DOI: 10.24996/ijs.2019.60.2.17

ISSN: 0067-2904

On μ-lifting Modules

Enas Mustafa Kamil*, Wasan Khalid
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Abstract
Let *R* be a ring with identity and let *M* be a left *R*-module. *M* is called μ-lifting modulei f for every sub module *A* of *M*, There exists a direct summand *D* of *M* such that $M = D \oplus D'$, for some sub module *D'* of *M* such that $A \leq D$ and $A \cap D' <<_\mu D'$. The aim of this paper is to introduce properties of μ-lifting modules. Especially, we give characterizations of μ-lifting modules. On the other hand, the notion of amply μ-supplemented is studied as a generalization of amply supplemented modules, we show that if *M* is amply μ-supplemented such that every μ-supplement sub module of *M* is a direct summand, then *M* is μ-lifting module. Finally, we give some conditions under which the quotient and direct sum of μ-lifting modules is μ-lifting.

Keywords: μ-coessential sub modules, μ-lifting modules.

حول مقاسات الرفع من النمط-μ

إيناس مصطفى كامل*، وسن خالد
قسم الرياضيات، كلية العلوم ، جامعة بغداد، بغداد، العراق

الخلاصة
لتكن *R* حلقة ذات عنصر محايد و ليكن *M* مقاسا ايسر معرف عليها. يدعى *M* مقاس رفع من النمط- μ اذا كان لكل مقاس جزئي *A* من *M* يوجد مركبة جداء مباشر *D* من *M* بحيث ان $M = D \oplus D'$ و *D'* هو مقاس جزئي من *M* و *A* مقاس جزئي من *D* و $A \cap D' <<_\mu D'$. الغرض من هذا البحث هو تقديم خواص مقاسات الرفع من النمط-μ.سوف نعطي مكافئات لمقاسات الرفع من النمط -μ, من جهة اخرى موضوع المقاسات المكملة من النمط (μ- amply) دُرِسَت كتعميم للمقاسات المكملة من النمط (amply) . سوف نبرهن انه اذا كان *M* مقاس مكمل من النمط (μ-amply) بحيث ان كل مقاس جزئي مكمل من النمط -μ من *M* هو مركبة جداء مباشر , فأن *M* هو مقاس رفع من النمط-μ. أخيرا نعطي بعض الشروط لتكن القسمة و الجمع المباشر لمقاسات الرفع من النمط -μ هي ايضا مقاسات رفع من النمط -μ.

1. Introduction

Throughout this paper all rings have an identity and all modules will be until left *R*- module Let *M* be a module and let *A* be a sub module of *M*, *A* is called small in *M* (notation $A << M$) if $M \neq A + B$, for any proper sub module *B* of *M*, see [1] and [2]. As a generalization of small sub modules we introduced the concept of μ- small sub modules, *A* sub module *A* of *M* is called μ- small sub module of *M* (denoted by $A <<_\mu M$) if whenever $M = A + X$, $\frac{M}{X}$ is cosingular, then $M = X$. See [3]. Let $A \leq B \leq M$. if $\frac{B}{A} << \frac{M}{A}$, then *A* is called a coessential sub module of *B* in M. We introduce μ-coessential sub module as a generalization of coessential sub module as follows. Let *M* be an *R*- module and let *X* and

*Email: enasmustafa90@yahoo.com

A be sub modules of *M* such that $X \leq A \leq M$, then *X* is Following , Oshiro [4], An *R*- module *M* is called lifting module if for every sub module *A* of *M* , there exists a direct summand *D* of *M* such that *D* is coessential sub module of *A* in *M*. This concept leads us to introduce the following concept. An *R*- module *M* is called µ-lifting module if for every sub module *A* of *M*, there exists a direct summand *D* of *M* such that $D \leq_{\mu ce} A$ in*M*.In this paper,we investigate characterizations and properties of µ-lifting modules.

In section2. We define µ-coessential sub modules. Also we list some of their important properties that are related to our work.

In section3. We define the notion of µ-lifting modules as a generalization of lifting modules with some examples and characterizations and basic properties. Also, we investigate various conditions for a direct sum of µ-lifting modules to be µ-lifting.

2. µ-Coessential submodules

Definition 2.1: Let M be an R- module and let X and A be sub modules of M such that $X \leq A \leq M$, then X is called **µ-coessential sub module** of A in M (briefly $X \leq_{\mu ce}$ A in M) if $\frac{A}{X} \ll_{\mu} \frac{M}{X}$.

Examples and Remarks 2.2

(1) Consider Z_8 as Z- module. It is easy to see that $\{\bar{0}, \bar{4}\} \leq_{\mu ce} \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ in Z_8.

(2) ConsidewZ_6 as *Z*- module. It is clear that $\{\bar{0}\}$ is not µ-coessential sub module of $\{\bar{0}, \bar{3}\}$. Suppose that $\frac{\{\bar{0},\bar{3}\}}{\{\bar{0}\}} \ll_{\mu} \frac{Z_6}{\{\bar{0}\}}$, it is easy to see that $\frac{\{\bar{0},\bar{3}\}}{\{\bar{0}\}} \cong \{\bar{0}, \bar{3}\}$ and $\frac{Z_6}{\{\bar{0}\}} \cong Z_6$ and hence $\{\bar{0}, \bar{3}\} \ll_{\mu} Z_6$ which is a contradiction.

