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Global Stability of an epidemic model with vaccine involving stage structure

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Abstract

In this paper a mathematical model that analytically as well as numerically the flow of infection disease in a population is proposed and studied. It is assumed that the disease divided the population into five classes: immature susceptible individuals (S_1), mature individuals (S_2), infectious individual (I), removal individuals (R) and vaccine population (V). The existence, uniqueness and boundedness of the solution of the model are discussed. The local and global stability of the model is studied. Finally the global dynamics of the proposed model is studied numerically.

Keywords: Epidemic model, Stage Structures, Vaccination, Local and Global Sta+bility.

الاستقرارية الشاملة لنموذج وبائي مع اللقاح متضمنا مراحل عمرية

حسن فاضل الحسيني

قسم الرياضيات ، كلية العلوم ، جامعة بغداد ، بغداد ، العراق

الخلاصة

في هذا البحث أقترحنا وحللنا نموذج رياضي لمرض معدي. أفترضنا أن المرض المعدي يقسم المجتمع الى خمسة أصناف : مجتمع سليم غير بالغ (S_1)، مجتمع سليم بالغ (S_2)، مجتمع مصابين (I)، مجتمع مشافين (R) ومجتمع ملقحين (V) . بحثنا كلا من وجود ، وحدانية وقيود الحل للنموذج ، وحددنا شروط الاستقرارية المحلية والشاملة لكل نقاط التوزان . اخيرا ، أستخدمنا المحاكاة العددية لدراسة الديناميكية الشاملة للنموذج

1. Introduction

Epidemic diseases that spread among communities severity vary depending on the age of the infected person, there are diseases of children without admiration and diseases afflict only adults, and also there are illnesses they suffer together. Therefore, the age a major impact on the spread of the disease usually is immune to the highest adult children and thus less morbidity. In this research, we studied the effect of growth stages in the spread of disease. There are many studies in this area see [1-5]. Hence also it appeared another problem is the diversity and vaccination, treatment and that have a significant impact on the spread of disease, or eliminate them. And here it came the importance of mathematical models to help understand and explain and give appropriate solutions and perceptions.

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There are several studies presented in this field, for example, Kribs-Zaleta and Velasco-Hernandez in 2000 [6] have been proposed and studied the *SIS* epidemic model with vaccine for the diseases such as pertussis and tuberculosis, later on Arino et al. [7], generalized this model by allowing individuals recovering from the diseases to go into a temporarily immune class rather than directly back in to the susceptible class. Kribs-Zaleta and Martcheva [8] investigated the effects of a vaccination campaign upon the spread of a non-fatal disease such as Hepatitis A, B. Alexander et al. [9] and Shim [10] are discussing the transmission dynamics of influenza with vaccination through using *SVIR* models. d' Onofrio et al. [11] gave a family of models for information related vaccinating behavior. In this paper, a mathematical model of epidemic disease, in which it is assumed that the disease transmitted by contact with stage structures and involving vaccination is proposed and studied. The local as well as global stability analysis of this model is investigated.

2. The derivation model

Consider an epidemic model in which the total population (say N(t)) at time t are divided into five sub classes the immature susceptible individuals $S_1(t)$, mature susceptible individuals $S_2(t)$, vaccinated individuals $V(t)$, at time t, infected individuals $I(t)$ at time t and removed individuals $R(t)$ at time t. such that model can be represented as follows :-

$$\begin{aligned}
\frac{dS_1(t)}{dt} &= \Lambda-(\alpha+\psi+\mu)S_1(t)-\beta_1 S_1(t)I(t)\\
\frac{dS_2(t)}{dt} &= \alpha S_1(t)-\beta_2 S_2(t)I(t)-\mu S_2(t)\\
\frac{dV(t)}{dt} &= \psi S_1(t)-\theta V(t)I(t)-\mu V(t)\\
\frac{dI(t)}{dt} &= \beta_1 S_1(t)I(t)+\beta_2 S_2(t)I(t)+\theta V(t)I(t)-(\mu+\gamma)I(t)\\
\frac{dR}{dt} &= \gamma I(t)-\mu R(t)
\end{aligned} \qquad (1)$$

