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Fully Prime Semimodule, Fully Essential Semimodule and Semi-Complement Subsemimodules

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Abstract

The concept of semi-essential semimodule has been studied by many researchers.

In this paper, we will develop these results by setting appropriate conditions, and defining new properties, relating to our concept, for example (fully prime semimodule, fully essential semimodule and semi-complement subsemimodule) such that: if for each subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$ is prime, then $\mathcal{W}$ is fully prime. If every semi-essential subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$ is essential then $\mathcal{W}$ is fully essential. Finally, a prime subsemimodule $\mathcal{H}$ of $\mathcal{W}$ is called semi-relative intersection complement (briefly, semi-complement) of subsemimodule $\mathcal{V}$ in $\mathcal{W}$, if $\mathcal{V} \cap \mathcal{H} = 0$, and whenever $\mathcal{V} \cap \mathcal{B} = 0$ with $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, $\mathcal{H} \subseteq \mathcal{B}$, then $\mathcal{H} = \mathcal{B}$. Furthermore, some results about the above concepts have been mentioned.

Keywords: Prime subsemimodule, Semi-essential subsemimodule, Multiplication semimodule, Fully prime semimodule, Fully essential semimodule and Semi-complement subsemimodule.

شبه المقاس الاولي والاساسي التام وشبه المقاسات الجزئية الشبه مكملة

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الخلاصة

تمت دراسة مفهوم شبه الأساسي على شبه المقاس من قبل باحثين. في هذا البحث سنناقش بعض النتائج حول شبه الأساسي من خلال إضافة بعض الشروط للحصول على نتائج جديدة، بالإضافة الى مفاهيم أخرى أطلقنا عليها حسب مفهومنا (شبه المقاس الاولي التام وشبه المقاس الأساسي التام وشبه مقاسات جزئية شبه مكملة) حيث: إذا كان كل شبه أساسي جزئي من شبه المقاس هو أساسي فنسمى شبه المقاس بـ أساسي تام. وإذا كان كل شبه مقاس جزئي من شبه المقاس هي شبه مقاس اولي فنسمى شبه المقاس بـ شبه المقاس الأولي التام، وأخيرا شبه المقاسات الجزئية الأولية من شبه المقاس تسمى بـ شبه مقاسات جزئية شبه مكملة لشبه مقاسات جزئية أخرى إذا كانت الأخيرة هي عظمى وان تقاطع المقاسين هو صفر. بالإضافة الى بعض النتائج حول المفاهيم الأخيرة.

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Introduction

When studying semimodules on semirings which are wider than modules on rings, we found that it has a long history in terms of building classes, which are important generalizations of loops, and at the same time we notice important differences between them. Among these differences there are the concepts of prime subsemimodule and semi-essential of semimodule, so in this research we will try to address some properties of what is related to the "semi-essential subsemimodule". The study of semimodules over semiring has been extensively considered, as reviewed by Golan [6].

In this paper, we mention some important definitions that we will rely obtain some important results on the topic (semi-essential subsemimodule). Assume $\mathcal{T}$ is semiring. A left $\mathcal{T}$-semimodule $\mathcal{W}$ is a commutative monoid $(\mathcal{W},+)$ for which we have a function $\mathcal{T}\times\mathcal{W}\to\mathcal{W}$ defined by $(t,w)\mapsto tw$ $(t\in\mathcal{T},\ w\in\mathcal{W})$ such that for all $t,t'\in\mathcal{T}$ and $w,v\in\mathcal{W}$, the following conditions are satisfied: $t(w+v)=tw+tv$, $(t+t')w=tw+t'w$, $(t.t')w=t(t'w), 0_{\mathcal{T}}w=0=t0_{\mathcal{W}}$. When $1w=w$ holds for each $w\in\mathcal{W}$ then a left $\mathcal{T}$-semimodule $\mathcal{W}$ is said to be unitary [6, p.148]. A subsemimodule $\mathcal{V}$ of $\mathcal{T}$-semimodule $\mathcal{W}$ is said to be subtractive if each of $w,w'\in\mathcal{W}$ with $w, w+w'\in\mathcal{V}$ implies $w'\in\mathcal{V}$ [6, p.154]. A $\mathcal{T}$-semimodule $\mathcal{W}$ is said to be subtractive if all its subsemimodules are subtractive. A $\mathcal{T}$-semimodule $\mathcal{W}$ is called semisubtractive if for every w,w' in $\mathcal{W}$, there exists h in $\mathcal{W}$ such that $w=w'+h$ or $w+h=w'$ [12]. A $\mathcal{T}$-semimodule $\mathcal{W}$ is additively cancellative if for all w,h,k in $\mathcal{W}$ with $w+h=w+k$ implies $h=k$ [6, p.172]. If for any subsemimodules $\mathcal{X},\mathcal{Y}$ and $\mathcal{Z}$ of a $\mathcal{T}$-semimodule , $\mathcal{X}\cap(\mathcal{Y}+\mathcal{Z})=\mathcal{X}\cap\mathcal{Y}+\mathcal{X}\cap\mathcal{Z}$, holds then $\mathcal{W}$ is called distributive. Let $\mathcal{V}$ and $\mathcal{H}$ be subsemimodules of a $\mathcal{T}$-semimodule $\mathcal{W}$, $\mathcal{W}$ is said to be a direct sum of $\mathcal{V}$ and $\mathcal{H}$, denoted by $\mathcal{W}=\mathcal{V}\oplus\mathcal{H}$, if each $w\in\mathcal{W}$ uniquely written as $w=v+h$ where $v\in\mathcal{V}$ and $h\in\mathcal{H}$. In this case then $\mathcal{V}$ (similarly $\mathcal{H}$) is called a direct summand of $\mathcal{W}$ [6, p.184].