(3) It is clear that µ-coessential sub module is a generalization of coessential sub module, but the converse is not true in general. For example. Consider Z_6 as Z_6- module. Since $\{\bar{0}, \bar{3}\} \ll_{\mu} Z_6$, then $\{\bar{0}\} \leq_{\mu ce} \{\bar{0}, \bar{3}\}$ in Z_6 but $\{\bar{0}\}$ is not coessential sub module of $\{\bar{0}, \bar{3}\}$ in Z_6. Note that they are equivalent when *M* is cosingular module

(4) Let *M* be an *R*- module. Then $A \ll_{\mu} M$ if and only if $\{0\} \leq_{\mu ce} A$ in *M*.

The following proposition gives a characterization of µ-coessential sub module.

Proposition 2.3: Let *M* be an *R*- module and let $A \leq B \leq M$ where $\frac{M}{A}$ is cosingular, then $A \leq_{\mu ce} B$ in *M* if and only if $M = B + X$ implies that $M = A + X$, for every sub module *X* of *M*.

Proof: Suppose that $A \leq_{\mu ce} B$ in *M* and let $M = B + X$, where $X \leq M$, then $\frac{M}{A} = \frac{B}{A} + \frac{X+A}{A}$. Since $\frac{M}{A}$ is cosingular, then $\frac{M}{A+X}$ is cosingular. But $\frac{\frac{M}{A}}{\frac{A+X}{A}} \cong \frac{M}{A+X}$, by third isomorphism theorem, therefore $\frac{\frac{M}{A}}{\frac{A+X}{A}}$ is cosingular. But $\frac{B}{A} \ll_{\mu} \frac{M}{A}$, therefore $\frac{M}{A} = \frac{X+A}{A}$. Thus $M = X + A$.

Conversely, let $\frac{M}{A} = \frac{B}{A} + \frac{U}{A}$, where $A \leq U$ with $\frac{M}{U}$ is cosingular. Then $M = B + U$ and $\frac{M}{U}$ is cosingular. So, by our assumption , we get $M = A + U$. But $A \leq U$, therefore $M = U$. Thus $A \leq_{\mu ce} B$ in *M*

The following propositions give some properties of µ-coessential sub modules which are needed later.

Proposition 2.4: Let *M* be an *R*- module and let $A \leq X \leq B \leq M$. Then $X \leq_{\mu ce} B$ in *M* if and only if $\frac{X}{A} \leq_{\mu ce} \frac{B}{A}$ in $\frac{M}{A}$.

Proof: Assume that $X \leq_{\mu ce} B$ in M. Since $\dfrac{B/A}{X/A} \cong \dfrac{B}{X}$ and $\dfrac{M/A}{X/A} \cong \dfrac{M}{X}$, by (the third isomorphism theorem) Then $\dfrac{B/A}{X/A} <<_{\mu} \dfrac{M/A}{X/A}$. Thus $\dfrac{X}{A} \leq_{\mu ce} \dfrac{B}{A}$ in $\dfrac{M}{A}$

Conversely, Suppose that $\dfrac{X}{A} \leq_{\mu ce} \dfrac{B}{A}$ in $\dfrac{M}{A}$, Since $\dfrac{B/A}{X/A} \cong \dfrac{B}{X}$ and $\dfrac{M/A}{X/A} \cong \dfrac{M}{X}$, by (the third isomorphism theorem) Then $\dfrac{B}{X} <<_{\mu} \dfrac{M}{X}$. Thus $X \leq_{\mu ce} B$ in M.

Proposition 2.5: Let M be an R- module and let $A \leq B \leq C \leq M$. Then $A \leq_{\mu ce} C$ in M if and only if $A \leq_{\mu ce} B$ in M and $B \leq_{\mu ce} C$ in M

Proof: Suppose that $A \leq_{\mu ce} C$ in M. Since $\dfrac{B}{A} \leq \dfrac{C}{A} \leq \dfrac{M}{A}$, then $\dfrac{B}{A} <<_{\mu} \dfrac{M}{A}$ and hence $A \leq_{\mu ce} B$ in M. Now, define g: $\dfrac{M}{A} \to \dfrac{M}{B}$ by g (m + A)= m + B for all $m \in M$. early g is an epimorphosis. Since $A \leq_{\mu ce} C$ in M, hence g $(\dfrac{C}{A}) = \dfrac{C}{B} <<_{\mu} \dfrac{M}{B}$. Thus $B \leq_{\mu ce} C$ in M

Conversely, assume that $A \leq_{\mu ce} B$ in M and $B \leq_{\mu ce} C$ in M, to show that $A \leq_{\mu ce} C$ in M. Let $\dfrac{M}{A} = \dfrac{C}{A} + \dfrac{U}{A}$, $\dfrac{M}{U}$ is cosingular, then M = C + U and hence $\dfrac{M}{B} = \dfrac{C+U}{B} = \dfrac{C}{B} + \dfrac{U+B}{B}$. Since $\dfrac{M}{U}$ is cosingular , then $\dfrac{M}{U+B}$ is cosingular . But $\dfrac{C}{B} <<_{\mu} \dfrac{M}{B}$, therefore $\dfrac{M}{B} = \dfrac{U+B}{B}$ and hence M = U + B , then $\dfrac{M}{A} = \dfrac{U}{A} + \dfrac{B}{A}$. Since $\dfrac{B}{A} <<_{\mu} \dfrac{M}{A}$ and $\dfrac{M}{U}$ is cosingular, then $\dfrac{M}{A} = \dfrac{U}{A}$. Thus M = U.