Here, all parameters in this model are positive and we can description in the following table

Table 1- Description of Parameters

Parameter	Description
Λ, α	**Recruitment of Population**
ψ	**Vaccination rate**
μ	**Natural death**
β_1	**Infection rate**
β_2	**Infection rate**
θ	**Failure Vaccine**
γ	**Recovery rate**

Theorem (1): All the solutions of system (1) are initiate in R_+^5, are uniformly bounde

Proof : Let $(S_1(t),S_2(t),V(t),I(t),R(t))$ be any solution of the system (1) with non-negative initial condition $(S_1(0),S_2(0),V(0),I(0),R(0))$, since $N(t)=S_1(t)+S_2(t)+V(t)+I(t)+R(t)$ and then:

$\frac{dN}{dt}=\frac{dS_1}{dt}+\frac{dS_2}{dt}+\frac{dV}{dt}+\frac{dI}{dt}+\frac{dR}{dt}$ which gives $\frac{dN}{dt}=\Lambda-\mu N$

Now, by using Gronweall lemma [3] , it obtains that $N(t)\leq\frac{\Lambda}{\mu}(1-e^{-\mu t})+N(0)e^{-\mu t}$

Therefore, $N(t)\leq\frac{\Lambda}{\mu}$ as $t\to\infty$, that is independent of the conditions and hence the proof is complete.

3. Existence of equilibrium points of system

There are three equilibrium points of system(1). First, because the removal class R is related with infected class only. In fact, if I= 0 then R= 0 . While, if $I=I_c$ is a positive constant, then R approaches to :-

$$R=\frac{\gamma I_c}{\mu} \tag{2}$$

Consequently, for simplifying, system (1) can be reduced to the following system, in which we can determine the value of I, by solving it, and then using Eq.(2).

$$\begin{aligned}
\frac{dS_1(t)}{dt}&=\Lambda-(\alpha+\psi+\mu)S_1(t)-\beta_1S_1(t)I(t)\\
\frac{dS_2(t)}{dt}&=\alpha S_1(t)-\beta_2S_2(t)I(t)-\mu S_2(t)\\
\frac{dV(t)}{dt}&=\psi S_1(t)-\theta V(t)I(t)-\mu V(t)\\
\frac{dI(t)}{dt}&=\beta_1S_1(t)I(t)+\beta_2S_2(t)I(t)+\theta V(t)I(t)-(\mu+\gamma)I(t)
\end{aligned} \tag{3}$$

Clearly, system(3) has at most two biologically feasible points. These two points can be described as follows:

In case of absence of disease ($I=0$), there is an equilibrium point represented by $E^o=\left(S_1^o,S_2^o,V^o,0\right)$ and it's called the disease free equilibrium point, where

$$S_1^o=\frac{\Lambda}{\alpha+\psi+\mu},\quad S_2^o=\frac{\alpha\Lambda}{\mu(\alpha+\psi+\mu)}\ \ and\ \ V^o=\frac{\psi\Lambda}{\mu(\alpha+\psi+\mu)} \tag{4}$$

In case of existence of disease ($I>0$), system (3) has an equilibrium point represented by $E^*=\left(S_1^*,S_2^*,V^*,I^*\right)$ and it's called endemic equilibrium point, where

$$\begin{aligned}
S_1^*&=\frac{\Lambda}{\alpha+\psi+\mu+\beta_1I^*},\ S_2^*=\frac{\alpha\Lambda}{(\mu+\beta_2I^*)(\alpha+\psi+\mu+\beta_1I^*)}\\
V^*&=\frac{\psi\Lambda}{(\theta I^*+\mu)(\alpha+\psi+\mu+\beta_1I^*)}
\end{aligned} \tag{5a}$$