Now, basic essential subsemimodule and semi-essential subsemimodule have been studied by many researchers. The semi-essential subsemimodule has been known in more than one style. In [12], by Tsiba, semi-essential subsemimodule was defined by "(Let $\mathcal{W}$ be a left $\mathcal{T}$-semimodule and $\mathcal{V}$ is a subsemimodule of $\mathcal{W}$, then $\mathcal{V}$ is said to be semi-essential in $\mathcal{W}$, written as $\mathcal{V}\trianglelefteq_s\mathcal{W}$ if for any subsemimodule $\mathcal{H}$ of $\mathcal{W}$, $\mathcal{H}\cap\mathcal{V}=0\Longrightarrow\mathcal{H}=0$)". Where this definition was given for essential subsemimodule in [9], by Pawar. But in this research we will rely on the semi-essential subsemimodule concept as stated in [10], by Pawar, let $\mathcal{V}$ be a non-zero subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$, then $\mathcal{V}$ is called semi-essential if for every non-zero prime subsemimodule $\mathcal{B}$ of $\mathcal{W}$ then $\mathcal{V}\cap\mathcal{B}\neq 0$, i.e. if $\mathcal{V}\cap\mathcal{B}=0$ implies $\mathcal{B}=0$ (expressed by $\mathcal{V}\leq_{sem}\mathcal{W}$).

This paper consists of three sections. The first section contains some of the definitions we need in this work (preliminaries). In Section two, some new results are studied via adding the necessary conditions. Finally, in the last section, some new concepts have been studied as (fully prime semimodule, fully essential semimodule and semi-complement subsemimodule).

1- Preliminaries

Definition 1.1: [7] Let $0\neq\mathcal{V}$ be a subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$. Then $\mathcal{V}$ is said to be essential in $\mathcal{W}$ if for each subsemimodule $\mathcal{H}$ of $\mathcal{W}$, $\mathcal{V}\cap\mathcal{H}=0$ implies $\mathcal{H}=0$ (expressed by $\mathcal{V}\leq_e\mathcal{W}$).
In this case $\mathcal{W}$ is said to be an essential extension of $\mathcal{V}$.

Definition 1.2: [4] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule and $\mathcal{U}$,$\mathcal{V}$ be subsemimodules of $\mathcal{W}$. Then $\mathcal{U}$ is said to be intersection complement of $\mathcal{V}$ if $\mathcal{U} \cap \mathcal{V} = 0$ and $\mathcal{U}$ is a maximal subsemimodule of $\mathcal{W}$ that have zero intersection with $\mathcal{V}$.
The subsemimodules $\mathcal{U}$ and $\mathcal{V}$ are said to be mutually complement if they are intersection complement of each other.

Definition 1.3:[11] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule and $w \in \mathcal{W}$. The left annihilator of w is defined by $ann_{\mathcal{T}}(w) = \{t \in \mathcal{T} | \, tw = 0\}$.
Note: $ann(w)$ is a left ideal of $\mathcal{T}$. If $\mathcal{V}$ is a subsemimodule of $\mathcal{W}$, then $ann_{\mathcal{T}}(\mathcal{V}) = \{ t \in \mathcal{T} \mid tw = 0, \forall\, w \in \mathcal{V}\}$.

Definition 1.4: [13] If $\mathcal{T}$ is a semiring, $\mathcal{W}$ is a $\mathcal{T}$-semimodule. A subsemimodule $\mathcal{B}$ of $\mathcal{W}$ is called prime if

1) $\mathcal{B}$ is a proper subsemimodule of $\mathcal{W}$ and
2) If for any $w \in \mathcal{W}, t \in \mathcal{T}$ and $tw \in \mathcal{B} \Longrightarrow w \in \mathcal{B}$ or $t \in [\mathcal{B}\colon \mathcal{W}]$, where $[\mathcal{B}\colon \mathcal{W}] = \{t \in \mathcal{T} | t\mathcal{W} \subseteq \mathcal{B}\}$.