Proposition 2. 6: Let M be an R- module. If $A \leq_{\mu ce} B$ in M and $X \leq_{\mu ce} C$ in M, then $A + X \leq_{\mu ce} B + C$ in M

Proof: Suppose that $A \leq_{\mu ce} B$ in M and $X \leq_{\mu ce} C$ in M. To show that $A + X \leq_{\mu ce} B + C$ in M, let f: $\dfrac{M}{A} \to \dfrac{M}{A+X}$ be a map defined by f (m + A) = m + (A+X) for each $m \in M$ and g : $\dfrac{M}{X} \to \dfrac{M}{A+X}$ be a map defined by g (m + X) = m + (A+X) for each $m \in M$. Clearly each f and g are epimorphosis. Since $\dfrac{B}{A} <<_{\mu} \dfrac{M}{A}$ and $\dfrac{C}{X} <<_{\mu} \dfrac{M}{X}$, then f $(\dfrac{B}{A}) = \dfrac{(B+X)}{(A+X)} <<_{\mu} \dfrac{M}{A+X}$ and g $(\dfrac{C}{X}) = \dfrac{(C+X)}{(A+X)} <<_{\mu} \dfrac{M}{A+X}$, by [3, prop. (2.14)] and hence $\dfrac{(B+X)}{(A+X)} + \dfrac{(C+X)}{(A+X)} = \dfrac{(B+C)}{(A+X)} <<_{\mu} \dfrac{M}{A+X}$, by [3,prop. (2.14)]. Thus $A + X \leq_{\mu ce} B + C$ in M

Proposition 2.7: Let M be an R- module. If $A \leq_{\mu ce} B$ in M and $X \leq M$, then $A+X \leq_{\mu ce} B+X$ in M. The converse is true if $X <<_{\mu} M$.

Proof: Assume that $A \leq_{\mu ce} B$ in M and $X \leq M$. Since $X \leq_{\mu ce} X$ in M, then $A+X \leq_{\mu ce} B+X$ in M , by prop. (2.6).

Conversely, Suppose that A+X $\leq_{\mu ce}$ B+X in M and X<<$_\mu$ M. To show that A $\leq_{\mu ce}$ B in M, let $\frac{M}{A}$ = $\frac{B}{A}$ + $\frac{U}{A}$, $\frac{M}{U}$ is cosingular, then M = B+U, hence $\frac{M}{A+X}=\frac{(B+X)}{(A+X)}+\frac{(U+X)}{(A+X)}$. Since $\frac{M}{U}$ is cosingular, then $\frac{M}{U+X}$ is cosingular. But $\frac{(B+X)}{(A+X)}$ <<$_\mu$ $\frac{M}{A+X}$, therefore $\frac{M}{A+X}$ = $\frac{(U+X)}{(A+X)}$ and hence M = U+X. Since $\frac{M}{U}$ is cosingular and X<<$_\mu$ M, then M = U. Thus A $\leq_{\mu ce}$ B in M

Proposition 2.8: Let M be an R- module and let X<<$_\mu$ M. If A $\leq_{\mu ce}$ B in M , then A $\leq_{\mu ce}$ B+X in M.

Proof: Suppose that A $\leq_{\mu ce}$ B in M and X<<$_\mu$ M. To show that A $\leq_{\mu ce}$ B+X in M let $\frac{M}{A}$ = $\frac{B+X}{A}$ + $\frac{U}{A}$, $\frac{M}{U}$ is cosingular. Hence M = B+X+U and $\frac{M}{U+B}$ is cosingular. Since X<<$_\mu$ M, then M = B+U, $\frac{M}{A}=\frac{B}{A}+\frac{U}{A}$. But $\frac{B}{A}$ <<$_\mu$ $\frac{M}{A}$ and $\frac{M}{U}$ is cosingular, therefore $\frac{M}{A}$ = $\frac{U}{A}$, hence M = U. Thus A $\leq_{\mu ce}$ B+X in M.

Proposition 2.9: Let M and M' be R- modules and let f: M $\rightarrow$ M' be an homomorphism, If A $\leq_{\mu ce}$ B in M , then f (A) $\leq_{\mu ce}$ f (B) in f (M).

Proof: Suppose that A $\leq_{\mu ce}$ B in M. To show that f (A) $\leq_{\mu ce}$ f (B) in M', Define φ: $\frac{M}{A}\rightarrow\frac{f(M)'}{f(A)}$ by φ (m + A) = f (m) + f (A), for each m$\in$M, Since $\frac{B}{A}$ <<$_\mu$ $\frac{M}{A}$, then φ ($\frac{B}{A}$) = $\frac{f(B)}{f(A)}$ <<$_\mu$ φ ($\frac{M}{A}$) = $\frac{f(M)}{f(A)}$. Thus we get the result.

Proposition 2.10: Let A, B, C and X be sub modules of an R-module M. The following statements are equivalent

(1) If A $\leq_{\mu ce}$ A+B in M, then A $\cap$ B $\leq_{\mu ce}$ B in M.
(2) If A $\leq_{\mu ce}$ B in M and Y $\leq$ M, then A $\cap$ Y $\leq_{\mu ce}$ B $\cap$ Y in M.
(3) If A $\leq_{\mu ce}$ B in M and X $\leq_{\mu ce}$ C in M, then A $\cap$ X $\leq_{\mu ce}$ B $\cap$ C in M.

Proof: (1)$\Rightarrow$(2) Let A $\leq_{\mu ce}$ B in M and Y $\leq$ M. Since A + (B $\cap$ Y) $\leq$ B, then A $\leq_{\mu ce}$ A + (B $\cap$ Y) in M by prop (2.5) Hence A $\cap$ (B $\cap$ Y) $\leq_{\mu ce}$ B $\cap$ Y in M, by (1). This implies that A $\cap$ Y $\leq_{\mu ce}$ B $\cap$ Y in M.
(2)$\Rightarrow$(3) Let A $\leq_{\mu ce}$ B in M and X $\leq_{\mu ce}$ C in M. By (2) , A $\cap$ X $\leq_{\mu ce}$ B $\cap$ X in M. Also X $\leq_{\mu ce}$ C in M and B $\leq$ M , then B $\cap$ X $\leq_{\mu ce}$ B $\cap$ C in M. Thus A $\cap$ X $\leq_{\mu ce}$ B $\cap$ C in M , by prop. (2.5).
(3)$\Rightarrow$(1) Let A $\leq_{\mu ce}$ A+B in M. Since B $\leq_{\mu ce}$ B in M , then by (3) A $\cap$ B $\leq_{\mu ce}$ (A+B) $\cap$ B. Thus A $\cap$ B $\leq_{\mu ce}$ B in M.