While I^* represents a positive root for the following equation

$$D_1{I^*}^4+D_2{I^*}^3+D_3{I^*}^2+D_4I^*=0 \tag{5b}$$

Here,

$$D_1 = -\theta\beta_1\beta_2(\mu+\gamma)$$
$$D_2 = \beta_1\beta_2\theta\Lambda - (\mu+\gamma)[\theta\beta_2(\alpha+\psi+\mu)+\beta_1\mu(\beta_2+\theta)]$$
$$D_3 = \beta_1\Lambda\mu(\beta_2+\theta)+\beta_2\Lambda\theta(\alpha+\psi)-\mu(\mu+\gamma)[(\beta_2+\theta)(\alpha+\psi+\mu)+\mu\beta_1]$$
$$D_4 = \Lambda\mu(\alpha\beta_2+\theta\psi+\beta_1\mu)-\mu^2(\mu+\gamma)(\alpha+\psi+\mu)$$

Obviously, the endemic equilibrium point exists unique in the interior of positive octant of S_1S_2VI-space if and only if the following condition holds

$$D_2>0,\ D_3>0 \ and \ \ D_4>0 \tag{5c}$$

4. Local and global stability of disease free equilibrium point

In this section, the local and global stable analysis of the disease free equilibrium point E^o is studied as shown in the following theorems.

Theorem (2): The disease free equilibrium point $E^o=\left(S_1^o,S_2^o,V^o,0\right)$ of system (3) is locally asymptotically stable provided that:-

$$\beta_1S_1^o+\beta_2S_2^o+\theta V^o \ < \ (\mu+\gamma) \tag{6}$$

Proof : Clearly the Jacobian matrix of system(3) at E^o which is denoted by $J\left(E^o\right)$ can be written

$$J(E^o)=\begin{bmatrix} -(\alpha+\psi+\mu) & 0 & 0 & \beta_1S_1^o \\ \alpha & -\mu & 0 & -\beta_2S_2^o \\ \psi & 0 & -\mu & -\theta V^o \\ 0 & 0 & 0 & \beta_1S_1^o+\beta_2S_2^o+\theta V^o-(\mu+\gamma) \end{bmatrix}$$

Accordingly, the eigenvalues of $J(E^o)$ are given by

$\lambda_{S_1}^o=-(\alpha+\psi+\mu)<0$, $\lambda_{S_2}^o=-\mu<0$, $\lambda_V^o=-\mu<0$, $\lambda_I^o=\beta_1S_1^o+\beta_2S_2^o+\theta V^o-(\mu+\gamma)$

Therefore, all the eigenvalues will be negative and hence the disease free equilibrium point is locally asymptotically stable if and only if condition (6) is holds

Theorem (3) : Assume that, the disease free equilibrium point E^o is locally asymptotically stable. Then it is a globally asymptotically stable in the sub region of R_+^4 that satisfies the following sufficient conditions:

$$\left(\frac{\alpha}{S_2^o}\right)^2 \le 2\left(\frac{\alpha+\psi+\mu}{S_1^o}\right)\left(\frac{\mu}{S_2^o}\right) \tag{7a}$$

$$\left(\frac{\psi}{V}\right)^2 \le 2\left(\frac{\alpha+\psi+\mu}{S_1^o}\right)\left(\frac{\mu}{V}\right) \tag{7b}$$

Proof: Consider the following positive definite function

$$L_1=\left(S_1-S_1^o-S_1^o\ln\frac{S_1}{S_1^o}\right)+\left(S_2-S_2^o-S_2^o\ln\frac{S_2}{S_2^o}\right)+\left(V-V^o-V^o\ln\frac{V}{V^o}\right)+I$$

