DOI: 10.24996/ijs.2023.64.2.24

ISSN: 0067-2904

Applications of the Finite Operator ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta\right)$ for the Polynomials $B_n(a,b,c,d,f,x,y|q)$

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Received: 28/1/2021 Accepted: 24/6/2022 Published: 28/2/2023

Abstract

In this work, polynomials $B_n(a,b,c,d,f,x,y|q)$ and the finite q-exponential operator ${}_3\mathcal{E}_2$ are constructed. The operator ${}_3\mathcal{E}_2$ is used to combine an operator proof of the generating function with its extension, Mehler's formula with its extension and Roger's formula for the polynomials $B_n(a,b,c,d,f,x,y|q)$. The generating function with its extension, Mehler's formula with its extension and Rogers formula for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ are deduced by giving special values to polynomials $B_n(a,b,c,d,f,x,y|q)$.

Keywords: Finite q-exponential operator, Generating function, Mehler's formula, Rogers formula, Al-Salam-Carlitz polynomials.

تطبيقات المؤثر المنتهي ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta\right)$ لمتعددات الحدود $B_n(a,b,c,d,f,x,y|q)$

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الخلاصة

نقوم ببناء متعددات الحدود $B_n(a,b,c,d,f,x,y|q)$ بالإضافة إلى المؤثرالأسي-q المنتهي ${}_3\mathcal{E}_2$. استخدمنا المؤثر ${}_3\mathcal{E}_2$ لأعطاء برهان المؤثر للدالة المولدة وتوسيعها، صيغة ملر وتوسيعها وصيغة روجرز لمتعددات الحدود $B_n(a,b,c,d,f,x,y|q)$. استنتجنا الدالة المولدة وتوسيعها، صيغة ملر وتوسيعها، وصيغة روجرز لمتعددات حدود السلام – كارلتز $U_n(x,y,a;q)$ بإعطاء قيم خاصة لمتعددات الحدود $B_n(a,b,c,d,f,x,y|q)$.

1. Introduction

In this paper, we use the conventional notations for basic hypergeometric series from [1], and we also suppose that $|q| < 1$.

Let a be a complex variable. The q-shifted factorial is described by the authors in [1] as follows:

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$$(a;q)_n=\begin{cases}1, & if\ \ n=0,\\ \prod_{k=0}^{n-1}(1-aq^k), & if\ \ \ n=1,2,\ldots.\end{cases}$$

We define

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),$$

for multiple q-shifted factorials, we use the following notations:

$$a_1,a_2,\cdots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \qquad n=0,1,2,\ldots.$$
$$(a_1,a_2,\cdots,a_m;q)_\infty=\ (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

The basic hypergeometric series are defined by the formula

$$\sum_{n=0}^{\infty}c_n$$

with ${c_{n+1}}/{c_n}$ is a rational function of q^n for a fixed parameter q, which is usually taken to satisfy $|q|<1$ [1].

The generalized basic hypergeometric series is stated by the authors in [1] as follows:

$${}_r\phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left[(-1)^n\,q^{\binom{n}{2}}\right]^{1+s-r}x^n,$$

where $r,s\in\mathbb{N}$; $a_1,\ldots,a_r\in C$; $b_1,\ldots,b_s\in C\backslash\{q^{-k},k\in N\}$ are assumed to be none of the denominator factors is evaluated to zero. This series converges absolutely for all x if rs and for $|x|<1$ if $r=s+1$.

The most essential case of series is $r=s+1$.

$${}_{s+1}\phi_s\left(\begin{matrix}a_1,\ldots,a_{s+1}\\ b_1,\ldots,b_s\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_{s+1};q)_n}{(q,b_1,\cdots,b_s;q)_n}\,x^n,\quad |x|<1.$$

For $n\in N$, the q-binomial coefficient is stated by

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{cases}\dfrac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, & if\ \ 0\ \le\ k\ \le n;\\ 0, & otherwise.\end{cases}$$

In this paper, we use the following identities [1]:

$$(aq^{-n};q)_n=(-1)^n\,q^{-\binom{n}{2}-n}a^n\left(\frac{q}{a};q\right)_n \tag{1.1}$$

$$(q^{-n};q)_k=\ (-1)^k\,q^{\binom{k}{2}-nk}\,\frac{(q;q)_n}{(q;q)_{n-k}}. \tag{1.2}$$

Euler identity is given in [1]:

$$\sum_{n=0}^{\infty}\frac{(-1)^n\,q^{\binom{n}{2}}x^n}{(q;q)_n}=(x;q)_\infty. \tag{1.3}$$

In 2014, Abdlhusein [2] provided the following identity:

$${}_1\phi_1\left(\begin{matrix}xt\\ yt\end{matrix};q,ys\right)=\frac{(xt,ys;q)_\infty}{(yt;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}\frac{y}{x},0\\ ys\end{matrix};q,xt\right). \tag{1.4}$$

Setting $s=t$ and $t=q^k s$ in (1.4), we obtain

$$ {}_2\phi_1\left(\begin{matrix}\frac{y}{x},0\\ yt\end{matrix};q,q^kys\right)=\frac{(q^kys;q)_\infty}{(q^kxs,yt;q)_\infty}\ {}_1\phi_1\left(\begin{matrix}q^kxs\\ q^kys\end{matrix};q,yt\right). \tag{1.5}$$

