Application of Mathematical 3D Surface Using Bi-cubic B-Spline Models

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Abstract

This paper describes a modified mathematical method that used for controlling and generating three dimensional surfaces based on different axes (X, Y, and Z) and free axis. The main aim of the proposed method is to allow the designer to change the shape of the surface to the desired one without changing the original data points which is presented in the earlier version of this paper under title "3D Surface Reconstruction of Mathematical Modelling Used for Controlling the Generation of Different Bi-cubic B-Spline in Matrix Form without Changing the Control Points". The proposed method has been done by changing the t and s, parameters value that are assigned secretly by the designer. Therefore, in case off the control points have been discovered by others, the same design will not be created because the secrete parameters values are known only by the designer. Besides, the proposed method can be used by various applications such as banknote design, shape design, decorations, and governmental documents. Finally, the proposed method shows high level of accuracy in generating 3D surfaces without changing the control points

Keywords: B-spline, blossoming, parametric, matrices, two dimensional, three dimensional surfaces.

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تطبيق رياضي لسطح ثلاثي الابعاد باستخدام نماذج Bi-cubic B-Spline

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الخلاصة

هذا البحث يقدم طريقة رياضية مطورة التي تستخدم للسيطرة على توليد سطوح ثلاثية الأبعاد معتمدا على المحاور (X, Y, Z) والمحور الحر.الهدف الرئيسي من هذه الطريقة المقترحة هو تمكين المصمم من السيطره على توليد اشكال ثلاثيه الابعاد دون تغيير نقاط البيانات الأصلية التي تم عرضها في إصدار سابق من هذه الورقة تحت عنوان "اعادة بناء سطح ثلاثي الابعاد باستخدام نموذج رياضي، للسيطرة على توليد نماذج مختلفة (B-Spline) من الدرجة الثالثة بصيغة المصفوفات من دون تغيير نقاط السيطرة". وقد تم تطبيق هذه الطريقة المقترحة من خلال تغيير قيمة المتغيرات (s, t) وهذه المتغيرات توضع بسرية تامة من قبل المصمم . لذلك، في حالة تم اكتشاف نقاط السيطرة من قبل الآخرين، فانه ليس من السهل اعاده بناء نفس التصميم لان قيم المعاملات معروفة من قبل المصمم فقط. الى جانب ذلك، الطريقة المقترحة يمكن استخدامها من قبل العديد من التطبيقات مثل تصميم الأوراق النقدية، وتصميم الاشكال، والزينة، والوثائق الحكومية. وأخيرا، تظهر الطريقة المقترحة مستوى عال من الدقة في توليد سطوح ثلاثي الأبعاد دون تغيير نقاط السيطرة

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1. Introduction

This work gives an application of Abdul-Hossen`s [1] which has been used in the field of computer graphic. The proposed method work by using the parameters
(t, s) coefficients of the control points. These parameters have been widely used to generate and control 3D graphics, the proposed method gives different surfaces based on different axes(x, y, z), and free axis. Basically, the proposed method will be used to control and produce different 3D modelling surfaces by using blossoming. In fact, the mathematical model is used to generate blossoming of cubic B-spline surface in matrix form.

Apart from that, Gallier [2] has used arithmetical technique to generate B-spline curve by using de -Boor algorithm in blossoming, also Rahma, [3] developed the cubic B-spline scheme by using the blossoming method. However, our work done by using two parameters in different cases based on t_i and s_j (for i, j= 1, 2 and3) coefficient of the control points. The proposed model generates different 3D surfaces based on different coordinate axes. In fact, there are several case studies to generate different modelling designs which have been introduced in this research. In summary, the proposed mathematical model produces high level of performance and generates diverse 3D surfaces without changing the control points of surface.

2. Background

2.1 Bi-cubic B-spline Surfaces

Using, a sequence of 16 control points which are used to define bi-cubic B-spline surface is defined as: [4], [5], [6].

$$F\ (t,\ s) = \frac{1}{36}\,\text{TMP}_{ij}\text{M}^{\text{T}}\text{S}^{\text{T}} \tag{1}$$

Where, M^{T} is the transpose of matrix M. $\qquad M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix},$ $\qquad$ (2)

$$P_{ij} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix},\ T = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix}, S = \begin{pmatrix} s^3 & s^2 & s & 1 \end{pmatrix}, \tag{3}$$

Where $t \in [0, 1]$, $s \in [0, 1]$,
Eq (1) is called the original (classical) bi-cubic B-Spline surfaces in matrix form.

2.2 Rotations

Rotation in three dimensional requires the treatment of an angle of rotation and an axis is chosen as the axis of rotation. The canonical rotations are defined when one of the x, y, z and free axes, coordinate axis is chosen as the axis of rotation.
Note that the direction of rotation is chosen in accordance with the
Anticlockwise with respect to the axis of rotation as [7], [8], [9]:

2.2.1 Rotation about X-axis is Defined by:-

3×3	4×4
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x & 0 \\ 0 & -\sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2.2.2 Rotation about Y-axis is Defined by:-

3×3	4×4
$\begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$	$\begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2.2.3 Rotation about Z-axis is Defined by:-

3×3	4×4
$\begin{pmatrix} \cos\theta_z & \cos\theta_z & 0 \\ -\sin\theta_z & \sin\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \cos\theta_z & \sin\theta_z & 0 & 0 \\ -\sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2.2.4 Rotation about General –axis

The rotation about free axis (arbitrary axis) has end points (x_1, y_1, z_1) & (x_2, y_2, z_2).

Let $\nabla x = x_2 - x_1$, $\nabla y = y_2 - y_1$, $\nabla z = z_2 - z_1$, $d_1 = \sqrt{(\nabla y)^2 + (\nabla z)^2}$, $d_2 = \sqrt{(\nabla x)^2 + (d_1)^2}$.

The following steps of transform matrices are used to rotate by angle $\theta°$

Step.1 Translate the start(x_1, y_1, z_1) to origin by translation vector Tr $(-x_1, -y_1, -z_1)$ After this translation the direction vector $(\nabla x, \nabla y, \nabla z)$ defines the rotation axis as follows: [7], [9].

$$\textbf{Tr } (\textbf{-x}_1, \textbf{-y}_1, \textbf{-z}_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_1 & -y_1 & -z_1 & 1 \end{pmatrix}$$

Step.2 Rotate about the x-axis until the rotation axis corresponds to the z-axis.

When the rotation axis is projected onto the x, z plane, any point on it has x coordinate equal to zero. In particular $\nabla x = 0$. The point $(0, b, c)$ is rotated **Φ** degree until the line corresponds to the z-axis. It is been found with the sin **(Φ)** and cos **(Φ)** we find that distance from the origin to $(0, b, c)$: [7], [9].

$$\mathbf{d_1} = \sqrt{(\nabla y)^2 + (\nabla z)^2} \ , \quad \textbf{Sin (Φ)} = \nabla y / d_1 \ , \ \textbf{Cos (Φ)} = \nabla z / d_1$$

$$\textbf{Rx (Φ)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \nabla z / d_1 & \nabla y / d_1 & 0 \\ 0 & -\nabla y / d_1 & \nabla z / d_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now the point $(0, \nabla y, \nabla z)$ has been transformed to the point $(0, 0, d_1)$ but since the rotation about the x-axis does not change the x coordinate value the point $(\nabla x, \nabla y, \nabla z)$ becomes at location $(\nabla x, 0, d_1)$.

Step.3 Rotate about the y-axis until the rotation axis corresponds to the z-axis.