Recall that, in semimodules to define the quotient $\mathcal{W}/\mathcal{V}$, $\mathcal{V}$ must be a subtractive subsemimodule of $\mathcal{W}$. [6, p.165].
Now, if $\mathcal{B}$ is a subtractive subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$, then
$ann_{\mathcal{T}}(\mathcal{W}/\mathcal{B}) = \{t \in \mathcal{T} | t(\mathcal{W}/\mathcal{B}) = 0\} = \{t \in \mathcal{T} | t\mathcal{W}/\mathcal{B} = 0\} = \{t \in \mathcal{T} | t\mathcal{W} \subseteq \mathcal{B}\} = [\mathcal{B}\colon \mathcal{W}]$

Definition 1.5: [10] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $0 \neq \mathcal{V} \leq \mathcal{W}$, then $\mathcal{V}$ is called semi-essential subsemimodule in $\mathcal{W}$ if for every nonzero prime subsemimodule $\mathcal{B}$ of $\mathcal{W}$ then $\mathcal{V} \cap \mathcal{B} \neq 0$, i.e., if $\mathcal{V} \cap \mathcal{B} = 0$ then $\mathcal{B} = 0$ (expressed by $\mathcal{V} \leq_{sem} \mathcal{W}$).
Note, every essential subsemimodule of $\mathcal{W}$ is semi-essential subsemimodule in $\mathcal{W}$.

Definition 1.6: [6] A $\mathcal{T}$-semimodule $\mathcal{W}$ is called multiplication if for each subsemimodule of $\mathcal{W}$ is of the form $I\mathcal{W}$, for some ideal I of $\mathcal{T}$.
Note, if $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule then $\mathcal{V} = [\mathcal{V}\colon \mathcal{W}]\mathcal{W}, \forall\, \mathcal{V} \leq \mathcal{W}$.

Definition 1.7: [6, p.153] A $\mathcal{T}$-semimodule $\mathcal{W}$ is called finitely generated by $\mathcal{A}$ if $\mathcal{A}$ is finite and $\mathcal{W}$ is the intersection of all subsemimodules of $\mathcal{W}$ containing $\mathcal{A}$.
Note, $\mathcal{T}\mathcal{A} = \{t_1 a_1 + t_2 a_2 + \cdots\cdots\cdots + t_n a_n | \, t_i \in \mathcal{T} \text{ and } a_i \in \mathcal{A}, i = 1,2, \dots n\}$.

Definition 1.8: [10] Let $\mathcal{W}, \mathcal{W}'$ be a $\mathcal{T}$-semimodules. A $\mathcal{T}$-homomorphism $\Psi\colon \mathcal{W} \longrightarrow \mathcal{W}'$ is called semi-essential if $\Psi(\mathcal{W})$ is a semi-essential subsemimodule of $\mathcal{W}'$.

- The radical of a $\mathcal{T}$-semimodule $\mathcal{W}$ (denoted $rad(\mathcal{W})$) is the intersection of all prime subsemimodules $\mathcal{B}$ of $\mathcal{W}$, i.e. $rad(\mathcal{W}) = \bigcap_{b \in spec(\mathcal{W})} \mathcal{B}$ where, $spec(\mathcal{W}) = \{ \mathcal{B}\colon \mathcal{B} \text{ is a prime subsemimodule of } \mathcal{W}\}$.
- The $\mathcal{T}$-semimodule $\mathcal{W}$ is faithful if for any nonzero $t \in \mathcal{T}$ there is an element $w \in \mathcal{W}$ such that $tw \neq 0$.
- The $\mathcal{T}$-semimodule $\mathcal{W}$ be denoted by (s-P) $\mathcal{T}$-semimodule, if any prime subsemimodule of $\mathcal{W}$ is subtractive in $\mathcal{W}$.

2- Some new propositions and results on the semi-essential subsemimodule after adding the necessary conditions

Proposition 2.1: [10] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule and $\mathcal{V}_1, \mathcal{V}_2$ subsemimodules of $\mathcal{W}$ such that $\mathcal{V}_1 \leq \mathcal{V}_2$. If $\mathcal{V}_1 \leq_{sem} \mathcal{W}$, then $\mathcal{V}_2 \leq_{sem} \mathcal{W}$.