Jackson's transformation of ${}_2\phi_1$ series is [1, Appendix III, equation (III.4)]

$$ {}_2\phi_1\left(\begin{matrix}a,b\\ c\end{matrix};q,z\right)=\frac{(az;q)_\infty}{(z;q)_\infty}\ {}_2\phi_2\left(\begin{matrix}a,c/b\\ c,az\end{matrix};q,bz\right). \tag{1.6}$$

Setting $a=0$, $b=y/x$, $c=q^kys$ and $z=xt$ in (1.6), we get

$$ {}_1\phi_1\left(\begin{matrix}q^kxs\\ q^kys\end{matrix};q,yt\right)=(xt;q)_\infty\ {}_2\phi_1\left(\begin{matrix}\frac{y}{x},0^k\\ q^kys\end{matrix};q,xt\right). \tag{1.7}$$

The Cauchy polynomials are defined by the following [3, 4, 5]

$$ P_n(x,y)=\begin{cases}(x-y)(x-qy)(x-q^2y)\cdots(x-q^{n-1}y), & if\ \ n>0;\\ 1, & if\ \ n=0.\end{cases} \tag{1.8}$$

Theorem 1.1 *Let* $P_n(x,y)$ *be the polynomials that defined in* (1.8), *then*

- *The generating function for Cauchy polynomials* $P_n(x,y)$ [6, 7, 8] *is*

$$ \sum_{k=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(xt;q)_\infty},\quad |xt|<1. \tag{1.9}$$

- *The Mehler's formula for Cauchy polynomials* $P_n(x,y)$ [9] *is*

$$ \sum_{n=0}^{\infty}P_n(x,y)P_n(z,w)\frac{t^n}{(q;q)_n}=\frac{(xwt;q)_\infty}{(xzt;q)_\infty}\ \ {}_1\phi_1\left(\begin{matrix}w/z\\ xwt\end{matrix};q,yzt\right), \tag{1.10}$$

provided $|xzt|<1$.

The Al-Salam-Carlitz polynomials are first introduced in 1965 by Al-Salam and Carlitz [10] as follows:

$$ u_n^{(a)}(x;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}(-1)^k\,q^{\binom{k}{2}}a^kP_{n-k}(x,1). $$

The operator θ is defined in [11, 12] by

$$ \theta\{f(x)\}=\frac{f(q^{-1}x)-f(x)}{q^{-1}x}. \tag{1.11}$$

The θ for is used for acting on the variable *x*. Otherwise, we use the operator θ_y that acts on the variable y.

Theorem 1.2 [11, 12]. *The Leibniz rule for* θ *is*

$$ \theta^n\{f(x)g(x)\}=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}\theta^k\{f(x)\}\theta^{n-k}\{g(xq^{-k})\}. \tag{1.12}$$

Theorem 1.3 [11, 13]. *Let* θ *be defined as in* (1.11), *then*

$$ \theta^k\{x^n\}=\frac{(q;q)_n}{(q;q)_{n-k}}x^{n-k}\,q^{\binom{k}{2}-nk+k}. \tag{1.13}$$

$$ \theta^k\{(xt;q)_\infty\}=(-1)^kt^k(xt;q)_\infty. \tag{1.14}$$

$$ \theta^k\left\{\frac{(xv;q)_\infty}{(xt;q)_\infty}\right\}=\ q^{-\binom{k}{2}}t^k(v/t;q)_k\frac{(xv;q)_\infty}{(xtq^{-k};q)_\infty},\quad |xt|<1. \tag{1.15}$$

$$ \theta_y^k\{P_n(x,y)\}=(-1)^k\frac{(q;q)_n}{(q;q)_{n-k}}P_{n-k}(x,y). \tag{1.16}$$

In 2010, Chen et al. [14] extended the original definition of Al-Salam-Carlitz polynomials as follows:

$$U_n(x,y,a;q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k\,q^{\binom{k}{2}}a^kP_{n-k}(x,y). \tag{1.17}$$

Theorem 1.4 [14]. *Let the polynomials $U_n(x,y,a;q)$ be defined as in* (1.17), *then*

- *The generating function for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ is*

$$\sum_{n=0}^{\infty}U_n(x,y,a;q)\frac{t^n}{(q;q)_n}=\frac{(at,yt;q)_\infty}{(xt;q)_\infty},\qquad |xt|<1. \tag{1.18}$$

- *The Rogers formula for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ is*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}U_{n+m}(x,y,a;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\frac{(as,ys;q)_\infty}{(xs;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k\,q^{\binom{k}{2}}(xs;q)_k(at)^k}{(q;q)_k(as,ys;q)_k}\;{}_2\phi_1\left(\begin{matrix}\frac{y}{x},0\\ ysq^k\end{matrix};q,xt\right), \tag{1.19}$$

provided that $max\{|xs|,|xt|\}<1$.

In 2010, Zhang and Yang [15] considered the finite q-exponential operator ${}_2\mathcal{E}_1\left[\begin{matrix}q^{-N},w\\ v\end{matrix};q,d\theta\right]$ with two parameters as follows:

$${}_2\mathcal{E}_1\left[\begin{matrix}q^{-N},w\\ v\end{matrix};q,d\theta\right]=\sum_{n=0}^{N}\frac{(q^{-N},w;q)_n}{(q,v;q)_n}(d\theta)^n.$$

In 2016, inspired by the basic hypergeometric series ${}_2\phi_1$, Li and Tan [16] introduced the generalized q-exponential operator $\mathbb{E}\left[\begin{matrix}u,v\\ w\end{matrix}q;t\theta\right]$ with three parameters as follows [16]:

$$\mathbb{E}\left[\begin{matrix}u,v\\ w\end{matrix}q;t\theta\right]=\sum_{n=0}^{\infty}\frac{(u,v;q)_n}{(q,w;q)_n}(t\theta)^n.$$

Our paper is structured as follows: In section 2, we define a finite q-exponential operator ${}_3\mathcal{E}_2$ and then deduce some of its identities, which will be used in the next sections. In section 3, we introduce a new polynomials $B_n(a,b,c,d,f,x,y|q)$ and obtain the generating function and its extension, after which we infer the generating function and its extension for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$. Section 4 deduces Mehler's formula and its extension for $B_n(a,b,c,d,f,x,y|q)$, it followed by the Mehler formula and its extension for $U_n(x,y,a;q)$. Section 5 yields the Rogers formula for $B_n(a,b,c,d,f,x,y|q)$, from which we get the Rogers formula for $U_n(x,y,a;q)$.