Since (∇ x, 0, d_1) lies in the x, z plane, for a rotation of angle β, we need to compute sin (β), cos (β) [7], [9].where:

$$d_2 = \sqrt{(\nabla x)^2 + (d_1)^2} \quad , \quad \textbf{Sin (β)} = \nabla x / d_2 \quad , \quad \textbf{Cos (β)} = d_1 / d_2$$

Step. 4 Rotate about the z-axis angle θ. This requires the Rz (θ) matrix [7], [9].

$$\textbf{Rz}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step.5 performing the inverse rotation of step (3), requires **Ry(-β)**

Step.6 Performing the inverse rotation of step (2), requires **Rx(-β)**

Step. 7 performing the inverse translation of step (1), requires Tr (x_1,y_1,z_1)**.**

Note. 1

The composite transformation is given by*: [7], [9].*

Tr(-x₁, -y₁, -z₁) * Rx(Φ) * Ry(β) * Rz(θ) * Ry(-β) * Rx(-Φ) * Tr(x₁,y₁,z₁)

$$* \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \nabla z/d_1 & \nabla y/d_1 & 0 \\ 0 & -\nabla y/d_1 & \nabla z/d_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} d_1/d_2 & 0 & \nabla x/d_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\nabla x/d_2 & 0 & d_1/d_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_1 & -y_1 & -z_1 & 1 \end{pmatrix}$$

$$* \begin{pmatrix} d_1/d_2 & 0 & -\nabla x/d_2 & 0 \\ 0 & 1 & 0 & 0 \\ \nabla x/d_2 & 0 & d_1/d_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \nabla z/d_1 & -\nabla y/d_1 & 0 \\ 0 & \nabla y/d_1 & \nabla z/d_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_1 & y_1 & z_1 & 1 \end{pmatrix}$$

3   Cubic Spline Curves in Blossoming Form

Method of Blossom used to find cubic spline From, $f(t_1,t_2,t_3)$ is given by : [1],[2]

$F_k = f_k(t_1,t_2,t_3) = (1-\lambda_6)(1-\lambda_4)(1-\lambda_1)[f(u_{k-2}, u_{k-1}, u_k)] + \{(1-\lambda_4)(1-\lambda_6)\lambda_1 + (1-\lambda_2)(1-\lambda_6)\lambda_4 + (1-\lambda_2)(1-\lambda_5)\lambda_6\}f(u_{k-1}, u_k, u_{k+1}) + \{\lambda_2\lambda_4(1-\lambda_6) + \lambda_6(1-\lambda_5)\lambda_2 + (1\lambda_3)\lambda_5\lambda_6\}f(u_k,u_{k+1}, u_{k+2})$

$\qquad (4) \qquad\qquad + \lambda_3\lambda_5 \lambda_6 f(u_{k+1}, u_{k+2}, u_{k+3}).$

Suppose $f(u_{k-2}, u_{k-1}, u_k)]$, $f(u_{k-1}, u_k, u_{k+1})$, $f(u_k,u_{k+1}, u_{k+2})$, $f(u_k,u_{k+1}, u_{k+2}) = p_0,p_1,p_2,p_3$ are control points and Eq (4) becomes.

$f_k(t_1,t_2,t_3) = (1-\lambda_6)(1-\lambda_4)(1-\lambda_1)p_0 + \{(1-\lambda_4)(1-\lambda_6)\lambda_1 + (1-\lambda_2)(1-\lambda_6)\lambda_4 + (1-\lambda_2)(1-\lambda_5)\lambda_6\}p_1 + \{\lambda_2\lambda_4(1-\lambda_6) + \lambda_6(1-\lambda_5)\lambda_2$

$\qquad (5) \qquad\qquad + (1-\lambda_3)\lambda_5\lambda_6\}p_2 + \lambda_3\lambda_5\lambda_6 p_3.$

Eq (5) is a formula of a modified cubic spline curve.

3.1 Generating Cubic Spline Curves in Matrices Form
When using, the following [1].

$$\lambda_1 = \frac{t_1 - u_{k-2}}{u_{k+1} - u_{k-2}}, \quad \lambda_2 = \frac{t_1 - u_{k-1}}{u_{k+2} - u_{k-1}}, \quad \lambda_3 = \frac{t_1 - u_k}{u_{k+3} - u_k},$$

$$\lambda_4 = \frac{t_2 - u_{k-1}}{u_{k+1} - u_{k-1}}, \quad \lambda_5 = \frac{t_2 - u_k}{u_{k+2} - u_k}, \quad \lambda_6 = \frac{t_3 - u_k}{u_{k+1} - u_k}.$$

And substituting, λ_i (for i= 1 to 6), in Eq (5) from the first point can see [1].

$$(1-\lambda_6)(1-\lambda_4)(1-\lambda_1)\,P_0 = \left\{ \frac{(u_{k+1} - t_1)(u_{k+1} - t_2)(u_{k+1} - t_3)}{(u_{k+1} - u_{k-2})(u_{k+1} - u_{k-1})(u_{k+1} - u_{k-2})} \right\} P_0$$

$$\left\{ \frac{-t_1 t_2 t_3 + u_{k+1}(t_1 t_2 + t_1 t_3 + t_2 t_3) - u^2{}_{k+1}(t_1 + t_2 + t_3) + u^3{}_{k+1}}{(u_{k+1} - u_{k-2})(u_{k+1} - u_{k-1})(u_{k+1} - u_k)} \right\} P_0$$

$$= \frac{1}{u_{k+1} - u_k} \left\{ \frac{-t_1 t_2 t_3 + u_{k+1}(t_1 t_2 + t_1 t_3 + t_2 t_3) - u^2{}_{k+1}(t_1 + t_2 + t_3) + u^3{}_{k+1}}{(u_{k+1} - u_{k-2})(u_{k+1} - u_{k-1})} \right\} P_0.$$

In the same way simple (Eq (5)) is used as [1]:

$$f_k(t_1, t_2, t_3) = \frac{1}{u_{k+1} - u_k} \Big\{ \; [-A_1 t_1 t_2 t_3 + A_1 u_{k+1}(t_1 t_2 + t_1 t_3 + t_2 t_3) - A_1 u^2{}_{k+1}(t_1 + t_2 + t_3) + A_1 u^3{}_{k+1}] P_0 +$$

$$[t_1 t_2 t_3 (A_1 + A_2 + A_3) + t_1 t_2(-u_{k+1}A_1 - u_{k+1}A_2 - u_k A_3) + t_1 t_3(-u_{k+1}A_1 - u_{k-1}A_2 - u_{k+2}A_3)$$

$$+ t_2 t_3(-u_{k-2}A_1 - u_{k+2}A_2 - u_{k+2}A_3) + t_1(u^2{}_{k+1}A_1 + u_{k+1}u_{k-1}A_2 + u_k u_{k+2}A_3) + t_2(u_{k-2}u_{k+1}A_1$$

$$+ u_{k+1}u_{k+2}A_2 + u_k u_{k+2}A_3) + t_3(u_{k-2}u_{k+1}A_1 + u_{k-1}u_{k+2}A_2 + u^2{}_{k+2}A_3)$$

$$- (u_{k-2}u^2{}_{k+1}A_1 + u_{k-1}u_{k+1}u_{k+2}A_2 + u_k u^2{}_{k+2}A_3)]P_1 +$$

$$[t_1 t_2 t_3(-A_4 - A_5 - A_6) + t_1 t_2(u_{k+1}A_4 + u_k A_5 + u_k A_6) + t_1 t_3(u_{k-1}A_4 + u_{k+2}A_5 + u_k A_6)$$

$$+ t_2 t_3(u_{k-1}A_4 + u_{k-1}A_5 + u_{k+3}A_6) + t_1(-u_{k-1}u_{k+1}A_4 - u_{k+2}u_k A_5 - u^2{}_k A_6)$$

$$+ t_2(-u_{k-1}u_{k+1}A_4 - u_k u_{k-1}A_5 - u_k u_{k+3}A_6) + t_3(-u^2{}_{k-1}A_4 - u_{k-1}u_{k+2}A_5 + u_k u_{k+3}A_6)$$

$$+ (u_{k+1}u^2{}_{k-1}A_4 + u_{k-1}u_k u_{k+2}A_5 + u_{k+3}u^2{}_k A_6)]P_2 +$$

(6)

$$[A_6 t_1 t_2 t_3 - A_6 u_k(t_1 t_2 + t_1 t_3 + t_2 t_3) + A_6 u^2{}_k(t_1 + t_2 + t_3) - A_6 u^3{}_k]P_3 \Big\}$$

Where

$$, \quad A_2 = \frac{1}{(u_{k+1} - u_{k-1})(u_{k+2} - u_{k-1})}, \quad A_1 = \frac{1}{(u_{k+1} - u_{k-2})(u_{k+1} - u_{k-1})}$$

$$, \quad A_4 = \frac{1}{(u_{k+2} - u_k)(u_{k+1} - u_{k-1})}, \quad A_3 = \frac{1}{(u_{k+2} - u_{k-1})(u_{k+2} - u_k)}$$

$$A_6 = \frac{1}{(u_{k+2} - u_k)(u_{k+3} - u_k)}. \quad A_5 = \frac{1}{(u_{k+2} - u_k)(u_{k+2} - u_{k-1})},$$