Corollary 2.2: [10] Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be subsemimodules of a $\mathcal{T}$-semimodule $\mathcal{W}$. If $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{sem} \mathcal{W}$, then $\mathcal{V}_i \leq_{sem} \mathcal{W}$, $(i = 1,2)$. But the convers is not true.
Example 2.3: In this example we will explain that the convers of Corollary 2.2 is not true.
We will consider the N-semimodule $\mathcal{W} = N_{30}$ (the semimodule of integers modulo 30). In this semimodule, there are eight subsemimodules which are $\mathcal{V}_1 =< \overline{0} >, \mathcal{V}_2 =< \overline{2} >, \mathcal{V}_3 =< \overline{3} >, \mathcal{V}_4 =< \overline{5} >, \mathcal{V}_5 =< \overline{6} >, \mathcal{V}_6 =< \overline{10} >, \mathcal{V}_7 =< \overline{15} >$ and $\mathcal{V}_8 = \mathcal{W}$. And there are three prime subsemimodules which are $\mathcal{V}_2, \mathcal{V}_3,$ and $\mathcal{V}_4$. Now, $\mathcal{V}_2 \leq_{sem} \mathcal{W} = N_{30}$, since $\left[\begin{cases} 0 \neq \mathcal{V}_2 \cap \mathcal{V}_3 = \mathcal{V}_5 \\ 0 \neq \mathcal{V}_2 \cap \mathcal{V}_4 = \mathcal{V}_6 \end{cases}\right]$, also $\mathcal{V}_3 \leq_{sem} \mathcal{W} = N_{30}$, since $\left[\begin{cases} 0 \neq \mathcal{V}_3 \cap \mathcal{V}_2 = \mathcal{V}_5 \\ 0 \neq \mathcal{V}_3 \cap \mathcal{V}_4 = \mathcal{V}_7 \end{cases}\right]$. But $\mathcal{V}_2 \cap \mathcal{V}_3 = \mathcal{V}_5$ is not semi-essential subsemimodule, since $\mathcal{V}_5 \cap \mathcal{V}_4 =< \overline{0} >$.□

Proposition 2.4: [3] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, and $\mathcal{V}_1 \leq_{sem} \mathcal{W}, \mathcal{V}_2 \leq_{sem} \mathcal{W}$. If $(\mathcal{V}_2 \cap \mathcal{B})$ is a prime subsemimodule in $\mathcal{W}$ for every prime subsemimodules $\mathcal{B}$ in $\mathcal{W}$, then $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{sem} \mathcal{W}$.

Lemma 2.5: If $\mathcal{B}$ is a prime subsemimodule of a $\mathcal{T}$-semimodule $\mathcal{W}$ and $\mathcal{V} \leq \mathcal{W}$ such that $\mathcal{V} \nleq \mathcal{B}$. Then $(\mathcal{V} \cap \mathcal{B})$ is a prime subsemimodule of $\mathcal{V}$.

Proof: Let $v \in \mathcal{V} \leq \mathcal{W}$ and $t \in \mathcal{T}$ such that $tv \in (\mathcal{V} \cap \mathcal{B})$, then $tv \in \mathcal{V}$ and $tv \in \mathcal{B}$. Since $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, then $v \in \mathcal{B}$ or $t\mathcal{W} \subseteq \mathcal{B}$, but $t\mathcal{V} \subseteq t\mathcal{W}$ then $t\mathcal{V} \subseteq \mathcal{B}$. Now, either $(v \in \mathcal{V} \wedge v \in \mathcal{B})$ or $(t\mathcal{V} \subseteq \mathcal{V} \wedge t\mathcal{V} \subseteq \mathcal{B})$, consequently, $v \in (\mathcal{V} \cap \mathcal{B})$ or $t\mathcal{V} \subseteq (\mathcal{V} \cap \mathcal{B})$, hence $(\mathcal{V} \cap \mathcal{B})$ is a prime subsemimodule of $\mathcal{V}$.□

Proposition 2.6: [3] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V}_1, \mathcal{V}_2$ be a semi-essential subsemimodules of $\mathcal{W}$ such that $\mathcal{V}_1 \cap \mathcal{V}_2 \neq < 0 >$ and all a prime subsemimodules in $\mathcal{V}_1$ are prime subsemimodules in $\mathcal{W}$, then $(\mathcal{V}_1 \cap \mathcal{V}_2)$ is a semi-essential subsemimodule of $\mathcal{W}$.

Proposition 2.7: [3] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V}_i \leq \mathcal{W}, (i = 1,2,3)$ with $\mathcal{V}_1 \leq \mathcal{V}_2 \leq \mathcal{V}_3$. If $\mathcal{V}_1 \leq_{sem} \mathcal{V}_2$ and $\mathcal{V}_2 \leq_{sem} \mathcal{V}_3$ then $\mathcal{V}_1 \leq_{sem} \mathcal{V}_3$.
The converse is not true.

