if $\lambda_0 = 0$, then condition (3.5) takes on the form

$$\textstyle\sum_{i \in I} \sum_{j=1}^{\beta_i} \lambda_{ij} \nabla_x \emptyset_i(x_0, y_{ij}) = 0 .$$

Then it follows from condition (3.6) and assumption (A2) that all λ_{ij} are zero a contradiction with the left-hand inequality in (3.7) . Hence , $\lambda_0 > 0$, and we may assume $\lambda_0 = 1$ and $\alpha_1 = \alpha + 1$, where α is the number of nonzero λ_{ij}, $j = 1, \ldots, \beta_i$, $i \in I$, so that conditions (3.2)-(3.4) hold .

In a similar way as in section 2 , one can easily set , for problem (P3) the following notation: For given vectors $y_{ij} \in Y_i(x_0)$ together with scalars $\lambda_{ij} \geq 0$, $j = 1, \ldots, \beta_i, \beta_i \in \mathbb{N}, i \in I(x_0)$, we define the Lagrangian function

$$L(x) := f(x) + \sum_{i \in I(x_0)} \sum_{j=1}^{\beta_i} \lambda_{ij} \emptyset_i(x, y_{ij}),$$

and partition $I_i(x_0) := \{1, \ldots, \beta_i\}, i \in I(x_0)$, into the sets

$$J_i(x_0) := \{j \in I_i(x_0) | \lambda_{ij} > 0\}, i \in I(x_0),$$

And

$$Q_i(x_0) := \{j \in I_i(x_0) | \lambda_{ij} = 0\}, i \in I(x_0).$$

We also define the set of directions

$$D(x_0) := \left\{ \begin{matrix} d \in \mathbb{R}^n \mid \nabla_x \emptyset_i(x_0, y_{ij}).\ d \leq 0, j \in Q_i(x_0), \nabla_x \emptyset_i(x_0, y_{ij}). d = 0, j \in J_i(x_0), \\ i \in I(x_0) \end{matrix} \right\}.$$

Finally, Definition 5 , is stated as follows:

Definition 8. Let $\emptyset_i(\,.\,, y_{ij}), y_{ij} \in Y_i(x_0), j = 1, \ldots, \beta_i, \beta_i \in \mathbb{N}, i \in I(x_0)$ be C^1 at x_0. We say that the Strict Mangasarian-Fromowitz Constraint Qualification (SMFCQ) holds at x_0 if

(i) $\nabla_x \emptyset_i(x_0, y_{ij}), j \in J_i(x_0), i \in I(x_0)$, are linearly independent ;
(ii) there exist $d' \in \mathbb{R}^n$ such that

$$\nabla_x \emptyset_i(x_o, y_{ij}).\, d' < 0\,, \forall j \in Q_i(x_0), i \in I(x_0),$$
$$\nabla_x \emptyset_i(x_0, y_{ij}).\, d' = 0\,, \forall j \in J_i(x_0), i \in I(x_0).$$

4. Characterization of strict local minimizers of order two

In this section, we present a characterization of strict local minimizers of order two for problem (P3) with $C^{1,1}$ data. This characterization is given in terms of the Lagrangian function .

To prove Theorem 11, we need the following lemma which is a parametric version of [6, Lemma 3.2].

Lemma 9. Let x_0 be a strict local minimizer of order two for (P3). Let $y_{ij} \in Y_i(x_0), \boldsymbol{\lambda_{ij}} \geq 0, \mathrm{j} = 1, \ldots, \beta_\mathrm{i}, \beta_\mathrm{i} \in \mathbb{N}, \mathrm{i} \in I(x_0)$. Suppose that the following conditions hold:

(a) $\lambda_{ij}\emptyset_i(x_0, y_{ij}) = 0$, for all $j = 1, \ldots, \beta_i, i \in I(x_0)$;

(b) there exists $\delta > 0$ such that $Y_i(x) \subset \{y_{ij} \mid j = 1, \ldots, \beta_i\}$, for all $i \in I(x_0)$ and $x \in B(x_0, \delta)\backslash S$.

Then x_0 is a strict local minimizer of order two for L on the set

$$C := \{x \mid \emptyset_i(x, y_{ij}) \leq 0, j = 1, \ldots, \beta_i, \beta_i \in \mathbb{N}, \emptyset_i(x, y_{ij}) = 0, j \in J_i(x_0), i \in I(x_0)\}.$$

Proof. Since x_0 is a strict local minimizer of order two for (P3), then there exist $\varepsilon > 0, \beta > 0$, such that

$$f(x) \geq f(x_0) + \beta\|x - x_0\|^2, \forall\, x \in S \cap B(x_0, \varepsilon) \tag{3.8}$$

By [11,Theorem 6.7.2], the functions $g_i, i \in I$, defined in (3.1), are continuous. Therefore, we may assume that ε Is so small that

$$g_i(x) < 0, \text{ for all } i \in I\backslash I(x_0) \text{ and } x \in B(x_0, \varepsilon). \tag{3.9}$$

Let δ be chosen according to assumption (b), and let $\eta := \min\ \{\varepsilon, \delta\}$. Take any $x \in C \cap B(x_0, \eta)$. We will show that $x \in S$.

Indeed, if $x \notin S$, then there exist $i \in I$ and $y_i \in Y_i(x)$ such that $\emptyset_i(x, y_i) > 0$. Since, in view of (3.9), we can omit the constraint functions which are not active at x_0, it follows that $i \in I(x_0)$. By assumption (b), there exists $j \in \{1, \ldots, \beta_i\}$ such that $y_i = y_{ij}$. Hence, $\emptyset_i(x, y_{ij}) > 0$, which means that $x \notin C$.