Clearly, $L_1:R_+^4\to R$ is a continuously differentiable function such that $L_1\left(S_1^o,S_2^o,V^o,0\right)=0$ and $L_1\left(S_1,S_2,V,I\right)>0$, $\forall\left(S_1,S_2,V,I\right)\neq\left(S_1^o,S_2^o,V^o,0\right)$. Further, we have

$$\frac{dL_1}{dt} = -\frac{(\alpha+\psi+\mu)}{2S_1}(S_1-S_1^o)^2 + \frac{\alpha}{S_2}(S_1-S_1^o)(S_2-S_2^o) - \frac{\mu}{S_2}(S_2-S_2^o)^2$$
$$-\frac{(\alpha+\psi+\mu)}{2S_1}(S_1-S_1^o)^2 + \frac{\psi}{V}(S_1-S_1^o)(V-V^o) - \frac{\mu}{V}(V-V^o)^2$$
$$+(\beta_1 S_1^o + \beta_2 S_2^o + \theta V^o)I - (\mu+\gamma)I$$

Now, by doing some algebraic manipulation and using the condition (6), we get

$$\frac{dL_1}{dt} \le -\left[\sqrt{\frac{\alpha+\psi+\mu}{2S_1}}(S_1-S_1^o) - \sqrt{\frac{\mu}{S_2}}(S_2-S_2^o)\right]^2$$
$$-\left[\sqrt{\frac{\alpha+\psi+\mu}{2S_1}}(S_1-S_1^o) - \sqrt{\frac{\mu}{V}}(V-V^o)\right]^2$$
$$+(\beta_1 S_1^o + \beta_2 S_2^o + \theta V^o)I - (\mu+\gamma)I$$

Consequently, due to condition (7a)-(7b), $\frac{dL_1}{dt} < 0$ is negative definite and hence L_1 is a Lyapunov function with respect to E^o in the region that satisfies the given condition. Thus E^o is a globally asymptotically stable and the proof is complete.

The next theorem deals with the stability of the positive equilibrium point using the Lyapunov function.

Theorem (4): Assume that the endemic equilibrium point $E^* = \left(S_1^*, S_2^*, V^*, I^*\right)$ exists then it is a asymptotically stable in the sub region $\Omega_2 \subseteq R_+^4$ that satisfy the following sufficient conditions

$$I^* < S_1 \quad (8a)$$
$$\beta_1 S_1 + \beta_2 S_2 + \theta V < \mu + \gamma \quad (8b)$$
$$\alpha^2 < d_{11} d_{22} \quad (8c)$$
$$\psi^2 < d_{11} d_{33} \quad (8d)$$
$$d_{24}^2 < d_{22} d_{44} \quad (8e)$$
$$d_{34}^2 < d_{33} d_{44} \quad (8f)$$

Where

$$d_{11} = \alpha + \psi + \mu + \beta_1 I^*, \quad d_{22} = \beta_2 I^* + \mu, \quad d_{33} = \theta I^* + \mu, \quad d_{34} = \theta V - \theta I^*$$
$$d_{44} = \mu + \gamma - (\beta_1 S_1 + \beta_2 S_2 + \theta V), \quad d_{14} = \beta_1 S_1 - \beta_1 I^*, \quad d_{24} = \beta_2 S_2 - \beta_2 I^*$$

Proof: Consider the following function

$$L_2(S_1, S_2, V, I) = \frac{1}{2}(S_1 - S_1^*)^2 + \frac{1}{2}(S_2 - S_2^*)^2 + \frac{1}{2}(V - V^*)^2 + \frac{1}{2}(I - I^*)^2$$