Example 2.8: [3] In this examples we will explain that the convers of Proposition 2.7 is not true.

We will consider the Z-semimodule $\mathcal{W} = Z_8 \oplus Z_2$. In this semimodule, there are eleven subsemimodules which are $\mathcal{V}_1 =< (\overline{0}, \overline{0}) >, \mathcal{V}_2 =< (\overline{1}, \overline{0}) >, \mathcal{V}_3 =< (\overline{0}, \overline{1}) >, \mathcal{V}_4 =< (\overline{1}, \overline{1}) >, \mathcal{V}_5 =< (\overline{2}, \overline{0}) >, \mathcal{V}_6 =< (\overline{2}, \overline{1}) >, \mathcal{V}_7 =< (\overline{4}, \overline{0}) >, \mathcal{V}_8 =< (\overline{4}, \overline{1}) >, \mathcal{V}_9 =< (\overline{0}, \overline{1}), (\overline{4}, \overline{0}) >, \mathcal{V}_{10} =< (\overline{2}, \overline{0}), (\overline{4}, \overline{1}) >$ and $\mathcal{V}_{11} = \mathcal{W}$. There are four prime subsemimodules of $\mathcal{W}$ which are $\mathcal{V}_2, \mathcal{V}_4, \mathcal{V}_5$ and $\mathcal{V}_{10}$, and three prime subsemimodules of the subsemimodule $\mathcal{V}_9$ which are $\mathcal{V}_3, \mathcal{V}_7$ and $\mathcal{V}_8$. Now, $\mathcal{V}_7 \leq \mathcal{V}_9 \leq \mathcal{W}$ and $\mathcal{V}_7 \leq_{sem} \mathcal{W}$, but $\mathcal{V}_7$ is not semi-essential in $\mathcal{V}_9$.□

Lemma 2.9: If $\mathcal{W}$ is a faithful multiplication $\mathcal{T}$-semimodule. Then $\mathcal{V} \leq_e \mathcal{W}$ if and only if $[\mathcal{V}: y] \leq_e \mathcal{T}$ for any $y \in \mathcal{W}$.

Proof: Since $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule, then $\mathcal{V} = I\mathcal{W}$ for some ideal I in $\mathcal{T}$ and $I \subseteq [\mathcal{V}: y]\ \forall\ y \in \mathcal{W}$. Now, assume that $[\mathcal{V}: y] \cap J = 0$ for some ideal J of $\mathcal{T}$. Thus $I \cap J = 0$,

so $(I \cap J)\mathcal{W} =< 0 >$ then $\mathcal{V} \cap J\mathcal{W} =< 0 >$. Since $\mathcal{V} \leq_e \mathcal{W}$, it follows $J\mathcal{W} =< 0 >$, but $\mathcal{W}$ is a faithful $\mathcal{T}$-semimodule, then $J = 0$. Therefore, $[\mathcal{V}: y] \leq_e \mathcal{T}$.
Conversely, if $[\mathcal{V}: y] \leq_e \mathcal{T}$, $\mathcal{V} = I\mathcal{W}$ and $\mathcal{V} \cap \mathcal{N} =< 0 >$ for some subsemimodule $\mathcal{N}$ of $\mathcal{W}$, then $\mathcal{N} = J\mathcal{W}$ for some $J \leq \mathcal{T}$ then $I\mathcal{W} \cap J\mathcal{W} =< 0 >$, thus $(I \cap J)\mathcal{W} =< 0 >$, but $\mathcal{W}$ is faithful $\mathcal{T}$-semimodule, then $I \cap J = 0$.

- If $[\mathcal{V}: y] \cap J \neq 0 \; \forall \, y \in \mathcal{W}$, then $\forall \, y \in \mathcal{W}$ then $\exists \, t \in J$ such that $0 \neq ty \in \mathcal{V}$ then $\mathcal{V} \leq_e \mathcal{W}$.
- If $[\mathcal{V}: y] \cap J = 0$ for some $y \in \mathcal{W}$, then $J = 0$, hence $\mathcal{N} =< 0 >$. Consequently, $\mathcal{V} \leq_e \mathcal{W}$.□

Lemma 2.10: Let E be a maximal ideal of $\mathcal{T}$, then $\mathcal{T}\backslash E$ is closed under multiplication.

Proof: Let $s_1, s_2 \in \mathcal{T}\backslash E$, then $\mathcal{T}s_1 + E = \mathcal{T} \longrightarrow \exists \, t \in \mathcal{T}$ such that $ts_1 + e = 1$ for some $t \in \mathcal{T}, e \in E$, consequently, $ts_1 s_2 + es_2 = s_2$. If $s_1 s_2 \in E$, then $s_2 \in E$, hence $s_1 s_2 \notin E$. That is $s_1 s_2 \in \mathcal{T}\backslash E$. □

Lemma 2.11: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, E a maximal ideal of $\mathcal{T}$, then: $T_E(\mathcal{W}) = ann_{\mathcal{W}}(\mathcal{T}\backslash E) = \{w \in \mathcal{W} | sw = 0 \text{ for some } s \in \mathcal{T}\backslash E\}$ is a subtractive subsemimodule of $\mathcal{W}$.