Proposition 2.11: Let M be an R- module and let A $\leq$ B $\leq$ M. If B = A+S and S<<$_\mu$ M , then A $\leq_{\mu ce}$ B in M.

Proof: Assume that B = A+S an S<<$_\mu$ M. Let $\frac{M}{A}$= $\frac{B}{A}$ + $\frac{U}{A}$, $\frac{M}{U}$ is cosingular, then M = B+U = A+S+U = S+U. But S<<$_\mu$ M and $\frac{M}{U}$ is cosingular, therefore M = U. Thus A $\leq_{\mu ce}$ B in M.

Lemma 2.12: Let M be a module such that M = A + B and M = (A $\cap$ B) + C for sub modules A, B and C of M. Then M = (B $\cap$ C)+ A = (A $\cap$ C) + B.

Proof: See [5, Lemma 1.2]

Theorem 2.13: Let M = A + B be a module with $\frac{M}{B}$ cosingular. Let B $\leq$ C and B $\leq_{\mu ce}$ C in M. Then A $\cap$ B $\leq_{\mu ce}$ A $\cap$ C in M.

Proof: Let $\frac{M}{(A\cap B)} = \frac{(A\cap C)}{(A\cap B)} + \frac{U}{(A\cap B)}$, $\frac{M}{U}$ is cosingular , then M = (A$\cap$C) + U , implies that M = C + U. By Lemma (2.12) , M = (A$\cap$U) + C , $\frac{M}{B} = \frac{(A\cap U)+B}{B} + \frac{C}{B}$. Since $\frac{M}{B}$ is cosingular, then $\frac{(A\cap U)+B}{B}$ is cosingular. But $\frac{C}{B} \ll_{\mu} \frac{M}{B}$, therefore M = (A$\cap$U) + B. Again by Lemma (2.12) , M = (A$\cap$B) + U = U. Thus $\frac{(A\cap C)}{(A\cap B)} \ll_{\mu} \frac{M}{(A\cap B)}$.

Let M be an R- module and let A , B be sub modules of M, B is called μ- supplement of A in M , if M = A+B and A$\cap$B $\ll_{\mu}$ B. If every sub module of M has a μ- supplement, then M is called μ-supplemented module. See [6].

We end this section by the following proposition.

Proposition 2.14: Let A, B and C be sub modules of an R- module M. If A is a μ-supplement of B in M and B is a μ-supplement of C in M with A ≤ C , then A $\leq_{\mu ce}$ C in M.

Proof: Assume that A is a μ-supplement of B in M and B is a μ-supplement of C in M with A ≤ C. To show that A $\leq_{\mu ce}$ C in M, let $\frac{M}{A} = \frac{C}{A} + \frac{Y}{A}$, where $\frac{M}{Y}$ is cosingular, then M = C+Y. By modular law Y = Y$\cap$M = Y$\cap$(A+B) = A+(Y$\cap$B). Hence M = C+Y = C+ A+(Y$\cap$B). So by (the second isomorphism theorem) $\frac{B}{Y\cap B} \cong \frac{B+Y}{Y} \leq \frac{M}{Y}$ But $\frac{M}{Y}$ isocosingular, therefore $\frac{B}{Y\cap B}$ is cosingular. Since B is a μ-supplement of C in M that is B$\cap$C$\ll_{\mu}$ B, then Y$\cap$B = B and hence Y = A+B = M. Thus A $\leq_{\mu ce}$ C in M.

3. μ-Lifting modules

Definition 3.1: An R- module M is called **μ-lifting module** if for every sub module A of M, there exists a sub module D of A such that M = D$\oplus$D' , D' ≤ M and A$\cap$D'$\ll_{\mu}$ D'

Remarks and Examples 3.2:

(1) It is clear that every lifting module is μ-lifting. The converse is not true in general, see [3, example (3.17)]

(2) Let M be a cosingular module. Then M is lifting if and only if M is μ-lifting module.

(3) Every μ-hollow is μ-lifting. The converse is not true in general. For example Z_6 as Z- module.

(4) Z_4 as Z- module is μ-lifting.

(5) Z as Z- module is not μ-lifting.

(6) Every μ-lifting is $\oplus$-μ-supplemented module. The converse is not true in general, for example $Z_8\oplus Z_2$ as Z- module.

(7) μ- lifting modules are closed under isomorphism.

The following propositions give characterizations of μ-lifting modules.

Proposition 3. 3: Let M be an R- module. Then M is μ-lifting if and only if for every sub module A of M , there exists a sub module D of M such that M = D$\oplus$ D' , D' ≤ M and A$\cap$D'$\ll_{\mu}$ M.

Proof: Clear.

Proposition 3. 4: Let M be an R-module. The following statements are equivalent.

(1) M is μ-lifting module.

(2) Every sub module A of M can be written as A = D$\oplus$S , where D is a direct summand of M and S$\ll_{\mu}$ M.

(3) For every sub module A of M, there exists a direct summand D of M such that D ≤ A and D$\leq_{\mu ce}$ A in M .