We have thus verified that $x \in S$. Then, we can use assumption (a) and condition (3.8) to obtain.

$$L(x) - L(x_0) = f(x) - f(x_0) + \textstyle\sum_{i\in I(x_0)}\sum_{j\in J_i(x_0)} \lambda_{ij}\left(\emptyset_i(x, y_{ij}) - \emptyset_i(x_0, y_{ij})\right)$$
$$= f(x) - f(x_0) \geq \beta\|x - x_0\|^2$$

Hence, x_0 is a strict local minimizer of order two for L on C . ■

Remark 10. Assumption (b) of Lemma 9 is satisfied, in particular, if all the sets $Y_i, i \in I$, are finite, and $Y_i(x_0) = \{y_{ij} \mid j = 1, \ldots, \beta_i\}$. This assumption is also valid for some examples where the sets Y_i are infinite; see, for instance, [11, Example 10.3.1].

The following theorem is a generalization of Theorem 6 .

Theorem 11. Let $f, \emptyset_i(.\,, y_{ij}), y_{ij} \in Y_i(x_0), j = 1, \ldots, \beta_i, \beta_i \in \mathbb{N}, i \in I(x_0)$, be $C^{1,1}$ at x_0 . Suppose that SMFCQ is satisfied at x_0 for some $\lambda_{ij} \geq 0, j = 1, \ldots, \beta_i, \beta_i \in \mathbb{N}, i \in I(x_0)$, such that $\nabla L(x_0) = 0$ and $\lambda_{ij}\emptyset_i(x_0, y_{ij}) = 0, j = 1, \ldots, \beta_i, i \in I(x_0)$. Then, x_0 is a strict local minimizer of order two for (P3) if and only if

$$d^2L^k(x_0; d) > 0, \forall\, d \in D(x_0)\backslash\{0\}. \tag{3.10}$$

Proof, (i) *Necessity*. Suppose that x_0 is a strict local minimizer of order two for (P3) . Then by Lemma 9, L has a strict local minimizer of order two on C , so by Proposition 2, we have

$$d^2L^{IK}(x_0; d) > 0, \forall\, d \in k(C, x_0)\{0\}. \tag{3.11}$$

It is known (see a remark on p. 565 of [6]) that, under assumption SMFCQ, we have

$k(C, x_0) = D(x_0).$ (3.12)
Combining (3.11) and (3.12) yields
$d^2 L^{IK}(x_0; d) > 0, \forall\, d \in D(x_0)\backslash\{0\}.$ (3.13)
By Lemma 4, we get
$d^2 L^{IK}(x_0; d) = d^2 L^{K}(x_0; d).$ (3.14)
Condition (3.10) then follows from (3.13) and (3.14) .

(ii) *Sufficiency*. Suppose that condition (3.10) holds and suppose that x_0 is not a strict local minimizer of order two for (P3). Then, there exists a sequence $\{x_n\}$ of feasible points for (P3) such that $\{x_n\} \to x_0, x_n \neq x_0$ and

$$f(x_n) - f(x_0) < \|x_n - x_0\|^2/n .$$

Define

$$t_n := \|x_n - x_0\|, d_n := (x_n - x_0)/t_n.$$

Then $t_n \to 0^+$, and we may assume, taking a subsequence if necessary, that $\{d_n\} \to d$, for some $d \neq 0$. We will prove that $d \in D(x_0)$.

By the definition of S (in problem (P3)) and $Y_i(x_0), i \in I(x_0)$, we get

$$\emptyset_i(x_n, y_{ij}) - \emptyset_i(x_0, y_{ij}) \leq 0, j = 1, \ldots, \beta_i, \beta_i \in \mathbb{N}, i \in I(x_0),$$

so that

$$\nabla_x \emptyset_i(x_0, y_{ij}).\, d = \lim_{n\to\infty} \left(\emptyset_i(x_n, y_{ij}) - \emptyset_i(x_0 - y_{ij})\right) / \, t_n \leq 0,$$
$$j = 1, \ldots, \beta_i, \beta_i \in \mathbb{N}, i \in I(x_0), \quad (3.15)$$

and
$\nabla f(x_0).\, d = \lim_{n\to\infty}(f(x_n) - f(x_0)) \,/ t_{\mathrm{n}} \leq \lim_{n\to\infty} t_n \,/ n \; = 0 .$ (3.16)
Since

$$\nabla L(x_0) = \nabla f(x_0) + \sum_{i\in I(x_0)} \sum_{j=1}^{\beta_i} \lambda_{ij} \nabla_{\mathrm{x}} \emptyset_{\mathrm{i}}(\mathrm{x}_0, \mathrm{y}_{\mathrm{ij}}) = 0,$$

by taking the inner product of both sides with d , we obtain

$$\nabla f(x_0).\, d + \sum_{i\in I(x_0)} \sum_{j=1}^{\beta_i} \lambda_{ij} \nabla_x \emptyset_i(x_0, y_{ij}).\, d = 0.$$

Using (3.15) and (3.16), we conclude that,

$$\nabla f(x_0).\, d = 0, \nabla_x \emptyset_i(x_0, y_{ij}).\, d = 0, \text{ for } j \in J_i(x_0), i \in I(x_0),$$
$$\nabla_x \emptyset_i(x_0, y_{ij}).\, d \leq 0, \text{ for } j \in Q_i(x_0), i \in I(x_0).$$

Hence, $d \in D(x_0)$. Then for all n ,

$$(L(x_n) - L(x_0))/t_n^2 \leq (f(x_n) - f(x_0))/t_n^2 \leq 1/n .$$

Therefore

$$d^2 f^K(x_0; d) \leq \lim_{n\to\infty} \inf\left(L(x_n) - L(x_0)\right) / t_n^2 \leq 0,$$

a contradiction to condition (3.10) . ■

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