2. The Finite q-Exponential Operator and its Identities

In this section, we construct the finite q-exponential operator ${}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\ c,d\end{matrix};q,-f\theta\right)$ and obtain some its identities based on the finite basic hypergeometric series ${}_3\phi_2$.

We define the finite q-exponential operator ${}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\ c,d\end{matrix};q,-f\theta\right)$ as follows*:*

$${}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\ c,d\end{matrix};q,-f\theta\right)=\sum_{k=0}^{\infty}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(-f\theta)^k. \tag{2.1}$$

- The finite q-exponential operator ${}_2\mathcal{E}_1\left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, d\theta\right]$ developed by Zhang and Yang [15] can be regarded as special case of the operator ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta\right)$ for $a = w$, $c = v$ and $b = d = 0$, $f = -d$.
- The generalized q-exponential operator $\mathbb{E}\left[\begin{matrix} u, v \\ w \end{matrix} | q; t\theta\right]$ with three parameters proposed by Li and Tan [16] can be thought as specific instance of the operator ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta\right)$
  For $a = u$, $b = v$, $c = w$, $d = q^{-N}$ and $f = -t$.

Theorem 2.1 *Let* ${}_3\mathcal{E}_2\left(\begin{matrix} \mathrm{q}^{-\mathrm{N}}, \mathrm{a}, \mathrm{b} \\ \mathrm{c}, \mathrm{d} \end{matrix}; \mathrm{q}, -\mathrm{f}\theta\right)$ *be the operator that defined in* (2.1), *then*

$$ {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta\right)\left\{\frac{(ys;q)_\infty}{(yt;q)_\infty}\right\} = \frac{(ys;q)_\infty}{(yt;q)_\infty}\ {}_4\phi_3\left(\begin{matrix} q^{-N}, a, b, s/t \\ c, d, q/yt \end{matrix}; q, qf/y\right), $$

provided that $max\{|\mathrm{yt}|, |\mathrm{qf/y}|\} < 1$.

Proof. By definition of the finite operator ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)$, we have

$$ {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\left\{\frac{(ys;q)_\infty}{(yt;q)_\infty}\right\} $$

$$ = \sum_{k=0}^{\infty} \frac{(q^{-N}a, b; q)_k}{(q, c, d; q)_k} (-f)^k \theta_y^k \left\{\frac{(ys;q)_\infty}{(yt;q)_\infty}\right\} $$

$$ = \sum_{k=0}^{\infty} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (-f)^k\ q^{-\binom{k}{2}} t^k (st;q)_k \frac{(ys;q)_\infty}{(ytq^{-k};q)_\infty} \qquad (by\ \text{using}\ (1.15)) $$

$$ = \frac{(ys;q)_\infty}{(yt;q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (-f)^k q^{-\binom{k}{2}} t^k (s/t;q)_k $$

$$ \times \frac{(ys;q)_\infty}{(-1)^k q^{-\binom{k}{2}-k} (yt)^k (q/yt;q)_k} \qquad (\text{by using } (1.1)) $$

$$ = \frac{(ys;q)_\infty}{(yt;q)_\infty}\ {}_4\phi_3\left(\begin{matrix} q^{-N}, a, b, s/t \\ c, d, q/yt \end{matrix}; q, qf/y\right). $$

Theorem 2.2 *Let* ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta\right)$ *be the operator that defined in* (2.1), *then*

$$ {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\{P_n(x,y)y^j\} $$

$$ = \sum_{k=0}^{N} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(q^{-N}, a, b; q)_{k+i}}{(c, d; q)_{k+i}} f^{k+i} \frac{(q^{-j};q)_k}{(q;q)_k} P_{n-i}(x,y) q^{-ij+k} y^{j-k}. \qquad (2.2) $$

Proof.

$$ {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\{P_n(x,y)y^j\} $$

$$ = \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (-f)^k {\theta_y}^k \{P_n(x,y)y^j\} $$

$$ = \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (-f)^k \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} {\theta_y}^i \{P_n(x,y)\} {\theta_y}^{k-i} \{(yq^{-i})^j\} $$

$$= \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (-f)^k \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} (-1)^i \frac{(q;q)_n}{(q;q)_{n-i}} P_{n-i}(x,y)$$
$$\times q^{-ij}(-1)^{k-i} y^{j-k+i} q^{k-i} (q^{-j}; q)_{k-i} \qquad \text{(by using (1.13) and (1.16))}$$
$$= \sum_{i=0}^{k} \sum_{k=i}^{N} \frac{(q^{-N}, a, b; q)_k}{(c, d; q)_k} f^k \frac{1}{(q;q)_{k-i}(q;q)_i} \frac{(q;q)_n}{(q;q)_{n-i}} P_{n-i}(x,y)$$
$$\times q^{-ij+k-i} y^{j-k+i} (q^{-j}; q)_{k-j}$$
$$= \sum_{k=0}^{N} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(q^{-N}, a, b; q)_{k+i}}{(c, d; q)_{k+i}} f^{k+i} \frac{(q^{-j}; q)_k}{(q;q)_k} P_{n-i}(x,y) q^{-ij+k} y^{j-k}.$$