Proof: (1)$\Rightarrow$(2) Suppose that M is a μ-lifting module and let A be a sub module of M , then there exists a sub module D of A such that M = D$\oplus$D' , D' ≤ M and A$\cap$D' $\ll_{\mu}$ M , by prop. (3.3). Now, A = A$\cap$M = A$\cap$(D$\oplus$D') = D$\oplus$(A$\cap$D'). Thus we get the result.

(2)$\Rightarrow$(3) Let A be a sub module of M. By (2) , A = D$\oplus$S , where D is a direct summand of M and S$<<_\mu$ M. We have to show that $\frac{A}{D} <<_\mu \frac{M}{D}$, let $\frac{M}{D} = \frac{A}{D} + \frac{U}{D}$, $\frac{M}{U}$ is cosingular, then $\frac{M}{D}$ = $\frac{D+S}{D} + \frac{U}{D}$ and hence M = D+S+U = S+U. But S$<<_\mu$ M, therefore M =U.

(3)$\Rightarrow$(1) Let A be a sub module of M. By (3), there exists a direct summand D of M such that D $\le$ A and D $\le_{\mu ce}$ A in M . We want to show that A$\cap$D' $<<_\mu$ D' , let D' = (A$\cap$D')+U , $\frac{D'}{U}$ is cosingular. Since M = D+D' = D+(A$\cap$D')+U , then $\frac{M}{D} = \frac{D+(A\cap D')+U}{D} = \frac{D+(A\cap D')}{D} + \frac{U+D}{D}$. Since D$\le$ D+(A$\cap$D')$\le$ A and D$\le_{\mu ce}$ A in M , then D$\le_{\mu ce}$ D+(A$\cap$D') in M , by prop. (2.5) and $\frac{M}{U+D} = \frac{D+D'}{U+D} = \frac{(D+U)+D'}{U+D} \cong \frac{D'}{D'\cap(U+D)} = \frac{D'}{U}$ which is cosingular, hence $\frac{M}{D} = \frac{U+D}{D}$, implies that M = U+D and clearly that U$\cap$D = 0, then M = U$\oplus$D, that is U = D'. Thus M is a μ-lifting module.

Theorem 3.5: Let M be an R-module. The following statements are equivalent.

(1) M is μ-lifting module.

(2) Every sub module A of M has a μ-supplement B in M such that A$\cap$B is a direct summand of A.

Proof: (1)$\Rightarrow$(2) Let M be μ-lifting module and let A be a sub module of M. By prop. 3. 4, there exists a direct summand D of M such that D $\le$ A and D$\le_{\mu ce}$ A in M. Now, A = A$\cap$M = A $\cap$(D$\oplus$D') = D $\oplus$(A$\cap$D'). Since D $\le$ A, then M = A+D', and A$\cap$D'$<<_\mu$ D'. Hence D' is μ-supplement of A and A $\cap$D' is a direct summand of A.

(2)$\Rightarrow$(1) Let A be a sub module of M . By (2) A has a μ-supplement B in M such that A$\cap$B is a direct summand of A. Then M = A+B, A$\cap$B $<<_\mu$ B and A = (A$\cap$B)$\oplus$Y , Y$\le$ A. Since M = A+B = (A$\cap$B) + Y+ B = Y+B and A$\cap$B$\cap$Y = B$\cap$Y ={0} , then M = B$\oplus$Y. It is sufficient to show that Y$\le_{\mu ce}$ A in M. Let $\frac{M}{Y} = \frac{A}{Y} + \frac{U}{Y}$, $\frac{M}{U}$ is cosingular, then M = A+U = (A$\cap$B) + Y + U = (A$\cap$B) + U. Since A$\cap$B$<<_\mu$ M, then M = U, implies that $\frac{A}{Y} <<_\mu \frac{M}{Y}$. Thus M is μ-lifting module.

The following proposition gives another characterization of μ-lifting module.

Proposition 3.6: Let M be an R-module. Then M is μ-lifting module if and only if for every sub module A of M , there exists an idempotent f $\in$ End(M) such that f (M) $\le$ A and (I-f)(A) $<<_\mu$ (I-f)(M).

Proof: ($\Rightarrow$) Assume that M is a μ-lifting module and let A be a sub module of M. By characterization (3.5) A has a μ-supplement B in M such that A$\cap$B is a direct summand of A , then M = A+B , A$\cap$B$<<_\mu$ B and A = (A$\cap$B)$\oplus$X , X$\le$ A. Note M = A+B = (A$\cap$B)+X +B = X+B and A$\cap$B$\cap$X = B$\cap$X = {0} , implies that M = B$\oplus$X. Now define the following map f : M $\rightarrow$X , it is clear that f is an idempotent and f (M) $\le$ X $\le$ A . It is sufficient to prove that (I-f) (A) $<<_\mu$ (I-f)(M). One can easily show that (I-f) (A) = A$\cap$(I-f) (M) = A$\cap$B$<<_\mu$ B = (I-f) (M).

($\Leftarrow$) Let A be a sub module of M. By our assumption, there exists an idempotent f $\in$End (M) such that f (M) $\le$ A and (I-f) (A) $<<_\mu$ (I-f) (M) , clearly that M = f (M)$\oplus$(I-f)(M) and A$\cap$(I-f)(M) = (I-f) (A) $<<_\mu$ (I-f) (M). Thus M is μ-lifting.

Proposition 3.7: Let M be an indecomposable module. Then M is μ-lifting if and only if M is μ-hollow

Proof: Let M be a μ-lifting indecomposable module and let A be a proper sub module of M. Since M is μ-lifting, there exists a sub module D of A such that M = D$\oplus$D' and A$\cap$D' $<<_\mu$ D'. But M is indecomposable, therefore either D = M or D = 0. If D = M, then A= M which is a contradiction, then D = 0 and hence M = D', implies A$\cap$D' =A $<<_\mu$ D' = M. Thus M is μ-hollow. The converse is clear.