Clearly $L_2 : R_+^4 \to R$ and it is a continuously differentiable function, in addition, $L_2(S_1^*, S_2^*, V^*, I^*) = 0$ while $L_2(S_1, S_2, V, I) > 0$, $\forall\, (S_1, S_2, V, I) \neq (S_1^*, S_2^*, V^*, I^*)$. Further by taking the derivative with respect to the time and simplifying the resulting terms, we get that

$$\frac{dL_2}{dt} = -\left[\frac{d_{11}}{2}(S_1 - S_1^*)^2 - \alpha(S_1 - S_1^*)(S_2 - S_2^*) + \frac{d_{22}}{2}(S_2 - S_2^*)^2\right]$$
$$-\left[\frac{d_{11}}{2}(S_1 - S_1^*)^2 - \psi(S_1 - S_1^*)(V - V^*) + \frac{d_{33}}{2}(V - V^*)^2\right]$$
$$-\left[\frac{d_{22}}{2}(S_2 - S_2^*)^2 + d_{24}(S_2 - S_2^*)(I - I^*) + \frac{d_{44}}{2}(I - I^*)^2\right]$$
$$-\left[\frac{d_{33}}{2}(V - V^*)^2 + d_{34}(V - V^*)(I - I^*) + \frac{d_{44}}{2}(I - I^*)^2\right]$$
$$-d_{14}(S_1 - S_1^*)(I - I^*)$$

It is easy to verify that, d_{14} *and* d_{44} are positive provided that conditions (8a)-(8b) are satisfied respectively. Consequently, due to conditions (8c)-(8f), we have

$$\frac{dL_2}{dt} < -\left[\sqrt{\frac{d_{11}}{2}}(S_1 - S_1^*) - \sqrt{\frac{d_{22}}{2}}(S_2 - S_2^*)\right]^2 - \left[\sqrt{\frac{d_{11}}{2}}(S_1 - S_1^*) - \sqrt{\frac{d_{33}}{2}}(V - V^*)\right]^2$$
$$-\left[\sqrt{\frac{d_{33}}{2}}(V - V^*) + \sqrt{\frac{d_{44}}{2}}(I - I^*)\right]^2 - \left[\sqrt{\frac{d_{22}}{2}}(S_2 - S_2^*) + \sqrt{\frac{d_{44}}{2}}(I - I^*)\right]^2$$
$$-d_{14}(S_1 - S_1^*)(I - I^*)$$

Therefore, $\frac{dL_2}{dt}$ is negative definite and hence L_2 is a Lyapunov function with respect to E^* in the sub region Ω_2. So E^* is a asymptotically stable.

Note that the function L_2 is approaching to infinity as any of its components do the same and its positive definite on R_+^4, however its derivative is negative definite on the sub region Ω_2 due to the given sufficient conditions. Therefore E^* is a globally asymptotically stable within Ω_2.

5. Numerical Simulations

In this section, the global dynamics of system (1) is investigated numerically for different sets of initial values and different sets of parameters values. The objectives of such investigation are determine the effect of varying the parameters values and confirm our obtained results. It is observed that, for the following biologically feasible set of hypothetical parameters values:

$$\Lambda = 100,\ \alpha = 0.61,\ \psi = 0.1,\ \mu = 0.1,\ \beta_1 = 0.0001,$$
$$\beta_2 = 0.001,\ \theta = 0.0001,\ \gamma = 0.11 \qquad (9)$$

The solution of system (1) approaches asymptotically to the endemic equilibrium point $E^* = (321, 149.1, 287.8, 115.2, 126.7)$ as shown in Figure- 1, started from different sets of initial points.