Theorem 2.3 *Let* ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)$ *be the operator that defined in* (2.1), *then*

$$ {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\{P_n(x,y)(yt;q)_\infty\}$$
$$= (yt;q)_\infty \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ft)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(n-k)} (q^{-n}; q)_j P_{n-j}(x,y)$$
$$\times (q/yt; q)_j y^j. \tag{2.3}$$

Proof.

$$ {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\{P_n(x,y)(yt;q)_\infty\}$$
$$= \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (-f)^k {\theta_y}^k \{P_n(x,y)(yt;q)_\infty\}$$
$$= \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (-f)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} {\theta_y}^j \{P_n(x,y)\} {\theta_y}^{k-j} \{(ytq^{-j}; q)_\infty\}$$
$$= \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (-f)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j \frac{(q;q)_n}{(q;q)_{n-j}} P_{n-j}(x,y)$$
$$\times (-1)^{k-j} (tq^{-j})^{k-j} (ytq^{-j}; q)_\infty \qquad \text{(by using (1.13) and (1.16))}$$
$$= (yt;q)_\infty \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ft)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} (-1)^j q^{-\binom{j}{2}+nj} (q^{-n}; q)_j P_{n-j}(x,y)$$
$$\times t^{-j} q^{-j(k-j)} (-1)^j (yt)^j q^{-\binom{j}{2}-j} (q/yt; q)_j \quad \text{(by using (1.1) and (1.2))}$$
$$= (yt;q)_\infty \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ft)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(n-k)} (q^{-n}; q)_j P_{n-j}(x,y) (q/yt; q)_j y^j.$$

3. The Generating Function for $\boldsymbol{B_n(a,b,c,d,f,x,y|q)}$

In this section, the polynomials $B_n(a,b,c,d,f,x,y|q)$ are defined . We also use the operator ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta\right)$ to obtain the generating function and its extension for the polynomials B_n. We give some special values for the parameters in the generating function as well as its extension for B_n, in order to recover the generating function and obtain its extension for the polynomials $U_n(x,y,a;q)$. We define the following polynomials:

$$B_n(a,b,c,d,f,x,y|q)=\sum_{k=0}^{N}\begin{bmatrix} n\\ k\end{bmatrix}\frac{(q^{-N},a,b;q)_k}{(c,d;q)_k}f^kP_{n-k}(x,y). \qquad (3.1)$$

- Establishing $c=q^{-N}$, $b=d=0$, $f\to f/a$, then $a\to\infty$ and then $f=a$, we get Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$.
- Setting $c=q^{-N}$, $a=b=d=0$ and $f=1$, so that we get the bivariate Rogers-Szegö polynomials $h_n(x,y|q)$, see [16].

Theorem 3.1 *Let* ${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N},a,b\\ c,d\end{matrix};q,-f\theta_y\right)$ *be the operator that defined in* (2.1), *then*

$${}_3\mathcal{E}_2\left(\begin{matrix} q^{-N},a,b\\ c,d\end{matrix};q,-f\theta_y\right)\{P_n(x,y)\}=B_n(a,b,c,d,f,x,y|q). \qquad (3.2)$$

Proof.

$$\begin{aligned}&{}_3\mathcal{E}_2\left(\begin{matrix} q^{-N},a,b\\ c,d\end{matrix};q,-f\theta_y\right)\{P_n(x,y)\}\\ &=\sum_{k=0}^{N}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(-f)^k{\theta_y}^k\{P_n(x,y)\}\\ &=\sum_{k=0}^{N}\begin{bmatrix} n\\ k\end{bmatrix}\frac{(q^{-N},a,b;q)_k}{(c,d;q)_k}f^kP_{n-k}(x,y) \qquad \text{(by using (1.16))}\\ &=B_n(a,b,c,d,f,x,y|q).\end{aligned}$$

Theorem 3.2 (The Generating Function for $B_n(a,b,c,d,f,x,y|q)$). *Let* $B_n(a,b,c,d,f,x,y|q)$ *be the polynomials that defined in* (3.1), *then*

$$\sum_{n=0}^{\infty}B_n(a,b,c,d,f,x,y|q)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(xt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} q^{-N},a,b\\ c,d\end{matrix};q,ft\right),\quad |xt|<1. \qquad (3.3)$$

Proof.

$$\begin{aligned}&\sum_{n=0}^{\infty}B_n(a,b,c,d,f,x,y|q)\frac{t^n}{(q;q)_n}\\ &=\sum_{n=0}^{\infty}{}_3\mathcal{E}_2\left(\begin{matrix} q^{-N},a,b\\ c,d\end{matrix};q,-f\theta_y\right)\{P_n(x,y)\}\frac{t^n}{(q;q)_n} \qquad \text{(by using (3.2))}\\ &={}_3\mathcal{E}_2\left(\begin{matrix} q^{-N},a,b\\ c,d\end{matrix};q,-f\theta_y\right)\left\{\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}\right\}\\ &=\frac{1}{(xt;q)_\infty}\,{}_3\mathcal{E}_2\left(\begin{matrix} q^{-N},a,b\\ c,d\end{matrix};q,-f\theta_y\right)\{(yt;q)_\infty\} \qquad \text{(by using (1.9))}\\ &=\frac{1}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(-f)^k{\theta_y}^k\{(yt;q)_\infty\}\\ &=\frac{1}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(-f)^k(-t)^k(yt;q)_\infty \qquad \text{(by using (1.14))}\\ &=\frac{(yt;q)_\infty}{(xt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} q^{-N},a,b\\ c,d\end{matrix};q,ft\right).\end{aligned}$$

• Setting $c = q^{-N}$, $b = d = 0$, $f \to f/a$ and then $a \to \infty$ and then $f = a$ in (3.3), then using (1.3), so that we recover the generating function for Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ (1.18).