Putting Eq (6) in matrices form yleds.[1] : -

$$B_0(u_k) = (-1)A_1 \begin{pmatrix} t_1t_2t_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, B_1(u_k) = A_1 \begin{pmatrix} t_1t_2 & t_1t_3 & t_2t_3 \end{pmatrix} u_{k+1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$B_2(u_k) = (-1)A_1 \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} u^2{}_{k+1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, B_3(u_k) = A_1 \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} u^3{}_{k+1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$B_4(u_k) = \begin{pmatrix} t_1t_2t_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 + A_2 + A_3 \\ 0 \\ 0 \end{pmatrix},$$

$$B_5(u_k) = (-1) \begin{pmatrix} t_1t_2 & t_1t_3 & t_2t_3 \end{pmatrix} \begin{pmatrix} A_1 u_{k+1} + A_2 u_{k+1} + A_3 u_k \\ A_1 u_{k+1} + A_2 u_{k-1} + A_3 u_{k+2} \\ A_1 u_{k-2} + A_2 u_{k+2} + A_3 u_{k+2} \end{pmatrix},$$

$$B_6(u_k) = \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} A_1 u^2{}_{k+1} + A_2 u_{k+1} u_{k-1} + A_3 u_{k+1} u_k \\ A_1 u_{k+1} u_{k-2} + A_2 u_{k+1} u_{k+2} + A_3 u_{k+2} u_k \\ A_1 u_{k+1} u_{k-2} + A_2 u_{k+2} u_{k-1} + A_3 u^2{}_{k+2} \end{pmatrix},$$

$$B_7(u_k) = (-1) \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A_1 u^2{}_{k+1} u_{k-2} \\ A_2 u_{k+2} u_{k+1} u_{k-1} \\ A_3 u^2{}_{k+2} u_k \end{pmatrix}$$

$$B_{12}(u_k) = \begin{pmatrix} t_1t_2t_3 & 0 & 0 \end{pmatrix} A_6 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, B_{13}(u_k) = (-1)(A_6 u_k) \begin{pmatrix} t_1t_2 & t_1t_3 & t_2t_3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$B_{14}(u_k) = \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} (A_6 u^2{}_k) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, B_{15}(u_k) = (-1)(A_6 u^3{}_k) \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Putting Eq (6) in simple form gives:-

$$f_k(t_1, t_2, t_3) = \frac{1}{u_{k+1} - u_k} \{ [B_0 + B_1 + B_2 + B_3] p_0 + [B_4 + B_5 + B_6 + B_7] p_1$$
$$+ [B_8 + B_9 + B_{10} + B_{11}]\} p_2 + [B_{12} + B_{13} + B_{14} + B_{15}] p_3. \tag{7}$$

Now we can put Eq (7) in the matrices form [1]:-

$$f_k(t_1, t_2, t_3) = \frac{1}{u_{k+1} - u_k} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} B_0(u_k) & B_4(u_k) & B_8(u_k) & B_{12}(u_k) \\ B_1(u_k) & B_5(u_k) & B_9(u_k) & B_{13}(u_k) \\ B_2(u_k) & B_6(u_k) & B_{10}(u_k) & B_{14}(u_k) \\ B_3(u_k) & B_7(u_k) & B_{11}(u_k) & B_{15}(u_k) \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}, \tag{8}$$

Eq (8) is a formula of a modified mathematical modelling of B-spline curve in matrices form of degree three.

3.2 Bi-cubic Spline in Matrix Form

Consider 16 points in the form $f(u_{k-2+i}, u_{k-1+i}, u_{k+i}; v_{l-2+j}, v_{l-1+j}, v_{l+j})$, $(0 \leq i \leq 3)(0 \leq j \leq 3)$, k and l are integers, suppose the bidegree is $\langle 3, 3 \rangle$ where u_i and v_j are real numbers taken from the knot sequence. The same method in Eq (8) is used to find second parameters $f(s_1, s_2, s_3)$, in matrix form. From using the following [1]:

$$, E_2 = \frac{1}{(v_{l+1} - v_{l-1})(v_{l+2} - v_{l-1})}, \; E_1 = \frac{1}{(v_{l+1} - v_{l-2})(v_{l+1} - v_{l-1})}$$

$$E_4 = \frac{1}{(v_{l+2} - v_l)(v_{l+1} - v_{l-1})}, \; E_3 = \frac{1}{(v_{l+2} - v_{l-1})(v_{l+2} - v_l)},$$

$$E_6 = \frac{1}{(v_{l+2} - v_l)(v_{l+3} - v_l)} . \; E_5 = \frac{1}{(v_{l+2} - v_l)(v_{l+2} - v_{l-1})},$$

And in matrix form [1]:-

$$f_l(s_1, s_2, s_3) = \frac{1}{(v_{l+1} - v_l)} (1 \quad 1 \quad 1 \quad 1) \begin{bmatrix} Z_0(v_l) & Z_4(v_l) & Z_8(v_l) & Z_{12}(v_l) \\ Z_1(v_l) & Z_5(v_l) & Z_9(v_l) & Z_{13}(v_l) \\ Z_2(v_l) & Z_6(v_l) & Z_{10}(v_l) & Z_{14}(v_l) \\ Z_3(v_l) & Z_7(v_l) & Z_{11}(v_l) & Z_{15}(v_l) \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

The above equation is a formula of a modified cubic spline curve in matrix form of second variable *s*. The bi-polynomial surface, of degree $\langle 3, 3 \rangle$ mathematically comes from product of two matrices defined as: See [1]

$$f(t_1, t_2, t_3; s_1, s_2, s_3) = \frac{1}{(u_{k+1} - u_k)(v_{l+1} - v_l)} (1 \quad 1 \quad 1 \quad 1) M_t P_{ij} M_s^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{9}$$

where

$$M_t = \begin{bmatrix} B_0(u_k) & B_4(u_k) & B_8(u_k) & B_{12}(u_k) \\ B_1(u_k) & B_5(u_k) & B_9(u_k) & B_{13}(u_k) \\ B_2(u_k) & B_6(u_k) & B_{10}(u_k) & B_{14}(u_k) \\ B_3(u_k) & B_7(u_k) & B_{11}(u_k) & B_{15}(u_k) \end{bmatrix}, \; \text{and} \; M_s = \begin{bmatrix} Z_0(v_l) & Z_4(v_l) & Z_8(v_l) & Z_{12}(v_l) \\ Z_1(v_l) & Z_5(v_l) & Z_9(v_l) & Z_{13}(v_l) \\ Z_2(v_l) & Z_6(v_l) & Z_{10}(v_l) & Z_{14}(v_l) \\ Z_3(v_l) & Z_7(v_l) & Z_{11}(v_l) & Z_{15}(v_l) \end{bmatrix}$$

$$P_{ij} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix}, \text{ for i, j} = 0, 1, 2, 3.$$

It is an easy method used to upgrade Eq (9) to 3D of degree $\langle 3, 3 \rangle$ by using the set of points such as, [8], [9], [10].

$P_{ij} = (x_{ij}, y_{ij}, z_{ij})$ for i, j = *0, 1, 2, 3*.

The coordinates of each point are treated as a three-component vector. That is

$$P_{ij} = \begin{pmatrix} x_{i,j} \\ y_{i,j} \\ z_{i,j} \end{pmatrix} . \text{The set of points, in parametric form is}$$

$$F(t,s) = \begin{pmatrix} F_1(t,s) \\ F_2(t,s) \\ F_3(t,s) \end{pmatrix} = \begin{pmatrix} x(t,s) \\ y(t,s) \\ z(t,s) \end{pmatrix} \qquad u_k \le t \le u_{k+1} \text{ and } v_l \le s \le v_{l+1}.$$

Eq (9) i.e. is modified bi-cubic 3D B-spline in matrix form obtained by using blossoming. The surface is defined by de Boor control points for the surface segment $f(t_1, t_2, t_3; s_1, s_2, s_3)$ associated with the middle intervals $[u_k, u_{k+1}]$ and $[v_l, v_{l+1}]$.