Proof: Since $0 \in \mathcal{W}$, since $1.0 = 0 \wedge 1 \notin E \longrightarrow T_E(\mathcal{W}) \neq \phi$. Let $w_1, w_2 \in T_E(\mathcal{W})$, then $s_1 w_1 = 0$ and $s_2 w_2 = 0$ for some $s_1, s_2 \in \mathcal{T}\backslash E$, hence $s_1 s_2(w_1 + w_2) = 0$ and $s_1 s_2 \in \mathcal{T}\backslash E$ then by Lemma 2.10 we have $(w_1 + w_2) \in T_E(\mathcal{W})$. Let $w \in \mathcal{T}_E(\mathcal{W})$ and $t \in \mathcal{T}$, then $sw = 0$ for some $s \notin E$, hence $s(tw) = 0 \longrightarrow tw \in T_E(\mathcal{W})$. Now, if $w_1, w_1 + w_2 \in T_E(\mathcal{W})$, then $s_1 w_1 = 0 \wedge s_2(w_1 + w_2) = 0 \longrightarrow s_2 s_1 w = 0 \wedge s_1 s_2 w_1 + s_1 s_2 w_2 = 0 \longrightarrow s_1 s_2 w_2 = 0 \longrightarrow w_2 \in T_E(\mathcal{W})$. □

- We say $\mathcal{W}$ is E-cyclie provided $\exists \, s \in \mathcal{T}\backslash E$ and $w \in \mathcal{W}$ such that $s\mathcal{W} \subseteq \mathcal{T}w$.

Lemma 2.12: Let $\mathcal{T}$ be a semisubtractive commutative semiring with identity. Then a $\mathcal{T}$-semimodule $\mathcal{W}$ is cancellative multiplication semimodule if and only if for each maximal ideal E of $\mathcal{T}$ either $\mathcal{W} = T_E(\mathcal{W})$ or $\mathcal{W}$ is E-cyclic.

Proof:
$\Longrightarrow$) Assume that $\mathcal{W}$ is multiplication and E is a maximal ideal of $\mathcal{T}$. Suppose that $\mathcal{W} = E\mathcal{W}$. Let $w \in \mathcal{W}$, then $\mathcal{T}w = \mathcal{A}\mathcal{W}$ for some ideal $\mathcal{A}$ of $\mathcal{T}$. Hence $\mathcal{T}w = \mathcal{A}\mathcal{W} = \mathcal{A}E\mathcal{W} = E\mathcal{A}\mathcal{W} = E\mathcal{W} \longrightarrow w = ew$ for some $e \in E$. Since $\mathcal{T}$ is semisubtractive we have either $1 = s + e$ or $1 + s = e$ for some $s \in \mathcal{T}$ (then $s \in \mathcal{T}\backslash E$) $\Longrightarrow$ $\begin{cases} \text{either } sw + ew = ew \longrightarrow sw = 0 \\ \text{or } \quad w = w + sw \longrightarrow sw = 0 \end{cases}$, $\Longrightarrow w \in T_E(\mathcal{W}) \longrightarrow \mathcal{W} = T_E(\mathcal{W})$. If $\mathcal{W} \neq E\mathcal{W}$, there exists $x \in \mathcal{W}$ and $x \notin E\mathcal{W}$, then there exists $I \leq \mathcal{T}$ such that $\mathcal{T}x = I\mathcal{W}$. Clearly, $I \nsubseteq E$, hence there exists $s \in I \wedge s \notin E, s\mathcal{W} \subseteq \mathcal{T}x$ and $\mathcal{W}$ is E-cyclic.
Conversely, assume that for every maximal ideal E of $\mathcal{T}$ either $\mathcal{W} = T_E(\mathcal{W})$ or $\mathcal{W}$ is E-cyclic. Let $\mathcal{V} \leq \mathcal{W}$ and $I = ann(\mathcal{W}/\mathcal{V})$. Clearly, $I\mathcal{W} \subseteq \mathcal{V}$. Let $y \in \mathcal{V}$ and $K = \{t \in \mathcal{T} | ty \in I\mathcal{W}\}$, if $K = \mathcal{T}$, then $y \in I\mathcal{W}$. If not, then there exists a maximal ideal Q of $\mathcal{T}$ such that $K \subseteq Q$. If $\mathcal{W} = T_Q(\mathcal{W})$ then $sy = 0$ for some $s \notin Q$, hence $s \in K \subseteq Q$, which is a contradiction. Thus, by hypothesis there exists $r \notin Q, z \in \mathcal{W}$ such that $r\mathcal{W} \subseteq \mathcal{T}z$. It follows that $r\mathcal{V}$ is a subsemimodule of $\mathcal{T}z$. Hence $r\mathcal{V} = Jz$ where J is the ideal $\{t \in \mathcal{T} | tz \in r\mathcal{V}\}$ of $\mathcal{T}$. Note that $rJ\mathcal{W} = Jr\mathcal{W} \subseteq Jz \subseteq \mathcal{V} \longrightarrow rJ \subseteq I \longrightarrow r^2 y \in r^2\mathcal{V} = rJz \subseteq I\mathcal{W}$. But this gives a contradiction $r^2 \in K \subseteq Q$ by using Lemma 2.10, thus $K = \mathcal{T}$ and $y \in I\mathcal{W}$. Therefore, $\mathcal{V} = I\mathcal{W}$ and $\mathcal{W}$ is a multiplication semimodule. □