Theorem 3.3 (Extention of the Generating Function for $\mathrm{B_n(a, b, c, d, f, x, y|q)}$). *Let* $B_n(a, b, c, d, f, x, y|q)$ *be the polynomials that defined in* (3.1), *then*

$$\sum_{n=0}^{\infty} B_{n+m}(a, b, c, d, x, y|q) \frac{t^n}{(q;q)_n} = \frac{(ytq^m;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ftq^m)^k$$
$$\times \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(m-k)} (q^{-m}; q)_j P_{m-j}(x, y) (q^{1-m}/yt; q)_j y^j. \quad (3.4)$$

Proof.

$$\sum_{n=0}^{\infty} B_{n+m}(a, b, c, d, f, x, y|q) \frac{t^n}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty} {}_3\mathcal{E}_2 \left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y \right) \{P_{n+m}(x, y)\} \frac{t^n}{(q;q)_n} \qquad \text{(by using (3.2))}$$
$$= {}_3\mathcal{E}_2 \left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; \boldsymbol{q}, -f\theta_y \right) \left\{ P_m(x, y) \sum_{n=0}^{\infty} P_n(x, q^m y) \frac{t^n}{(q;q)_n} \right\}$$
$$= \frac{1}{(xt;q)_\infty} {}_3\mathcal{E}_2 \left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y \right) \{P_m(x, y)(q^m yt; q)_\infty\} \quad \text{(by using (1.9))}$$
$$= \frac{(ytq^m;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ftq^m)^k$$
$$\times \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(m-k)} (q^{-m}; q)_j P_{m-j}(x, y) (q^{1-m}/yt; q)_j y^j. \quad \text{(by using (2.3))}$$

• Letting $c = q^{-N}$, $b = d = 0$, $f \to f/a$, hence $a \to \infty$ and $f = a$ in (3.4), by using (1.3), we obtain an extension for the generating function for Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$.

Corollary 3.3.1 (Extention of the Generating Function for $\mathrm{U_n(x, y, a; q)}$). *Let* $U_n(x, y, a; q)$ *be the polynomials that defined in* (1.17), *then*

$$\sum_{n=0}^{\infty} U_{n+m}(x, y, a; q) \frac{t^n}{(q;q)_n} = \frac{(ytq^m;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (atq^m)^k}{(q;q)_k}$$
$$\times \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(m-k)} (q^{-m}; q)_j P_{m-j}(x, y) (q^{1-m}/yt; q)_j y^j.$$

4. Mehler's Formula for $B_n(a,b,c,d,f,x,y|q)$

In this part, we propose an operator approach to Mehler's formula and its extension for polynomials $B_n(a,b,c,d,f,x,y|q)$. The Miller formula and its extension for Al-Salam-Carlitz polynomials $U_n(x,y,a;q)$ are produced by providing specific values for variables in the Mehler's formula and its extension for $B_n(a,b,c,d,f,x,y|q)$.

Theorem 4.1 (The Mehler's Formula for $B_n(a,b,c,d,f,x,y|q)$). *Let* $B_n(a,b,c,d,f,x,y|q)$ *be the polynomials that defined in* (3.1), *then*

$$\sum_{n=0}^{\infty} B_n(a,b,c,d,f,x,y|q)B_n(a,b,c,d,f,z,w|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(xwt;q)_\infty}{(xzt;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(ft)^k\sum_{i=0}^{\infty}\frac{(w/z;q)_i}{(q,xwt;q)_i}(-1)^i q^{\binom{i}{2}}(q^k zt)^i$$
$$\times\sum_{j=0}^{i}\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}\frac{(q^{-N},a,b;q)_{j+s}}{(q,c,d;q)_{j+s}}f^{j+s}\frac{(q^{-i};q)_j}{(q;q)_j}P_{k-s}(x;y)q^{-si+j}y^{i-j}. \quad (4.1)$$