4. Application

By using 3D bi-cubic B-spline surfaces in matrices form (Eq (9)), different applications can be applied based on the proposed mathematical model in control and generating 3D surfaces by using parameters *t*, and *s* with different values in increasing (i.e. positive) and decreasing (i.e. negative) on different axes(x, y, z, and free axis) as follows:

A. First application, controlling the generated surfaces based on tuning the values of the parameters t_i or s_j (for i, j = 1or 2 or 3), at all parts of Eq (9). In fact, different surfaces will be produces based on different x, y, z, or free axes, as illustrated in the following sections:-

1. Using $(t_1=t_2=t_3=t$, $s_1=s_2=s_3=s)$, and $k=0,l=0$ the sequence $\{u_{k-2},\ u_{k-1},\ u_k,\ u_{k+1},\ u_{k+2},\ u_{k+3};\ v_{l-2},\ v_{l-1},\ v_l,\ v_{l+1},\ v_{l+2}\ v_{l+3}\}$ becomes $(u_{-2},\ u_{-1}\ u_0,\ u_1\ u_2,\ u_3;\ v_{-2},\ v_{-1}\ v_0,\ v_1\ v_2,\ v_3)$, using uniform knot sequence. $(u_{-2},\ u_{-1}\ u_0,\ u_1\ u_2,\ u_3)=(-2,\ -1,\ 0,\ 1,\ 2,\ 3)$, which implies $u_k = u_0 = 0$, and $u_{k+1} = u_1 = 1$, then $t \in [0,1]$, $(v_{-2},\ v_{-1}\ v_0,\ v_1\ v_2,\ v_3) =(-2,\ -1,\ 0,\ 1,\ 2,\ 3)$, implies $v_1 = v_0 = 0$, and $v_{l+1} = v_1 = 1$, then $s \in [0,1]$, and Eq (9) go to original bi-cubic spline surface based on frame $[0,\ 1]^2$. It is identical to Eq (1). Figure-1 gives a clear illustration to the original bi b- spline surface based on different coordinate axes.

2. By increasing the value of parameter t_i (i.e. positive $+t_1$) and decrease its value (i.e. negative $-t_1$) different surfaces will be generated based on different axes (x, y, z, and free axes). Figure-2 and Figure-3. gives an example to the increasing and decreasing of the t_i parameter. Fundamentally, the surface direction moves exterior when increasing the t_i and moves to the interior when decreasing the t_i value.

3. By increasing the value of parameter s_j (i.e. positive $+ s_1$) and decrease its value (i.e. negative $- s_1$) different surfaces will be generated based on different axes (x, y, z, and free axes). Figure-.4 and Figure-5 gives an example to the increasing and decreasing of the s_j parameter. Fundamentally, the surface direction moves exterior when increasing the s_j and moves to the interior when decreasing the s_j value.

B.

The second application is illustrated in this section. Fundamentally, controlling the generated surfaces based on tuning the values of the parameters t_i and s_j at the same time (for i, j = 1or 2 or 3), at all parts of Eq (9). Mainly, various models of bi b-spline design will be generated based on different axes (x,y,z, and free axes). In fact, the parameters t_i and s_j values are increased (i.e. positive $+ t_i$, and $+ s_j$) at the same time or decreased (i.e $- t_i$, and $- s_j$) at the same time will generate different surfaces model as shown in Figure-6 and 7.

C.

The third application, different models of bi b- spline surface, in different directions based on different coordinates is generated. In fact, in this case the t_i value is increased and s_j value is decreased at the same time (i.e. $+ t_i$ and $- s_j$) or decrease the t_i and increase the s_j at the same time (i.e. $- t_i$ and $+ s_j$). Fundamentally, various surfaces modelling will be generated based on the increasing and decreasing of the t_i and sj parameters on different coordinates (x, x, z, and free axes). Figure-8 and 9, gives an example to this case.
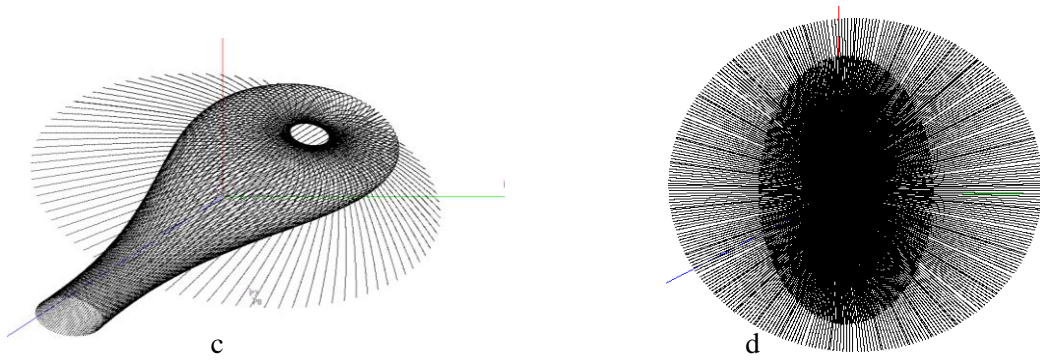
D.

Different surfaces will be generated based on quadrants of free axes coordinate with parameters t and/or s in different values when increases or decreases their values. Figure- 10 and 11 have shown a clear example to this case.
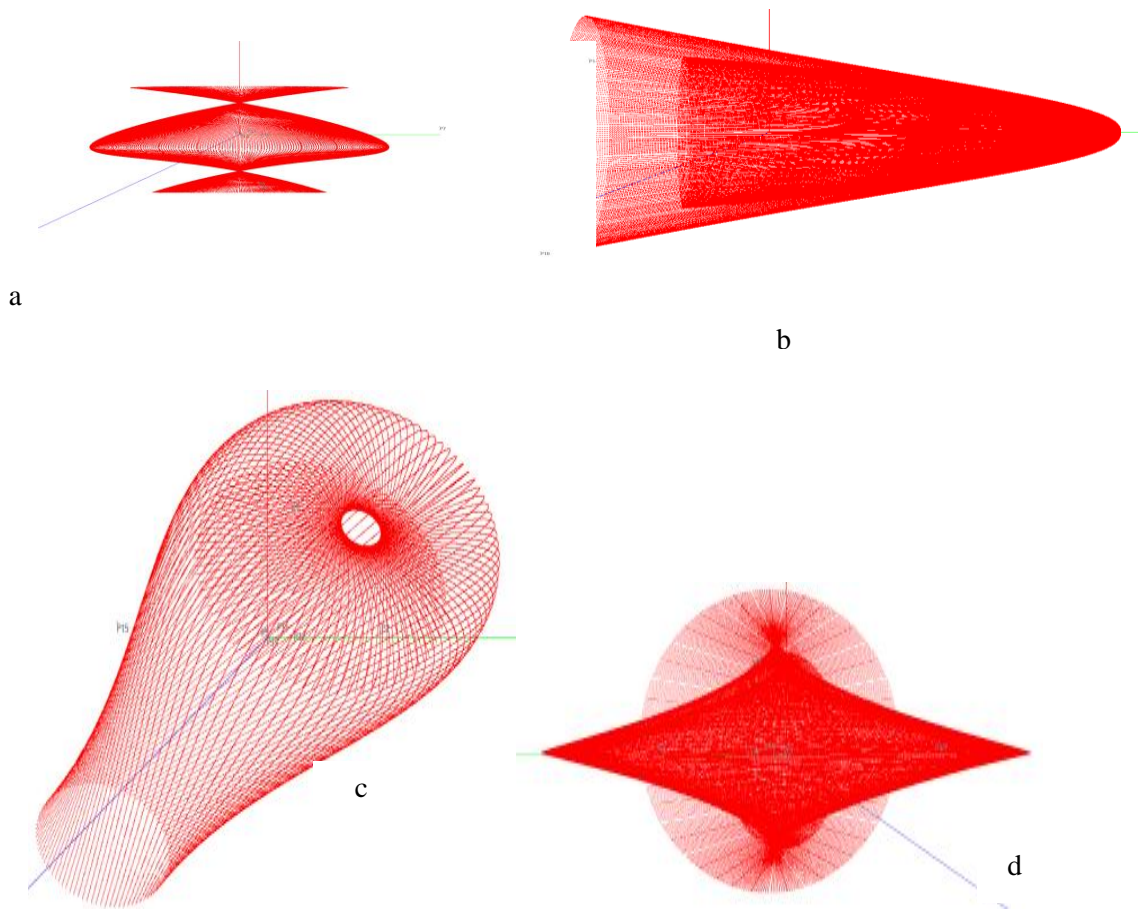
E.