Proposition 3.8: Any direct summand of μ-lifting is μ-lifting

Proof: Let M = $M_1 \oplus M_2$ be a μ-lifting and let A be a sub module of M_1, then A = D$\oplus$S, where D is a direct summand of M and S<<$_\mu$ M, by characterization (3.4). Since D is a direct summand of M contained in M_1, then D is a direct summand of M_1 and S<<$_\mu$ M, S ≤ M_1 and M_1 is a direct summand of M, then S<<$_\mu$ M_1. Thus M_1 is μ-lifting.

Note: Let A be a sub module of a μ-lifting R-module M. Then $\frac{M}{A}$ need not be μ-lifting. For example, Let M = $Z_8 \oplus Z_8$ as Z- module, clearly M is μ-lifting module. Let $\pi \oplus I$: $Z_8 \oplus Z_8 \rightarrow Z_2 \oplus Z_8$ is an epimorphism. So $\frac{Z_8 \oplus Z_8}{Ker(\pi \oplus I)} \cong Z_2 \oplus Z_8$ which is not μ-lifting

Next, we give some various conditions under which the quotient of μ-lifting module is μ-lifting.

Recall that an R- module M is called distributive if for all A, B and C ≤M, A$\cap$(B+C) = (A$\cap$B)+(A$\cap$C). See [7].

Proposition 3.9: Let M be a μ- lifting R- module and let A be a sub module of M. Then $\frac{M}{A}$ is μ-lifting in each of the following cases

(1) For every direct summand D of M, $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$.

(2) M is distributive module.

Proof: (1) Suppose that M is μ-lifting R- module and let $\frac{X}{A}$ be a sub module of $\frac{M}{A}$, then there exists D ≤ X such that M = D$\oplus$D', D' ≤ M and D $\leq_{\mu ce}$ X in M. By hypothesis, $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$. By prop. (2.4), $\frac{D+A}{A} \leq_{\mu ce} \frac{X}{A}$ in $\frac{M}{A}$. Thus $\frac{M}{A}$ is μ-lifting.

(2) Suppose that M is distributive module, we use (1) to show that $\frac{M}{A}$ is μ-lifting. Let D be a direct summand of M, M = D$\oplus$D', D' ≤ M, then $\frac{M}{A} = \frac{D+D'}{A} = \frac{D+A}{A} + \frac{D'+A}{A}$ and $\frac{D+A}{A} \cap \frac{D'+A}{A} = \frac{(D+A)\cap D'+[(D+A)\cap A]}{A} = \frac{(A\cap D')+(D\cap A)+A}{A}$ =A. Hence $\frac{D+A}{A}$ is a direct summand of $\frac{M}{A}$. So, by (1) M is μ-lifting module.

Let M be an R- module. Recall that a sub module A of M is called a fully invariant if g (A) ≤ A , for every g $\in$End(M) and M is called duo module if every sub module of M is fully invariant. See [8].

Lemma 3.10: [8, lemma 5-4]**:** Let M be an R-module, if M =$M_1 \oplus M_2$, then $\frac{M}{A} = \frac{A\oplus M_1}{A} \oplus \frac{A+M_2}{A}$, for every fully invariant sub module A of M.

Proposition 3.11: Let M be a μ-lifting module if A is a fully invariant sub module of M, then $\frac{M}{A}$ is a μ-lifting module.

Proof: Let $\frac{X}{A}$ be a sub module of $\frac{M}{A}$. Since M is μ-lifting, there exists a sub module D of X such that D$\leq_{\mu ce}$ X in M and M = D$\oplus$D', D'≤M. By lemma (3.10) we have $\frac{M}{A} = \frac{D+A}{A} \oplus \frac{D'+A}{A}$, let f: $\frac{M}{D} \rightarrow \frac{M}{D+A}$ be a map defined by f (m + D) = m + D + A, $\forall$ m$\in$M, it is clear that f is an epimorphosis. Now, since D$\leq_{\mu ce}$ X in M, $\frac{X}{D} \ll_{\mu} \frac{M}{D}$ and f ($\frac{X}{D}$) $\ll_{\mu}$ f ($\frac{M}{D}$), by [3, prop. (2.14)] which implies that $\frac{X}{D+A} \ll_{\mu} \frac{M}{D+A}$, then D+A $\leq_{\mu ce}$ X in M and hence $\frac{D+A}{A} \leq_{\mu ce} \frac{X}{A}$ in $\frac{M}{A}$, by prop. (2.4). Thus $\frac{M}{A}$ is a µ-lifting module.

Lemma 3.12: Let M = A+B be a µ-lifting module, if $\frac{M}{A}$ is cosingular, then there exists a direct summand D of M such that M = A+D and D $\leq_{\mu ce}$ B in M.

Proof: Let M = A+B be a µ-lifting and assume that $\frac{M}{A}$ is cosingular. Since B ≤ M and M is µ-lifting , there exists a direct summand D of M such that D$\leq_{\mu ce}$ B in M. Then $\frac{M}{D} = \frac{A+B}{D} = \frac{A+D}{D} + \frac{B}{D}$.

Since $\frac{M}{A}$ is cosingular, then $\frac{M}{A+D}$ is cosingular and hence $\frac{\frac{M}{D}}{\frac{A+D}{D}}$ is cosingular, by third isomorphism theorem. But $\frac{B}{D} \ll_{\mu} \frac{M}{D}$, then $\frac{M}{D} = \frac{A+D}{D}$ which implies that M = A+D. So, we get the result.

Let M be an R- module. M is called amply µ-supplemented if for any sub modules A and B of M with M = A+B , there exists a µ-supplement X of A contained in B. See [6].