Lemma 2.13: Let E be a prime ideal of a semiring $\mathcal{T}$, with $\mathcal{T}$ is semisubtractive and $\mathcal{W}$ is cancellative faithful multiplication $\mathcal{T}$-semimodule. Let $a \in \mathcal{T}, x \in \mathcal{W}$ satisfy $ax \in E\mathcal{W}$, then $a \in E$ or $x \in E\mathcal{W}$.

Proof: Assume that $a \notin E$. Let $K = \{t \in \mathcal{T} | tx \in E\mathcal{W}\}$. If $K = \mathcal{T}$, then $x \in E\mathcal{W}$. If not, then there exists a maximal ideal Q of $\mathcal{T}$ such that $K \subseteq Q$. Clearly $x \notin T_Q(\mathcal{W})$ (since if $sx = 0 \in E\mathcal{W} \longrightarrow s \in K \subseteq Q$). By Lemma. 2.12 we have $\mathcal{W}$ is Q-cyclic, that is $\exists\, w \in \mathcal{W}, q \notin Q$ such that $q\mathcal{W} \subseteq \mathcal{T}w$. In particular $qx = sw$, and $qax = ew$ for some $s \in \mathcal{T}$ and $e \in E$. Thus $saw = ew \longrightarrow \exists\, h \in ann_{\mathcal{T}}(w)$ such that $sa + h = e$ or $sa = e + h$. But $\mathcal{W}$ is faithful, hence $h = 0$, so $sa = e \in E \longrightarrow s \in E$ as E is prime ideal and $a \notin E$. Then $qx = sw \in E\mathcal{W} \longrightarrow q \in K \subseteq Q$, which is a contradiction. It follows that $K = \mathcal{T}$ and $x \in E\mathcal{W}$, as required.

Corollary 2.14: Let $\mathcal{W}$ be a faithful multiplication $\mathcal{T}$-semimodule, then E is a prime ideal of $\mathcal{T}$ with $\mathcal{W} \neq E\mathcal{W}$ if and only if $E\mathcal{W}$ is a prime subsemimodule of $\mathcal{W}$.

Proof: Since $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule and $\mathcal{W} \neq E\mathcal{W}$ then there exists $\mathcal{V} \leq \mathcal{W}$ such that $\mathcal{V} = E\mathcal{W}$.
$\Longrightarrow$) Assume that E is a prime ideal of $\mathcal{T}$, to prove $\mathcal{V} = E\mathcal{W}$ is a prime subsemimodule of $\mathcal{W}$.
Let $t \in \mathcal{T}, w \in \mathcal{W}$ and $tw \in \mathcal{V} = E\mathcal{W}$, (By Lemma 2.13) then $t \in E$ or $w \in E\mathcal{W} = \mathcal{V}$, but E is a prime ideal of $\mathcal{T}$, then $t.s \in E$ for any $s \in \mathcal{T} \Longrightarrow tw \in E\mathcal{W}$ for any $w \in \mathcal{W}$, therefore, $t\mathcal{W} \subseteq E\mathcal{W} = \mathcal{V} \Longrightarrow E\mathcal{W} = \mathcal{V}$ is a prime subsemimodule of $\mathcal{W}$.
$\Longleftarrow$) Suppose that $\mathcal{V} = E\mathcal{W}$ is a prime subsemimodule of $\mathcal{W}$, to prove E is a prime ideal of $\mathcal{T}$.
Let $t_1, t_2 \in \mathcal{T}$ and $t_1.t_2 \in E$, let $w \in \mathcal{W}$ then $(t_1.t_2)w \in E\mathcal{W} = \mathcal{V}$ implies $t_1(t_2w) \in E\mathcal{W} = \mathcal{V}$, then by using Lemma 2.13 we have $t_1 \in E$ or $t_2w \in E\mathcal{W} = \mathcal{V}$. But $\mathcal{V}$ is a prime subsemimodule in $\mathcal{W}$, then $w \in \mathcal{V}$ or $t_2 \in [\mathcal{V}: \mathcal{W}]$. Thus, E is a prime ideal of $\mathcal{T}$.□