All the above effects on the designs take place without changing the control points.
.

a   b
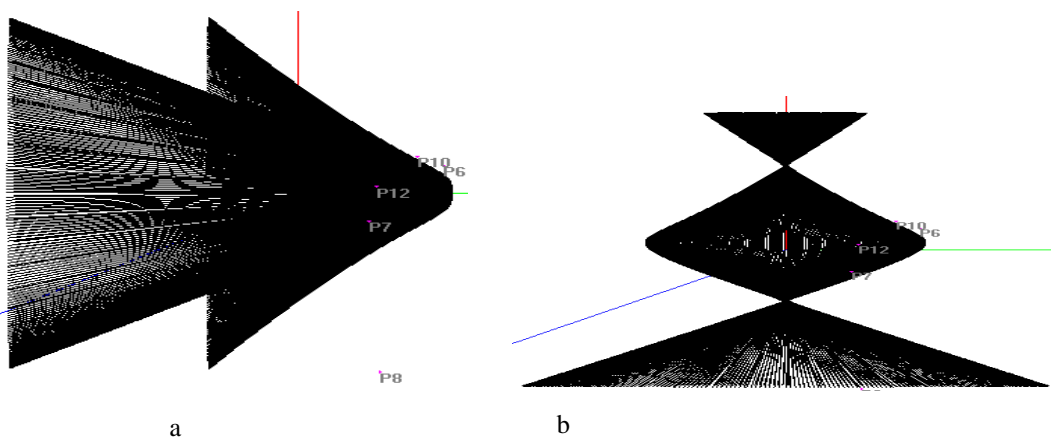
Figure- 1 Different models, of original Bib- spline design**. (a)** Direction of x- axis. **(b)** Direction of y-axis. **(c)** Direction of z- axis. **(d)** Direction of free- axis.
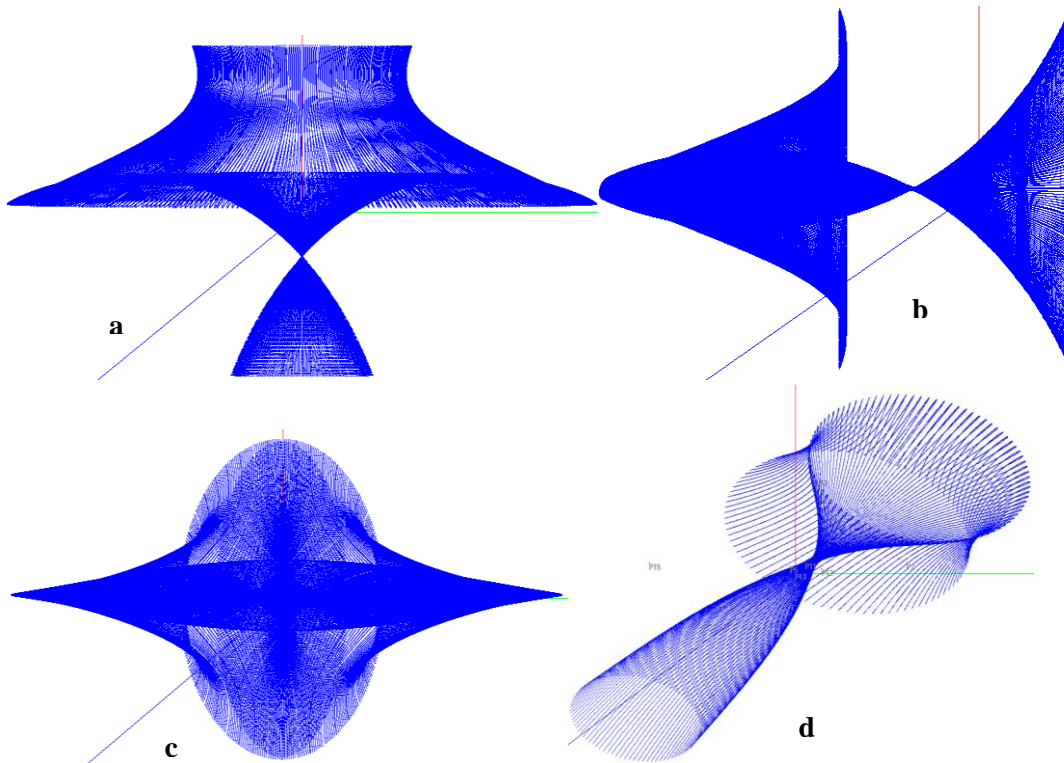

Figure 2-Different models of modified bi-polynomial B-Spline (3D) designs. The parameters t_1 is taken to be increased (use t_1 is to be increased i.e. $+ t_1$) and is taken in, (a) direction of x- axis, (b) direction of y- axis, (c) direction of z- axis (d) direction of free- axis.
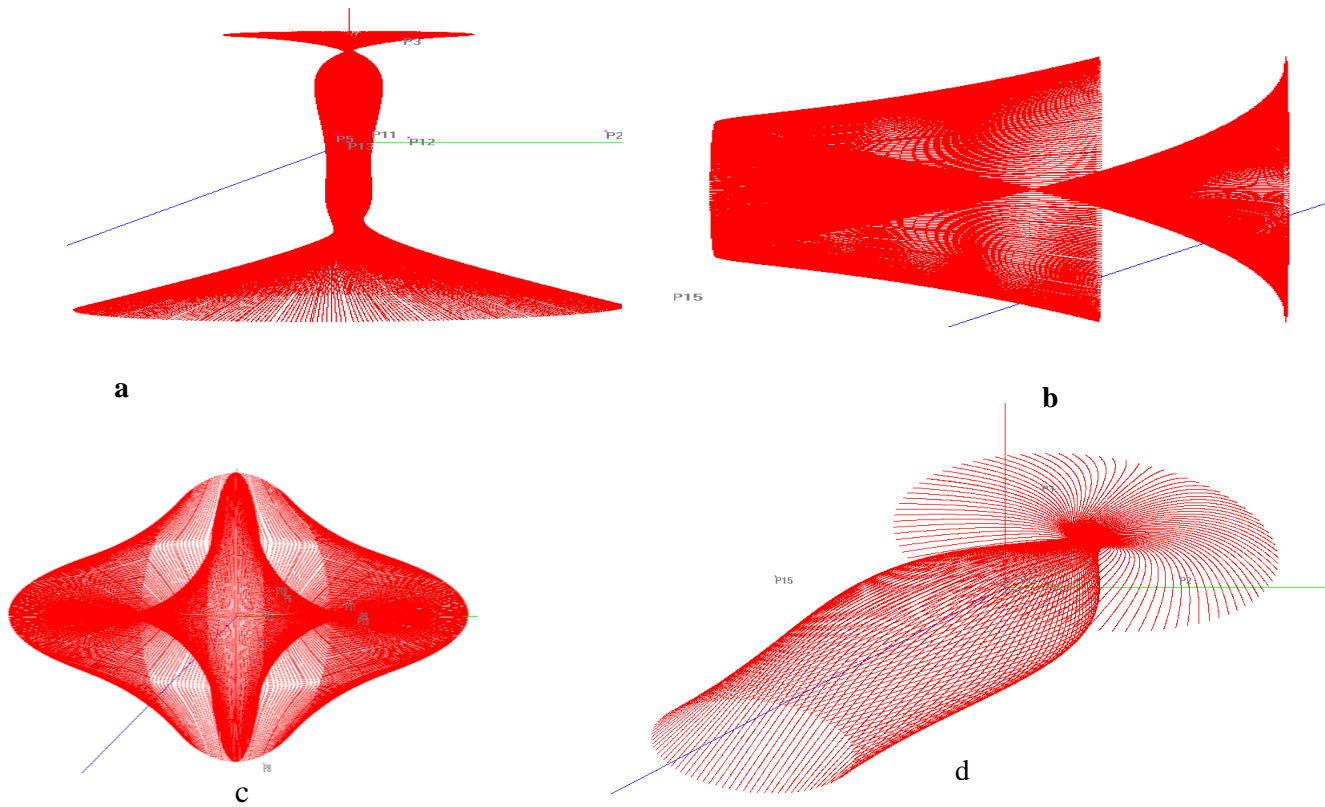
Figure 3- Different models of modified bi-polynomial B-Spline (3D) designs. T he parameters t_1 is taken to be decreased (by use t_1 is to be decreased i.e. - t_1) is taken in, (a) direction of x- axis,(b) direction of y- axis, (c) direction of z- axis, (d) direction of free- axis.
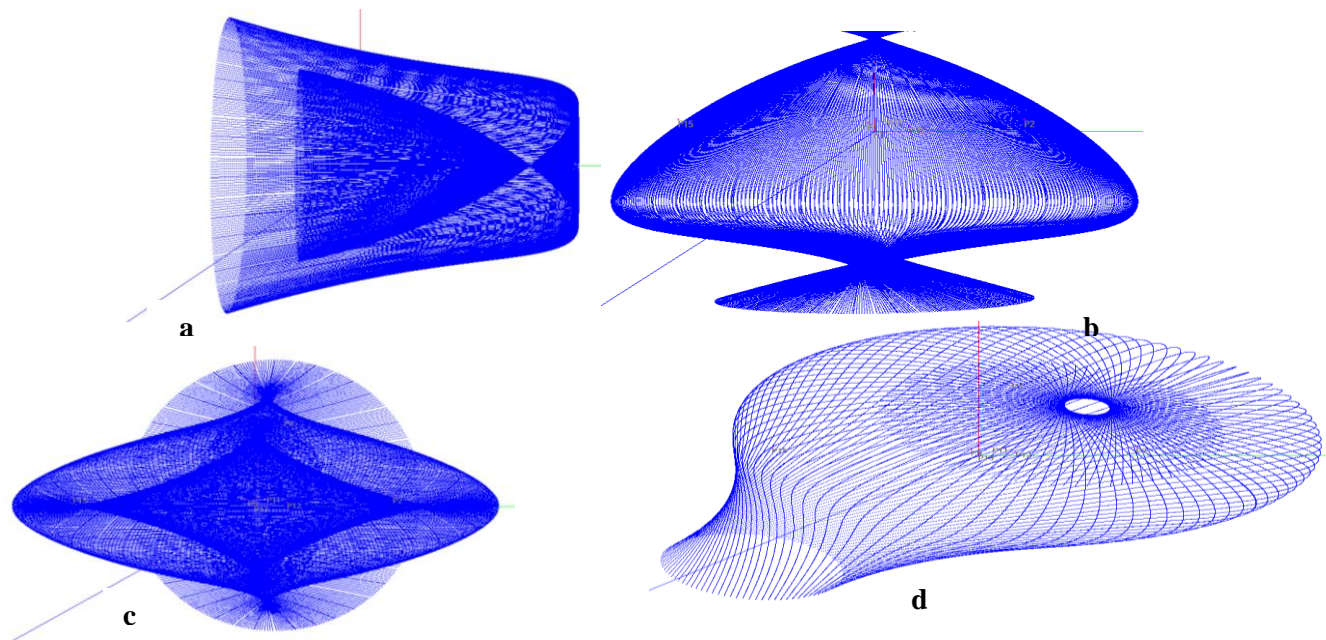
Figure 4- Different models of modified bi-polynomial B-Spline (3D) designs. The parameters s_1 is taken to be increased (by use s_1 is to be increased i.e. + s_1) is taken in, (a) Direction of x- axis, (b) direction of y- axis**.** (c) Direction of z- axis, (d) direction of free- axis.