Proposition 3.13: Let M be an amply µ-supplemented module such that every µ-supplement sub module of M is a direct summand, then M is a µ-lifting.

Proof: Suppose the M is amply µ-supplemented module and let A be a sub module of M, then A has a µ- supplement B in M , hence M = A+B and $A \cap B \ll_{\mu}$ B. Since M is amply µ-supplemented and M = A+B, then A contains a µ-supplement X of B. By our assumption, X is a direct summand of M, so M = $X \oplus Y$, Y ≤ M. Now, A = $A \cap M$ = $A \cap (X+Y)$ = X + $(A \cap Y)$, by modularity. Since X is a µ-supplement of B in M, then M = X+B, hence A = $A \cap M$ = $A \cap (X+B)$ = X+ $(A \cap B)$. Now, consider the projection map P : M $\rightarrow$ Y , P(A) = P (X+$(A \cap Y)$) = $A \cap Y$ and also P(A) = P(X+$(A \cap B)$) = P($A \cap B$), hence P($A \cap B$) = $A \cap Y$ Since $A \cap B \ll_{\mu}$ M , then P($A \cap B$) = $A \cap Y \ll_{\mu}$ Y. Thus M is µ-lifting module.

Let M be an R- module and let A be a sub module of M , we say that A is a µ-coclosed sub module of M denoted by (A$\leq_{\mu cc}$ M) if whenever $\frac{A}{X}$ is cosingular and X$\leq_{\mu ce}$ A in M for some sub module X of A , we have X = A. See [3].

Proposition 3.14: Let M be a µ-lifting module. Then every cosingular µ-coclosed sub module of M is a direct summand.

Proof: Let A be a cosingular µ-coclosed sub module of M. Since M is µ-lifting, there exists a sub module D of A such that M = $D \oplus D'$, D' ≤ M and $A \cap D' \ll_{\mu}$ M. Since A is µ-coclosed sub module of M, Then $A \cap D' \ll_{\mu}$ A, by [6, prop. (3.4)]. Now, A = $A \cap M$ = $A \cap (D+D')$ = D + $(A \cap D')$ and $\frac{A}{D}$ is cosingular , hence A = D. Thus A is a direct summand of M.

Remark A direct sum of µ-lifting modules need not be µ-lifting module as the following example shows.

Let M = $Z_8 \oplus Z_2$ as Z- module. It is clear that Z_8 and Z_2 are µ-lifting Z- modules, but M is not µ-lifting module.

Now, we give various conditions under which a direct sum of µ- lifting modules is µ- lifting.

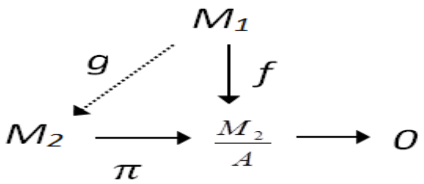
Proposition 3.15: Let M = $M_1 \oplus M_2$ be an R- module such that ann (M_1) + ann(M_2) = R, if M_1 and M_2 are µ-lifting , then M is µ-lifting.

Proof: Let A be a sub module of M. By [9, prop. 4.2], $A = A_1 \oplus A_2$, where $A_1 \leq M_1$ and $A_2 \leq M_2$. Since M_1 and M_2 are μ-lifting modules, then $A_1 = B_1 \oplus S_1$ and $A_2 = B_2 \oplus S_2$, where B_1 and B_2 are direct summands of M_1 and M_2 respectively and S_1, S_2 are μ-small sub modules of M_1 and M_2 respectively, by prop. 3.4. Note, $A = A_1 \oplus A_2 = B_1 \oplus S_1 \oplus B_2 \oplus S_2 = (B_1 \oplus B_2) \oplus (S_1 \oplus S_2)$, where $B_1 \oplus B_2$ is a direct summand of M and $S_1 \oplus S_2$ is μ-small sub module of M, by [3,prop. (2.14)]. Thus M is μ-lifting module.

Proposition 3.16: Let $M = M_1 \oplus M_2$ be a duo module such that M_1 and M_2 are μ-lifting modules, then M is μ-lifting.

Proof: Let $M = M_1 \oplus M_2$ be a duo module and let A be a sub module of M, then A is a fully invariant. Hence $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2)$. Since M_1 and M_2 are μ-lifting modules, then $A \cap M_1 = A_1 \oplus A_2$ and $A \cap M_2 = A_3 \oplus A_4$, where A_1 and A_3 are direct summands of M_1 and M_2 respectively and A_2, A_4 are μ-small sub modules of M_1 and M_2 respectively, by prop. (3.4). It is clear that $A_1 \oplus A_3$ is a direct summand of M and $A_2 \oplus A_4$ is μ-small sub module of M. Thus M is μ-lifting.

Let M_1 and M_2 be R-modules. Recall that M_1 is M_2-projective if for every sub module A of M_2 and any homomorphism $f : M_1 \to \frac{M_2}{A}$, there is a homomorphism: $M_1 \to M_2$ such that $\pi \circ g = f$, where $\pi : M_2 \to \frac{M_2}{A}$ is the natural epimorphosis, see [2].



M_1 and M_2 are said to be relatively projective if M_1 is M_2- projective and M_2 is M_1-projective.

Proposition 3.17: For $M = M_1 \oplus M_2$, where M_1 be a μ-lifting module and let M_2 is M_1-projective. Then the following statements are equivalent.

(1) M is μ-lifting module.

(2) For every sub module A of M such that $M \neq A+M_1$, there exists a direct summand D of M such that $D \leq_{\mu ce} A$ in M.

Proof: (1)$\Rightarrow$(2) Clear.