Proof. By using (3.2), we get

$$\sum_{n=0}^{\infty} B_n(a,b,c,d,f,x,y|q)B_n(a,b,c,d,f,z,w|q)\frac{t^n}{(q;q)_n}$$
$$=\sum_{n=0}^{\infty}\ {}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\{P_n(x,y)\}\,B_n(a,b,c,d,f,z,w;q)\frac{t^n}{(q;q)_n}$$
$$=\ {}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\left\{\sum_{n=0}^{\infty}P_n(x,y)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(q^{-N},a,b;q)_k}{(c,d;q)_k}f^kP_{n-k}(z,w)\frac{t^n}{(q;q)_n}\right\}$$
$$=\ {}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\left\{\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}P_n(x,y)\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}f^kP_{n-k}(z,w)\frac{t^n}{(q;q)_{n-k}}\right\}$$
$$=\ {}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\left\{\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}P_{n+k}(x,y)\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(ft)^kP_n(z,w)\frac{t^n}{(q;q)_n}\right\}$$
$$=\sum_{k=0}^{\infty}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(ft)^k$$
$$\times\ {}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\left\{P_k(x,y)\sum_{n=0}^{\infty}P_n(x,q^ky)P_n(z,w)\frac{t^n}{(q;q)_n}\right\}$$
$$=\frac{(xwt;q)_\infty}{(xzt;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(ft)^k$$
$$\times\ {}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\left\{P_k(x,y)\ {}_1\phi_1\left(\begin{matrix}w/z\\xwt\end{matrix};q,q^kyzt\right)\right\}$$
$$=\frac{(xwt;q)_\infty}{(xzt;q)_\infty}\sum_{k=0}^{\infty}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(ft)^k$$
$$\times\ {}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\left\{P_k(x,y)\sum_{i=0}^{\infty}\frac{(w/z;q)_i}{(q,xwt;q)_i}(-1)^iq^{\binom{i}{2}}(yztq^k)^i\right\}$$

$$= \frac{(xwt;q)_\infty}{(xzt;q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k} (ft)^k \sum_{i=0}^{\infty} \frac{(w/z;q)_i}{(q,xwt;q)_i} (-1)^i q^{\binom{i}{2}} (ztq^k)^i$$
$$\times \ {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N},a,b \\ c,d \end{matrix};q,-f\theta_y\right)\{P_k(x,y)y^i\}$$
$$= \frac{(xwt;q)_\infty}{(xzt;q)_\infty} \sum_{k=0}^{\infty} \frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k} (ft)^k \sum_{i=0}^{\infty} \frac{(w/z;q)_i}{(q,xwt;q)_i} (-1)^i q^{\binom{i}{2}} (q^k zt)^i$$
$$\times \sum_{j=0}^{i} \sum_{s=0}^{k} \begin{bmatrix} k \\ s \end{bmatrix} \frac{(q^{-N},a,b;q)_{j+s}}{(c,d;q)_{j+s}} f^{j+s} \frac{(q^{-i};q)_j}{(q;q)_j} P_{k-s}(x;y) q^{-si+j} y^{i-j}.$$

- When $c = q^{-N}$, $b = d = 0$, $f \to f/a$,then $a \to \infty$ and $f = a$ in (4.1), we get Mehler's formula for $U_n(x,y,a;q)$.

Corollary 4.1.1 (The Mehler's Formula for $U_n(x,y,a;q)$). *Let* $U_n(x,y,a;q)$ *be the polynomials that defined in* (1.17), *then*

$$\sum_{n=0}^{\infty} U_n(x,y,a;q) U_n(z,w,a|q) \frac{t^n}{(q;q)_n}$$
$$= \frac{(xwt;q)_\infty}{(xzt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (at)^k \sum_{i=0}^{\infty} \frac{(w/z;q)_i}{(q,xwt;q)_i} (-1)^i q^{\binom{i}{2}} (q^k zt)^i$$
$$\times \sum_{j=0}^{i} \sum_{s=0}^{k} \begin{bmatrix} k \\ s \end{bmatrix} (-1)^{j+s} q^{{j+s}_2} a^{j+s} \frac{(q^{-i};q)_j}{(q;q)_j} P_{k-s}(x;y) q^{-si+j} y^{i-j}.$$

Theorem 4.2 (Extention of Mehler's Formula for $B_n(a,b,c,d,f,x,y|q)$). *Let* $B_n(a,b,c,d,f,x,y|q)$ *be the polynomials that defined in* (3.1), *then*

$$\sum_{n=0}^{\infty} B_{n+m}(a,b,c,d,f,x,y|q) B_n(a,b,c,d,f,z,w|q) \frac{t^n}{(q;q)_n}$$
$$= \frac{(xwt;q)_\infty}{(xzt;q)_\infty} \sum_{k=0}^{N} \frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k} (ft)^k \sum_{j=0}^{\infty} \frac{(w/z;q)_j}{(q,xwt;q)_j} (q^{m+k} zt)^j (-1)^j q^{j_2}$$
$$\times \sum_{s=0}^{N} \sum_{i=0}^{m+k} \begin{bmatrix} m+k \\ i \end{bmatrix} \frac{(q^{-N},a,b;q)_{s+i}}{(c,d;q)_{s+i}} f^{s+i} \frac{(q^{-j};q)_s}{(q;q)_s} P_{m+k-i}(x,y) q^{-ij+s} y^{j-s}. \quad (4.2)$$

Proof. By using (3.2), we get

$$\sum_{n=0}^{\infty} B_{n+m}(a,b,c,d,f,x,y|q) B_n(a,b,c,d,f,z,w|q) \frac{t^n}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty} {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N},a,b \\ c,d \end{matrix};q,-f\theta_y\right)\{P_{n+m}(x,y)\} B_n(a,b,c,d,f,z,w|q) \frac{t^n}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty} {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N},a,b \\ c,d \end{matrix};q,-f\theta_y\right)\{P_{n+m}(x,y)\} \sum_{k=0}^{N} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q^{-N},a,b;q)_k}{(c,d;q)_k} f^k P_{n-k}(z,w) \frac{t^n}{(q;q)_n}$$

$$= \sum_{k=0}^{N}\sum_{n=k}^{\infty} {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\{P_{n+m}(x,y)\} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q^{-N}, a, b; q)_k}{(c, d; q)_k} f^k P_{n-k}(z,w) \frac{t^n}{(q;q)_n}$$