Figure 5- Different models of modified bi-polynomial B-Spline (3D) designs. The parameters s_1 is taken to be decreased (by use s_1 is to be decreased i.e. $+ s_1$) is taken in, (a) direction of x- axis (b) direction of y- axis. (c) Direction of z- axis. (d) Direction of free- axis.
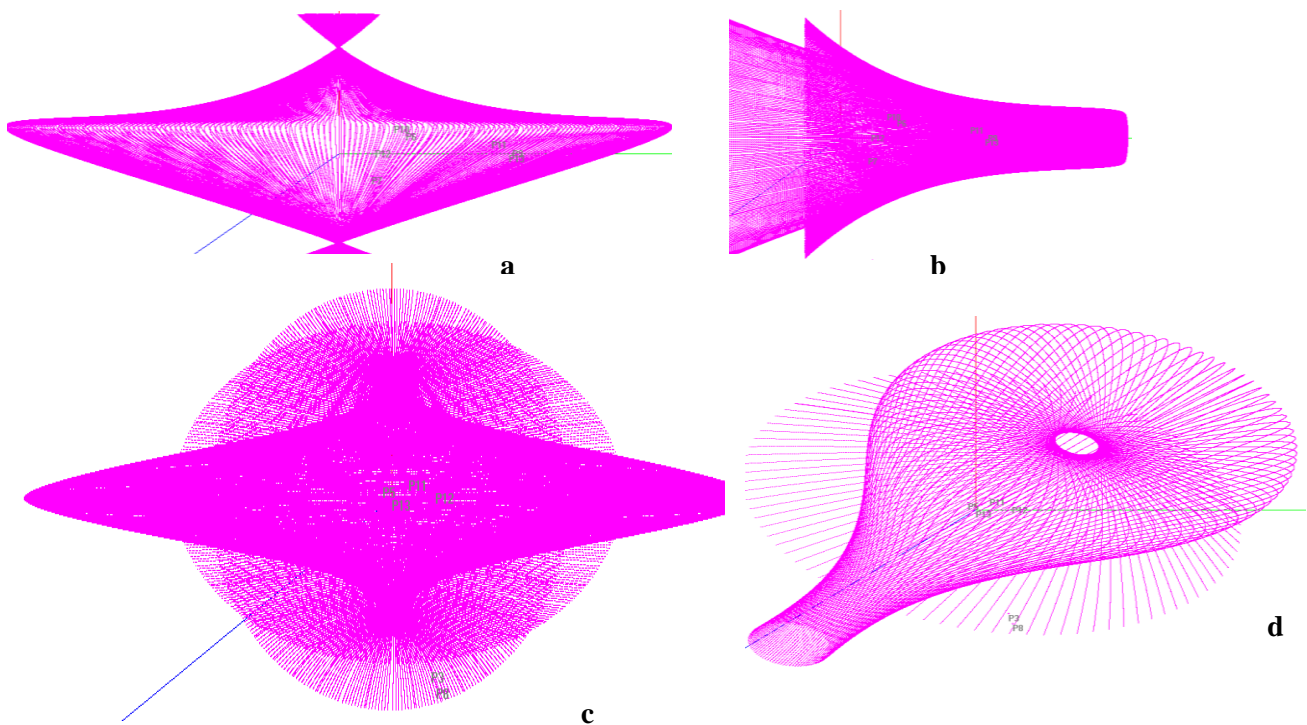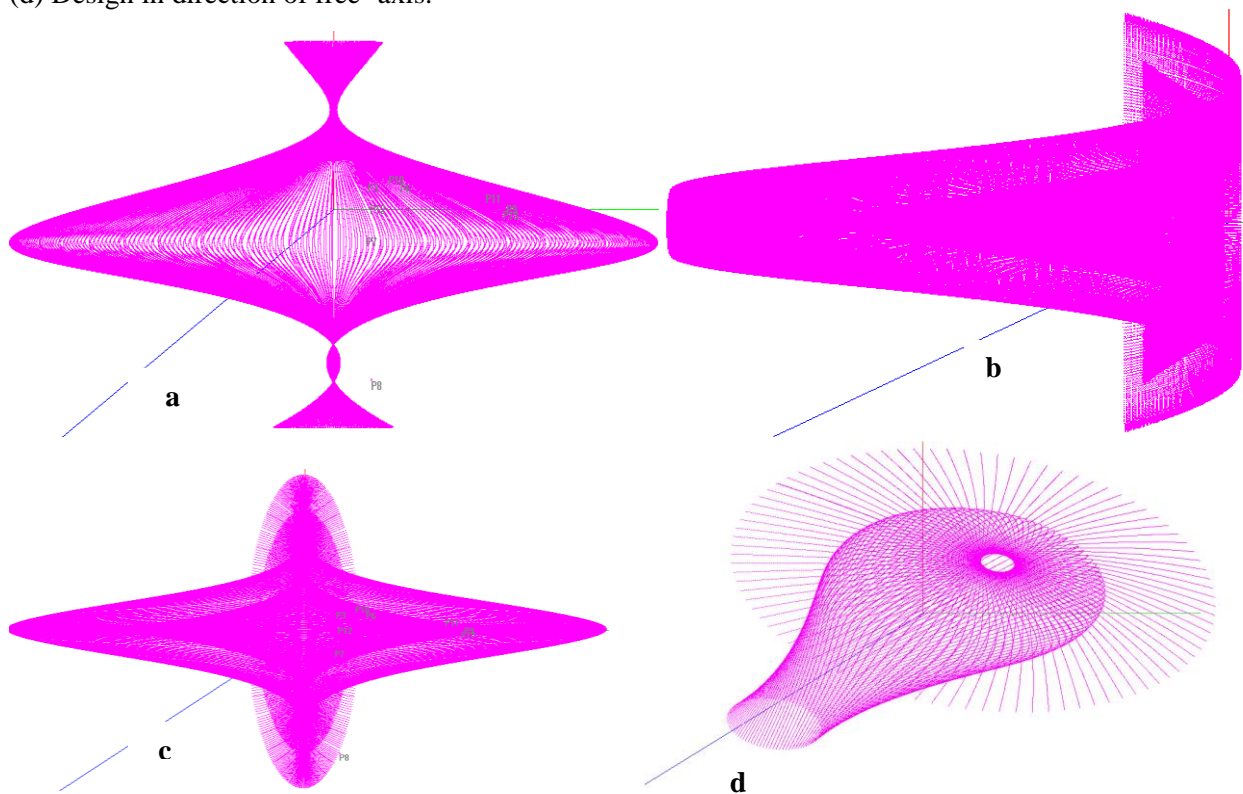