(2)$\Rightarrow$(1) Let A be a sub module of M and let $M = A+M_1$. Since M_2 is M_1-projective, then there exists a sub module $A_1 \leq A$ such that $M = A_1 \oplus M_1$, by [10, lemma 5]. But M_1 is μ-lifting and $\frac{M}{A_1} = \frac{A_1 + M_1}{A_1} \cong \frac{M_1}{A_1 \cap M_1} = M_1$, by (the second isomorphisim theorem), therefore $\frac{M}{A_1}$ is μ-lifting, so there exists a direct summand $\frac{D}{A_1}$ of $\frac{M}{A_1}$ such that $\frac{D}{A_1} \leq_{\mu ce} \frac{A}{A_1}$ in $\frac{M}{A_1}$. Hence $D \leq_{\mu ce} A$ in M, by prop. (2.5). Now, $D = D \cap M = D \cap (A_1 \oplus M_1) = A_1 \oplus (D \cap M_1)$, by modular law. But $\frac{D}{A_1}$ is a direct summand of $\frac{M}{A_1}$, so $\frac{A_1 \oplus (D \cap M_1)}{A_1}$ is a direct summand of $\frac{A_1 \oplus M_1}{A_1}$. Hence $D \cap M_1$ is a direct summand of M_1, by (the second isomorphism theorem). Let $M_1 = (D \cap M_1) \oplus Y$, for some sub module Y of M. Thus $M = A_1 \oplus M_1 = A_1 \oplus (D \cap M_1) \oplus Y = D \oplus Y$ and hence M is μ-lifting module.

Proposition (3.18): Let M_1 and M_2 be μ-lifting modules such that M_i is M_j-projective (i, j = 1, 2). Then $M = M_1 \oplus M_2$ is μ-lifting.

Proof: Assume that M_1 and M_2 are μ-lifting modules. To show that M is μ-lifting, let A be a sub module of M, Consider the sub module $M_1 \cap (A+M_2)$ of M_1. Since M_1 is μ-lifting, there exists decomposition $M_1 = A_1 \oplus B_1$ such that $A_1 \leq M_1 \cap (A+M_2)$ and $[M_1 \cap (A+M_2)] \cap B_1 = B_1 \cap (A+M_2) <<_{\mu} B_1$. Therefore $M = M_1 \oplus M_2 = A_1 \oplus B_1 \oplus M_2 = M_1 \cap (A+M_2) + B_1+M_2 = (A+M_2)+B_1+M_2 = A+(M_2 \oplus B_1)$. Since $M_2 \cap (A+B_1) \leq M_2$ and M_2 is μ-lifting , there exists a decomposition $M_2 = A_2 \oplus B_2$ such that $A_2 \leq M_2 \cap (A+B_1)$ and $B_2 \cap (M_2 \cap (A+B_1)) = B_2 \cap (A+B_1) <<_{\mu} B_2$. We have $M = A + (B_1 \oplus M_2) = A+B_1+A_2+B_2 = A + (B_1 \oplus B_2)$, so $M = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$. Since M_i is M_j- projective, then M_1 is M_j-projective and M_2 is M_j-projective (j = 1, 2) and hence A_1 is B_j-projective and A_2 is B_j-projective (j = 1,2) , by [11 , prop. 2-1-6]. So by [11, prop. 2-1-7] A_1 is $B_1 \oplus B_2$ –projective and A_2 is $B_1 \oplus B_2$ -projective. Hence $A_1 \oplus A_2$ is $B_1 \oplus B_2$ -projective, by [11, prop. 2-1-6]. Then there exists $Y \leq A$ such that $M = Y \oplus (B_1 \oplus B_2)$ by [10, lemma5]. Since $B_1 \cap (A+M_2) <<_{\mu} B_1$ and $B_2 \cap (A+B_1) <<_{\mu} B_2$, then $[B_1 \cap (A+M_2) \oplus B_2 \cap (A+B_1)] <<_{\mu} B_1 \oplus B_2$, Since $A \cap (B_1 \oplus B_2) \leq [B_1 \cap (A+M_2) \oplus B_2 \cap (A+B_1)] <<_{\mu} B_1 \oplus B_2$, then $A \cap (B_1 \oplus B_2) <<_{\mu} B_1 \oplus B_2$. Thus M is μ-lifting.

References

1. Inoue, T. **1983.** Sum of hollow modules, *Osaka J. Math*., : 331-336.
2. Mohamed S.H. and Muller B. J. **1990**. *Continuous and discrete Modules*, London Math. Soc. LNS 147 Cambridgeo University Press, Cambridge.
3. Khalid Kamil E. M. **2018.** On a generalization of small sub modules, *Sci. Int*. (Lahore), : 359-365.
4. Oshiro, K. **1984**. Lifting modules, Extending modules and their applications to QF-rings, *Hokkaido Math. J*., **13**: 3310-338.
5. Keskin, D. **2000.** On lifting modules, *Comm. Algebra*, : 3427-3440.
6. Kamil, E. M. and Khalid, W. **2018.** On μ-supplemented and cofinitely μ- supplemented, *Sci. Int.* (Lahore), : 567-572.
7. Erdogdu, V. **1987.** Distributive Modules, *Can. Math. Bull* , : 248-254,
8. Orhan, N., Tutuncu, D. K. and Tribak, R. **2007.** On Hollow-lifting Modules, *Taiwanese J. Math*, : 545-568.
9. Abass, M. S. **1991.** On fully stable modules, Ph.D. Thesis, University of Baghdad.
10. Kasch, F. **1982.** *Modules and Rings*, Acad. Press, London,
11. Abdelkader B. H. **2001.** On lifting modules, M. Sc. thesis, University of Baghdad .