$$= \sum_{k=0}^{N}\sum_{n=0}^{\infty} {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\{P_{n+m+k}(x,y)\} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} f^k P_n(z,w) \frac{t^{n+k}}{(q;q)_n}$$

$$= \sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ft)^k$$

$$\times {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\left\{P_{m+k}(x,y)\left(\sum_{n=0}^{\infty} P_n(x, q^{m+k}y) P_n(z,w)\frac{t^n}{(q;q)_n}\right)\right\}$$

$$= \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ft)^k$$

$$\times {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\left\{P_{m+k}(x,y)\ {}_1\phi_1\left(\begin{matrix} w/z \\ xwt \end{matrix}; q, yztq^{m+k}\right)\right\}$$

$$= \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ft)^k \sum_{j=0}^{\infty}\frac{(w/z;q)_j}{(q, xwt;q)_j}(q^{m+k}zt)^j(-1)^j q^{\binom{j}{2}}$$

$$\times {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\{P_{m+k}(x,y)y^j\}$$

$$= \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ft)^k \sum_{j=0}^{\infty}\frac{(w/z;q)_j}{(q, xwt;q)_j}(q^{m+k}zt)^j(-1)^j q^{\binom{j}{2}}$$

$$\times {}_3\mathcal{E}_2\left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta_y\right)\{P_{m+k}(x,y)y^j\}$$

$$= \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\sum_{k=0}^{N} \frac{(q^{-N}, a, b; q)_k}{(q, c, d; q)_k} (ft)^k \sum_{j=0}^{\infty}\frac{(w/z;q)_j}{(q, xwt;q)_j}(q^{m+k}zt)^j(-1)^j q^{\binom{j}{2}}$$

$$\times \sum_{s=0}^{N}\sum_{i=0}^{m+k} \begin{bmatrix} m+k \\ i \end{bmatrix} \frac{(q^{-N}, a, b; q)_{s+i}}{(c, d; q)_{s+i}} f^{s+i} \frac{(q^{-j};q)_s}{(q;q)_s} P_{m+k-i}(x,y) q^{-ij+s} y^{j-s}.$$

(by using (2.2))

• If $c = q^{-N}$, $b = d = 0$, $f \to f/a$ then $a \to \infty$ and then $f = a$ in (4.2), we acquire an

extension of the Mehler's formula for $U_n(x, y, a; q)$.

Corollary 4.2.1 (Extention of Mehler's Formula for $U_n(x, y, a; q)$). *Let* $U_n(x, y, a; q)$ *be the polynomials that defined in* (1.17), *then*

$$\sum_{n=0}^{\infty} U_{n+m}(x, y, a; q) U_n(z, w, a; q) \frac{t^n}{(q;q)_n}$$

$$= \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\sum_{k=0}^{N} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (at)^k \sum_{j=0}^{\infty}\frac{(w/z;q)_j}{(q, xwt;q)_j}(q^{m+k}zt)^j(-1)^j q^{\binom{j}{2}}$$

$$\times \sum_{s=0}^{N}\sum_{i=0}^{m+k} \begin{bmatrix} m+k \\ i \end{bmatrix} (-1)^{s+i} q^{{s+i}_2}\, a^{s+i} \frac{(q^{-j};q)_s}{(q;q)_s} P_{m+k-i}(x,y) q^{-ij+s} y^{j-s}.$$

5. The Rogers Formula for $B_n(a,b,c,d,f,x,y|q)$

In this section, we plan to provide an operator approach to Rogers formula for the polynomials $B_n(a,b,c,d,f,x,y|q)$. The Rogers formula for the polynomials $B_n(a,b,c,d,f,x,y|q)$ leads to Rogers formula for Al-Salam-Carlitz polynomials by including special values for variables in the Rogers formula for $B_n(a,b,c,d,f,x,y|q)$.

Theorem 5.1 (The Rogers Formula for $B_n(a,b,c,d,f,x,y|q)$). *Let* $B_n(a,b,c,d,f,x,y|q)$ *be the polynomials that defined in* (3.1), *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{n+m}(a,b,c,d,f,x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{N}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(q^m ft)^k$$
$$\times\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix} q^{j(m-k)}(q^{-m};q)_j P_{m-j}(x,y)(q/ytq^m;q)_j y^j\frac{s^m}{(q,yt;q)_m}. \tag{5.1}$$

Proof. By using (3.2), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{n+m}(a,b,c,d,f,x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\{P_{n+m}(x,y)\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}{}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\{P_m(x,y)P_n(x,q^m y)\}\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$=\sum_{m=0}^{\infty}{}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\left\{\left(\sum_{n=0}^{\infty}P_n(x,q^m y)\frac{t^n}{(q;q)_n}\right)P_m(x,y)\frac{s^m}{(q;q)_m}\right\}$$
$$=\sum_{m=0}^{\infty}{}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\left\{\frac{(q^m yt;q)_\infty}{(xt;q)_\infty}P_m(x,y)\frac{s^m}{(q;q)_m}\right\}$$
$$=\frac{1}{(xt;q)_\infty}\sum_{m=0}^{\infty}{}_3\mathcal{E}_2\left(\begin{matrix}q^{-N},a,b\\c,d\end{matrix};q,-f\theta_y\right)\{P_m(x,y)(q^m yt;q)_\infty\}\frac{s^m}{(q;q)_m}$$
$$=\frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{N}\frac{(q^{-N},a,b;q)_k}{(q,c,d;q)_k}(q^m ft)^k$$
$$\times\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix} q^{j(m-k)}(q^{-m};q)_j P_{m-j}(x,y)(q/ytq^m;q)_j y^j\frac{s^m}{(q,yt;q)_m}.$$

- Setting $c=q^{-N}$, $b=d=0$, $f\to f/a$ then $a\to\infty$ and then $f=a$ in (5.1), we regain Rogers formula for $U_n(x,y,a;q)$ (1.19).