Figure 6- The parameters t and s are taken to be increased at the same time *(*i.e.$+ t_1$and $+s_1$at the same time *)* are taken in (a) Direction of x- axis, (b) direction of y- axis, (c) direction of z- axis, (d) direction of free- axis.
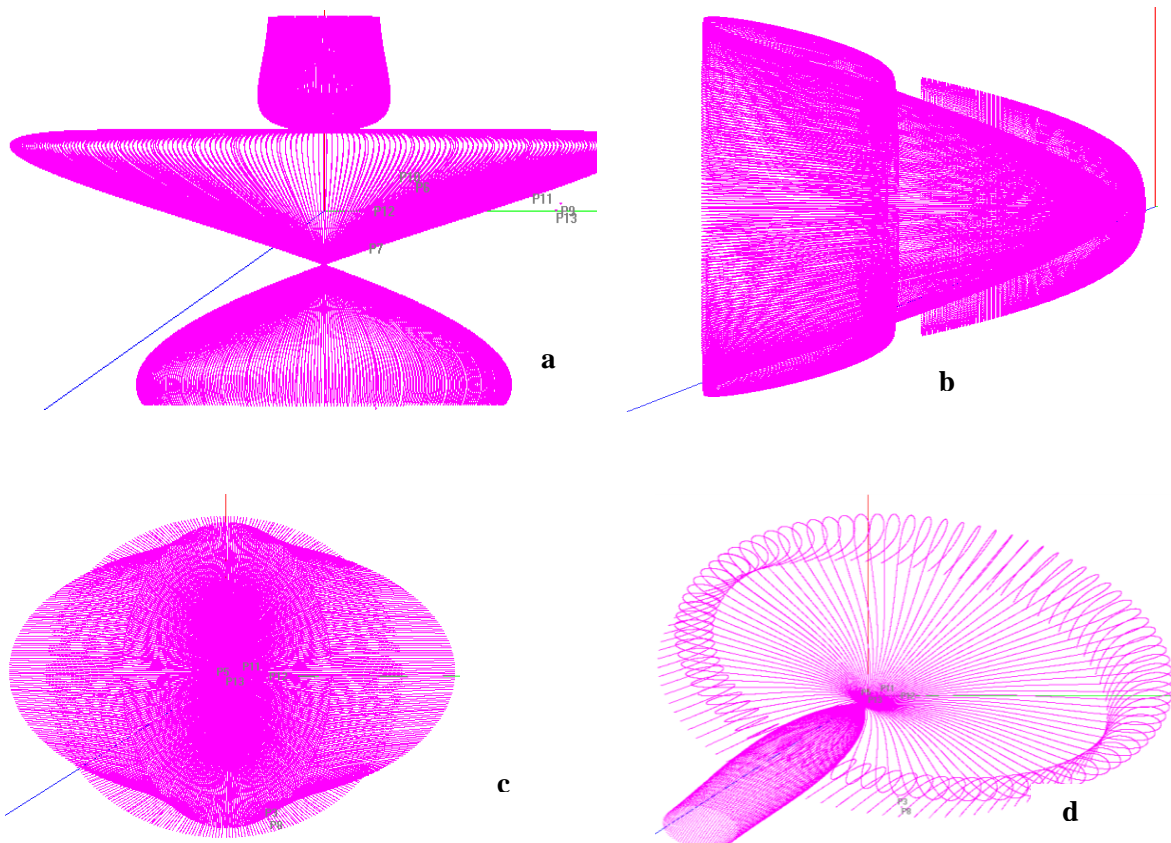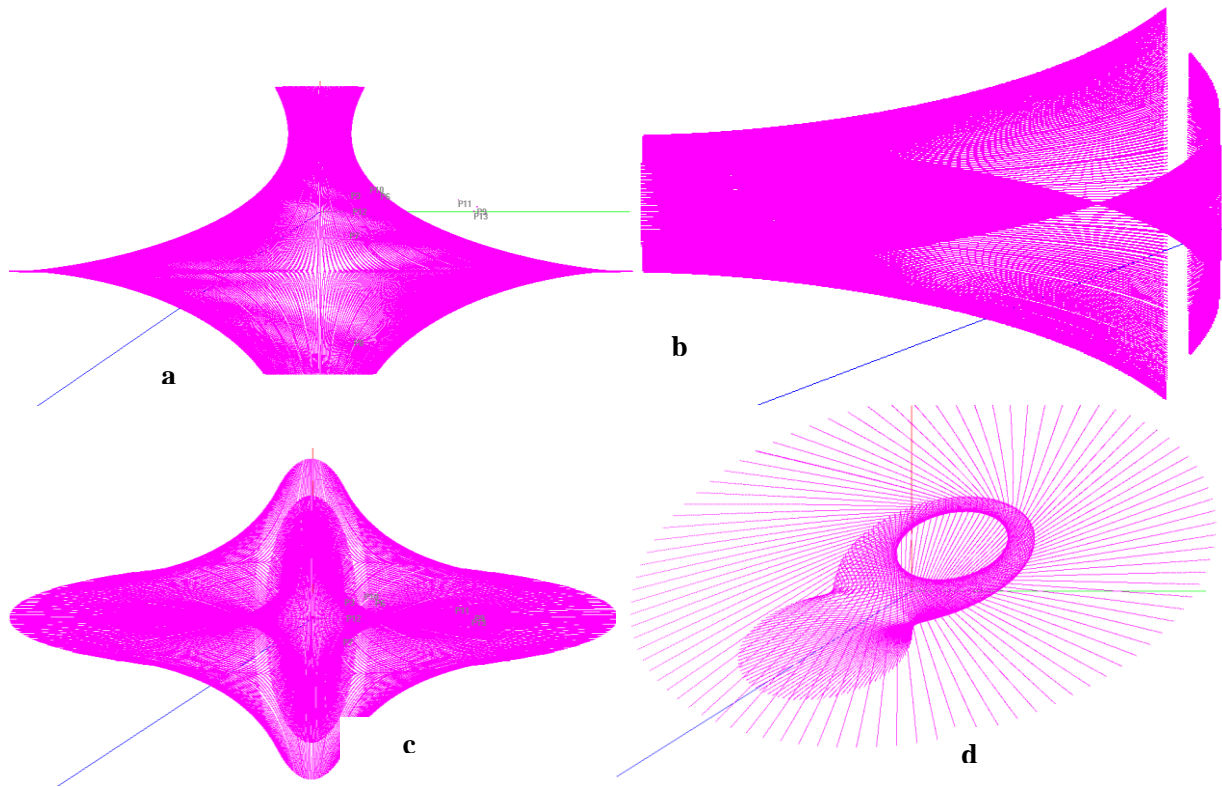
Figure7- The parameters t and   s are taken to be decreased at the same time *(*i.e.- $t_1$and - s_1 at the same time) and are taken in, (a) direction of x- axis,(b) direction of y- axis, (c) direction of z- axis, (d) Design in direction of free- axis.


Figure 8- The parameters t increase and s decrease and are taken at the same time *(*i.e.+ $t_1$and - $s_1$at the same time) are taken in, (a) direction of x- axis, (b) direction of y- axis, (c) direction of z- axis, (d) direction of free- axis.