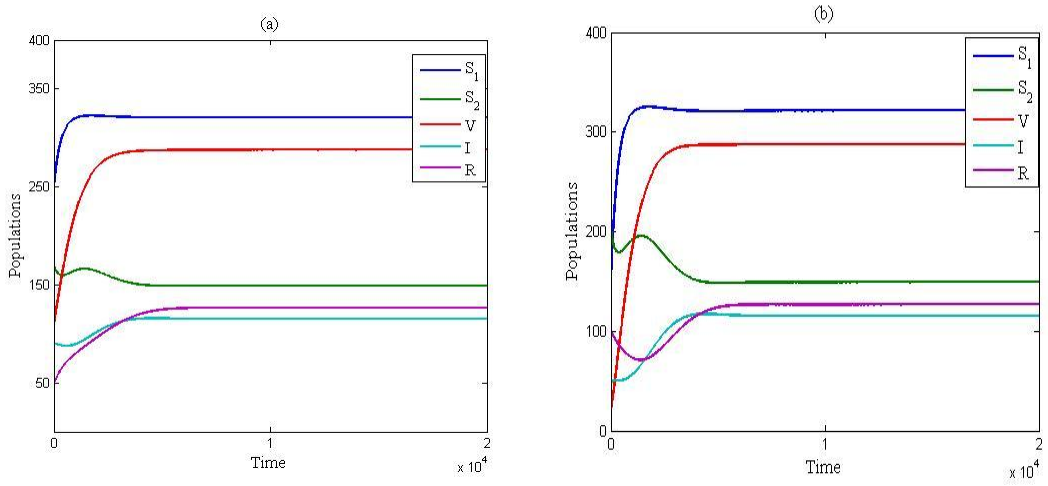

Figure 1- Globally asymptotically stable positive equilibrium point of system (1) for the parameters set (9), started from different sets of initial point.

Obviously, Figure-1 shows that, system (1) approaches asymptotically to the globally stable endemic equilibrium point E^* from different sets of initial conditions. This is indicates to the existence of globally asymptotically stable of system (1) in the interior of positive octant, which represents the persistence of all the species too.
On the other hand, system (1) for the following set of hypothetical data approaches asymptotically to the DFE as shown in Figure- 2

$$\Lambda = 100,\ \alpha = 0.61, \psi = 0.1,\ \mu = 0.1,\ \beta_1 = 0.0001,\ \beta_2 = 0.0001, \theta = 0.0001,\ \gamma = 0.11 \quad (10)$$

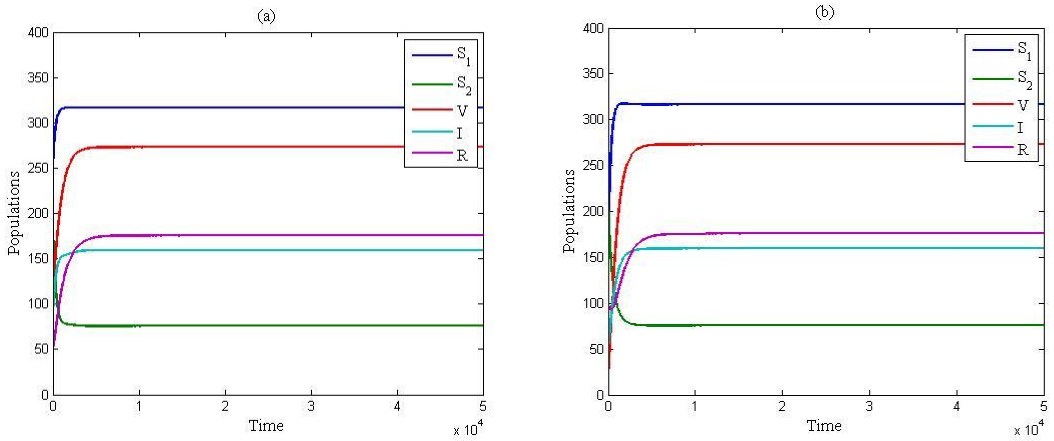

Figure 2- Globally asymptotically stable DFE of system (1) for the parameters set (10)

Clearly Figure- 2 shows the approaching of the solution of system (1) asymptotically to the free disease equilibrium point from different initial values as the infection rate reduced to $\beta_2 = 0.0001$.

Varying of the parameters values $(\Lambda, \alpha, \theta, \beta_1, \beta_2)$ don't have qualitative effects on the dynamics of system (1) rather than that they have quantitative effects on the value of positive equilibrium point.