Proof. Setting $c=q^{-N}$, $b=d=0$, $f\to f/a$ then $a\to\infty$ and then $f=a$ in (5.1), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} U_{n+m}(x,y,a;q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$

$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (q^m at)^k$$
$$\times \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{j(m-k)} (q^{-m};q)_j P_{m-j}(x,y) (q/ytq^m;q)_j y^j \frac{s^m}{(q,yt;q)_m}$$
$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (q^m at)^k \sum_{j=0}^{k} kj q^{j(m-k)} \frac{(q;q)_m}{(q;q)_{m-j}} (-1)^j q^{\binom{j}{2}-mj}$$
$$\times x^{m-j} (y/x;q)_{m-j} \frac{(q/yt;q)_j (yt;q)_m}{(q^{-j}yt;q)_m} q^{-mj} y^j \frac{s^m}{(q,yt;q)_m}$$
(by using (1.2) and [Appendix I, equation (I.13)]
$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (q^m at)^k \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} q^{-jk} \frac{(-1)^j q^{\binom{j}{2}}}{(q;q)_{m-j}}$$
$$\times x^{m-j} (y/x;q)_{m-j} \frac{(q/yt;q)_j}{(q^{-j}yt;q)_m} q^{-mj} y^j s^m$$
$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+j} q^{\binom{k+j}{2}}}{(q;q)_k} (q^m at)^{k+j} q^{-j(k+j)} \frac{(-1)^j q^{\binom{j}{2}}}{(q;q)_j (q;q)_{m-j}}$$
$$\times x^{m-j} (y/x;q)_{m-j} \frac{(q/yt;q)_j}{(q^{-j}yt;q)_m} q^{-mj} y^j s^m$$
$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}+mk-j}}{(q;q)_k} (at)^{k+j} \frac{x^{m-j} (y/x;q)_{m-j}}{(q;q)_j (q;q)_{m-j}} \frac{(q/yt;q)_j}{(q^{-j}yt)_m} y^j s^m$$
$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}+(m+j)k-j}}{(q;q)_k} (at)^{k+j} \frac{x^m (y/x;q)_m}{(q;q)_j (q;q)_m}$$
$$\times \frac{(q/yt;q)_j}{(q^{-j}yt;q)_j (yt;q)_m} y^j s^{m+j}$$
$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}+(m+j)k-j}}{(q;q)_k} (at)^{k+j} \frac{x^m (y/x;q)_m}{(q;q)_j (q;q)_m} \frac{(-1)^j q^{\binom{j}{2}+j}}{(yt)^j (yt;q)_m}$$
$\times y^j s^{m+j}$ (by using [1, Appendix I, equation (I.8)])
$$= \frac{(yt;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (at)^k \sum_{m=0}^{\infty} \frac{(y/x;q)_m}{(q,yt;q)_m} (q^k xs)^m \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}}}{(q;q)_j} (q^k as)^j$$
$$= \frac{(as,yt;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q,as;q)_k} (at)^k \ {}_2\phi_1 \left(\begin{matrix} \frac{y}{x}, 0 \\ yt \end{matrix} ; q, q^k xs \right) \qquad \text{(by using (1.3))}$$
$$= \frac{(as,yt;q)_\infty}{(xt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (at)^k}{(q,as;q)_k} \frac{(q^k ys;q)_\infty}{(q^k xs,yt;q)_\infty} \ {}_1\phi_1 \left(\begin{matrix} q^k xs \\ q^k ys \end{matrix} ; q, yt \right)$$
$$= \frac{(as,ys;q)_\infty}{(xs;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (xs;q)_k (at)^k}{(q,as,ys;q)_k} \frac{1}{(xt;q)_\infty} \ {}_1\phi_1 \left(\begin{matrix} q^k xs \\ q^k ys \end{matrix} ; q, yt \right)$$

$$= \frac{(as, ys; q)_\infty}{(xs; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (xs; q)_k (at)^k}{(q, as, ys; q)_k} \; {}_2\phi_1 \left(\begin{matrix} y/x, 0 \\ q^k ys \end{matrix}; q, xt \right).$$

Conclusions

1. The finite q-exponential operator ${}_2\mathcal{E}_1 \left[\begin{matrix} q^{-N}, w \\ v \end{matrix}; q, d\theta \right]$ and the generalized q-exponential operator $\mathbb{E} \left[\begin{matrix} u, v \\ w \end{matrix} q; t\theta \right]$ are special cases of the operator ${}_3\mathcal{E}_2 \left(\begin{matrix} q^{-N}, a, b \\ c, d \end{matrix}; q, -f\theta \right)$, see [15,16].
2. The Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ and the bivariate Rogers-Szego polynomials $h_n(x, y|q)$ are special cases of the polynomials $B_n(a, b, c, d, f, x, y|q)$, see [4,9,14,17].
3. The polynomials identities for $B_n(a, b, c, d, f, x, y|q)$ are an extension of the polynomials identities for the Al-Salam-Carlitz polynomials $U_n(x, y, a; q)$ and the bivariate Rogers-Szego polynomials $h_n(x, y|q)$.

Acknowledgement. Special thanks go out to the referees for their thoughtful remarks that are helped us to enhance an earlier manuscript.

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