Figure 9- The parameters t decrease and s increase taken at the same time *(*i.e.- t_1 and $+ s_1$ at the same time) taken in, (a) direction of x- axis,(b) direction of y- axis, (c) direction of z- axis, (d) Design in direction of free- axis.
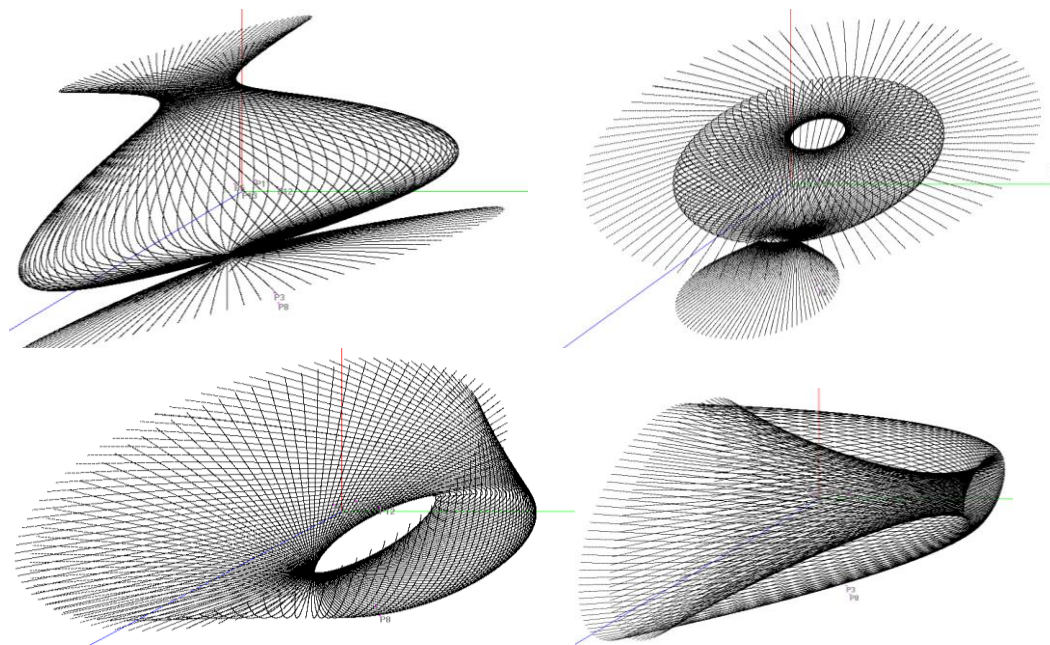
Figure 10- Different models by using original Bib- spline design. Dependents on coordinate axes by using free- axis.
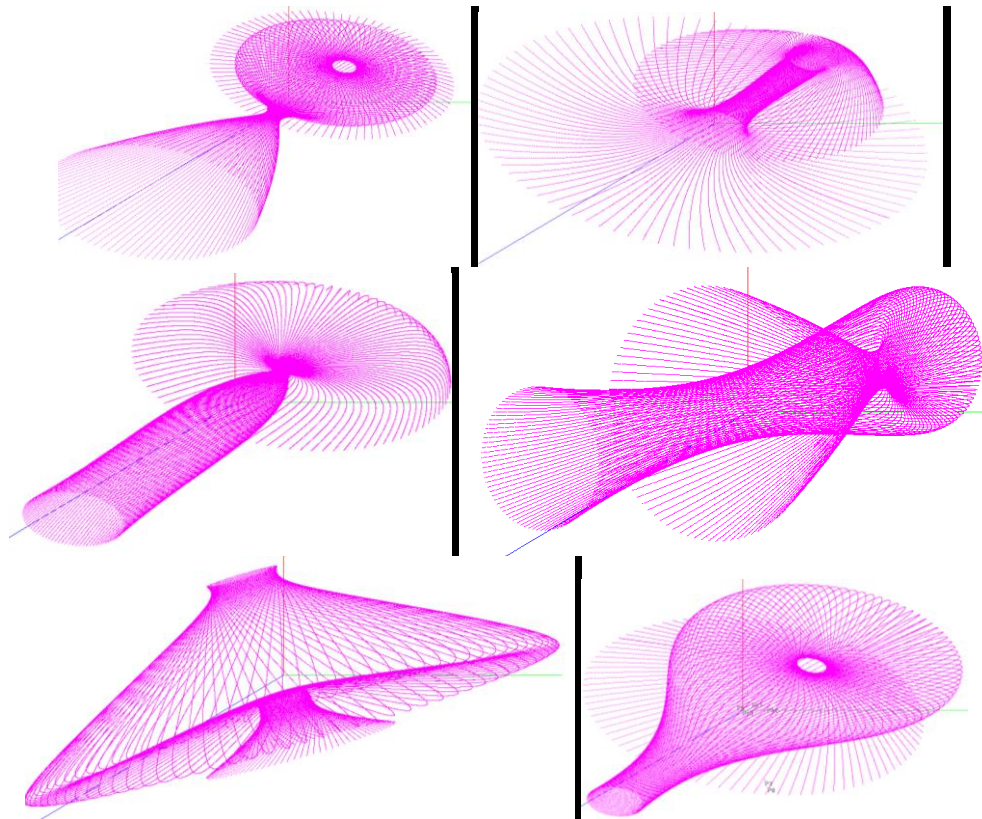
Figure 11-. Different models, and directions depending on coordinate axes by using free- axis, dependent on the values of parameters t and s to increase (i.e. + t_1, $s_{1,}$ and s_2) taken at the same time.

5. Conclusions
   The proposed mathematical model has been successfully controlling and generating different modelling surfaces by using 3D bi-cubic B-spline surfaces in matrices form (Eq (9)). In other words, by using parameters *t*, and *s* with different values in increasing (i.e. positive), and decreasing (i.e. negative) on different axes(x, y, z, and free axis). When applied this work the result that appeared may discuss in the following points:-
1. Different surfaces based on different values in parameters *t*, and *s* coefficient of control points with different axes.
2. The effects on the surfaces are clear in using the parameters $t_{1,}$ $t_{2,}$ and t_3 when increasing or decreasing their values. Basically, same as t parameter, the s parameter has direct effect on the generated surfaces when increasing or decreasing (s_1, s_2, and s_3).
3. Different cases of the surface based on different axes(x, or y, or z, or free axes).
4. When using free axis, different surfaces based on quadrants of coordinates and on parameters *t*, and/or *s* in different values, either increasing and/or decreasing.
5. Different surfaces depend on bi-cubic 3D B-spline surface in matrix form, based on 16 control points, if we change this surface, all the above designs change with new surface also.
6. One can use 2D for another aim of this work.
7. All changes in designs are made without any changing the control points.

References
1.   Abdul-Hossen, A.   M. J. **2016** . 3D Surface Reconstruction of Mathematical Modelling Used for Controlling the Generation of Different Bi-cubic B-Spline in Matrix Form Without Changing the Control Points. *Eng. & Tech. Journal*, 34 (1), part(B), pp:136-152.
2.   Gallier, J. **2000** . *Curves and Surfaces in Geometric Modelling Theory and Algorithms*. Morgan Kaufmann Publishers.

3.   Rahma, A. M. S. Abdulla, A. And Abdul-Hossen, A. M. J., **2012**. Modified Method for Generating B-Spline Curves of Degree Three and Their Controlling. *Eng. & Tech. Journal*, 30(1), pp: 109-125.
4.   Abdul-Hossen, A. M. J. Hammadie, A. H. and Saeid, A. **2012**. Modified Method for Controlling, and Generating Design by Using Cubic Bezier Surface in 3D. *Journal of Babylon University*, 20(1), pp: 53-65.
5.   Solomon, D. **2006**. *Curves and Surfaces for Computer Graphics*. Springer Science + Business Media. Inc.
6.   Zamani, M. **2009**. An Investigation of B-Spline and Bezier Method s for Interpolation of Data. *IEEE Contemporary Engineering Sciences*,.2(8), pp: 361-371.
7.   Eger ton, P. A. and Hall, W. S. **1989**. *Computer Graphics*. Prentice Hall Europe.
8.   Gerald, C. F and Wheatley, P. O. **1999**. *Applied Numerical Analysis*. Addison Wesley.
9.   Buss, S. R., **2003**. *3-D Computer Graphics A Mathematical Introduction*. With Open GL," Cambridge University Press.
10. Langley, E. **2004**. *Mathematics for 3D Gage Programming and Commuter Graphics*. Charles River Medal. Inc