For the parameters values given in Eq.(9) with $\psi \geq 2.4$ the solution of the system (1) approaches asymptotically to disease free equilibrium point $E^o = (38.4, 38.4, 923.02, 0, 0)$ as shown in the typical figure given by Figure- 3 below.

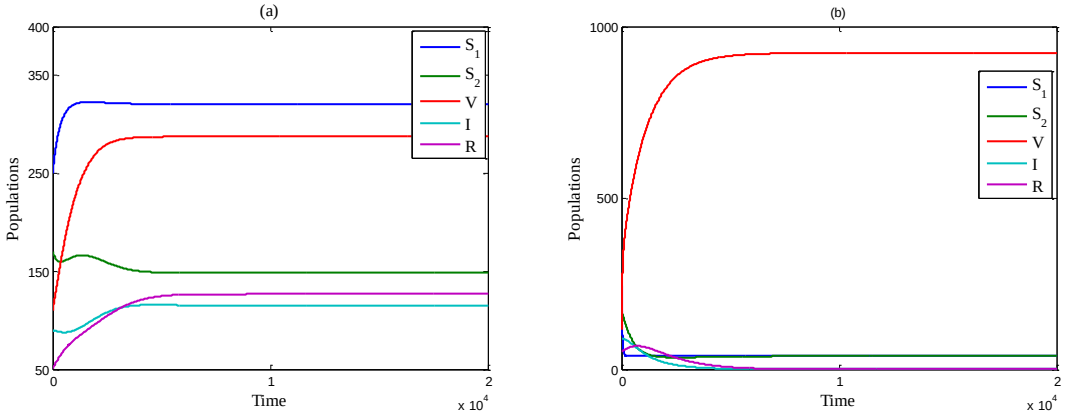

Figure 3- Time series of the solution of system (1) for the data (9) with different values of ψ **(a)** Globally asymptotically stable endemic equilibrium point for $\psi = 0.1$ **(b)** Globally asymptotically stable disease free equilibrium point E^o for $\psi = 2.5$.

According to the Figure- 3, it is clear that as the ψ increase the solution of system (1) approaches asymptotically to the E^o equilibrium point.

Now for the data given by Eq. (9) with $\mu \geq 0.17$, the solution of system (1) approaches asymptotically to disease free equilibrium point as shown in the following typical figure, Figure 4. However the system still approaches to the endemic equilibrium point for other values of μ

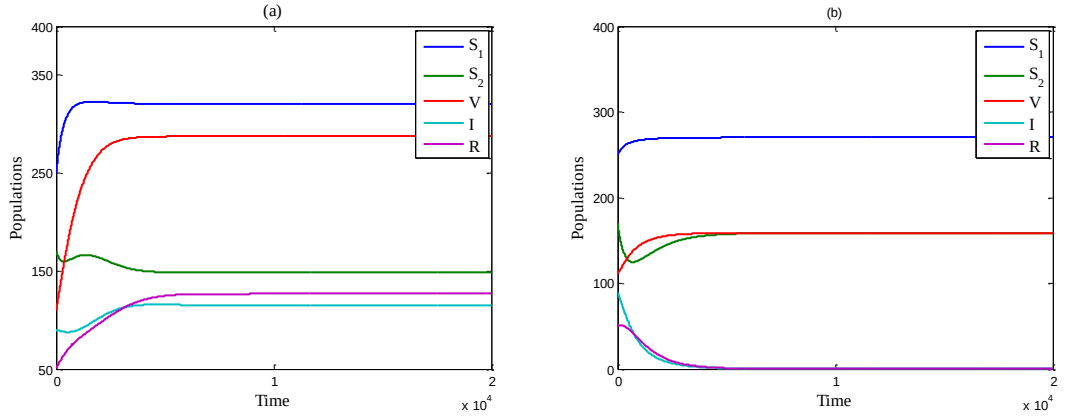

Figure 4- Time series of the solution of system (1) for the data (9) with different values of μ **(a)** Globally asymptotically stable endemic equilibrium point for $\mu = 0.1$ **(b)** Globally asymptotically stable disease free equilibrium point E^o for $\mu = 0.2$.

Now by varying the recovery rate of the population in the range $\gamma \geq 0.37$, keeping the rest of parameters values as in Eq. (9), system (1) approaches asymptotically to the disease free equilibrium point E^o as shown in the typical figure, Figure- 5

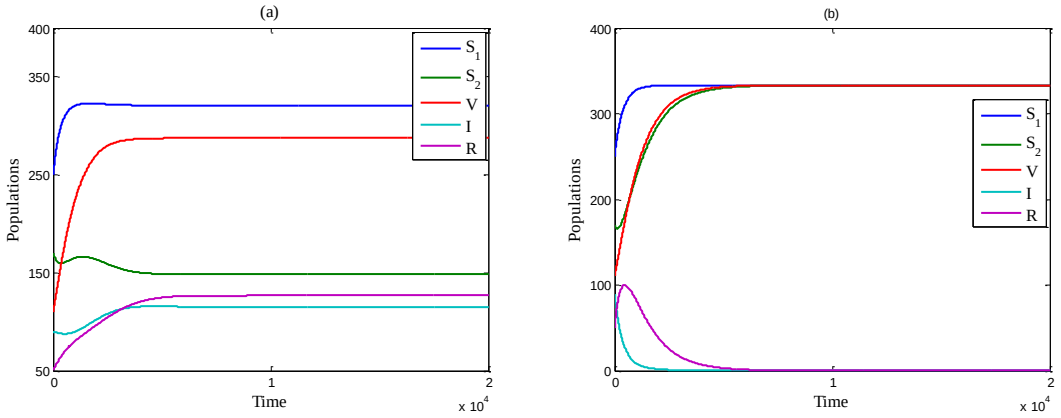

Figure 5- Time series of the solution of system (1) for the data (9) with different values of γ (a) Globally asymptotically stable endemic equilibrium point for $\gamma = 0.11$ (b) Globally asymptotically stable disease free equilibrium point E^o for $\gamma = 0.4$.

Clearly, Figure-5 explains the approaches of the solution of system (1) to the disease free equilibrium point as the recovery rate of the population increase.

6. Conclusions and discussion

The objective of this study is to understand the effects of all factors, which helping the spread of this type of disease and hence get the capability of control the disease.

The boundedness of the system has been discussed. The existence conditions of all possible equilibrium points of the system are established. All possible equilibrium points with their local and global stability are investigated. The qualitative dynamical behavior as a function of varying the parameters values is studied analytically as well as numerically. Finally, for the biologically feasible set of hypothetical data as given in Eq. (9), the system (1) is solved numerically and the obtained results are explained in some typical figures and we will summarize as follows.

1. System (1) does not have periodic dynamics; instead of that it approaches either to the disease free equilibrium point or else to endemic equilibrium point depending on the value of reproduction number.
2. For the set of hypothetical parameters values given by Eq.(9), the system (1) approached asymptotically to the global stable endemic equilibrium point E^*
3. Increasing the recovery rate ψ above a specific value causes bifurcation in the system and the trajectory transferred from the endemic equilibrium point to the disease free equilibrium point asymptotically. Otherwise the system still approaches to the endemic point.
4. Increasing the inverse of natural death μ above 0.17 in Eq. (9) caused destabilizing to the endemic equilibrium point and the trajectories of system (1) approached asymptotically to the free disease equilibrium point. Otherwise the system still has a globally asymptotically stable endemic equilibrium point.
5. Further increasing the recovery rate γ above 0.37 in Eq. (9) causes bifurcation in the system and the trajectory transferred from the endemic point to the disease free equilibrium point asymptotically an hence the system will losses the persistence. Otherwise the system still approaches to the endemic point.
6. Finally all the other parameters have quantitative change but note qualitative change on the stability of the positive equilibrium point.

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