Theorem 2.15: [7] If $\mathcal{W}$ is a faithful multiplication $\mathcal{T}$-semimodule. Then $\mathcal{V} \leq_{sem} \mathcal{W}$ if and only if $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$ for any $y \in \mathcal{W}$.
Proof:
$\Longrightarrow$) Let $\mathcal{V} \leq_{sem} \mathcal{W},\ \mathcal{V} = I\mathcal{W}$, then $I \subseteq [\mathcal{V}: y]$ for each $y \in \mathcal{W}$. To prove $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$. Suppose that $[\mathcal{V}: y] \cap E = 0$ for some prime ideal E of $\mathcal{T}$, then $I \cap E = 0$ (since $I \subseteq [\mathcal{V}: y]$), then $(I \cap E)\mathcal{W} =< 0 >$, thus $\mathcal{V} \cap E\mathcal{W} =< 0 >$. But by Corollary 2.14 we obtain $E\mathcal{W}$ is a prime subsemimodule of $\mathcal{W}$, hence $E\mathcal{W} =< 0 >$, as $\mathcal{W}$ is faithful $\mathcal{T}$-semimodule, thus $E = 0$. Therefore, $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$.
$\Longleftarrow$) If $[\mathcal{V}: y] \leq_{sem} \mathcal{T}\ \forall\, y \in \mathcal{W}, \mathcal{V} = I\mathcal{W}$ and $\mathcal{V} \cap \mathcal{B} =< 0 >$ for some prime subsemimodule $\mathcal{B}$ of $\mathcal{W}$, then $\mathcal{B} = E\mathcal{W}$ for some prime ideal E of $\mathcal{T}$. Now, $\mathcal{V} \cap \mathcal{B} =< 0 >$ then $I\mathcal{W} \cap E\mathcal{W} =< 0 >$, implies $(I \cap E)\mathcal{W} =< 0 >$, since $\mathcal{W}$ is faithful $\mathcal{T}$-semimodule, thus $I \cap E = 0$.
- If $[\mathcal{V}: y] \cap E = 0$ for some $y \in \mathcal{W}$, then $E = 0$ (since $[\mathcal{V}: y] \leq_{sem} \mathcal{T}$).
- If $[\mathcal{V}: y] \cap E \neq 0$ for all $y \in \mathcal{W}$, this mean $\forall\, y \in \mathcal{W},\ \exists\, t \in E$ such that $0 \neq ty \in \mathcal{V}$, then $\mathcal{V} \leq_{sem} \mathcal{W}$.

In any case $\mathcal{V} \leq_{sem} \mathcal{W}$.□

Lemma 2.16: [10] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V} \leq \mathcal{W}$ and $\mathcal{B}$ is a prime subsemimodule of $\mathcal{W}$. If $(\mathcal{V} \cap \mathcal{B}: y) = ann(\mathcal{W})$, for each $y \in \mathcal{W}$ and $y \notin (\mathcal{V} \cap \mathcal{B})$, then $(\mathcal{V} \cap \mathcal{B})$ is a prime subsemimodule of $\mathcal{W}$.

Proposition 2.17: If $\mathcal{W}$ is a $\mathcal{T}$-semimodule, $\mathcal{V}_1, \mathcal{V}_2$ semi-essential subsemimodules of $\mathcal{W}$, if $(\mathcal{V}_1 \cap \mathcal{B}: y) = ann(\mathcal{W})$ for each prime subsemimodule $\mathcal{B}$ of $\mathcal{W}$ for each $y \in \mathcal{W}$ and $y \notin (\mathcal{V}_1 \cap \mathcal{B})$, then $(\mathcal{V}_1 \cap \mathcal{V}_2)$ is a semi-essential subsemimodule of $\mathcal{W}$.

Proof: Let $\mathcal{B}$ be a prime subsemimodule of $\mathcal{W}$ such that $(\mathcal{V}_1 \cap \mathcal{V}_2) \cap \mathcal{B} =< 0 >$, implies $\mathcal{V}_2 \cap (\mathcal{V}_1 \cap \mathcal{B}) =< 0 >$. Then by Lemma 2.16 we get $\mathcal{V}_1 \cap \mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, since $\mathcal{V}_2 \leq_{sem} \mathcal{W}$, then $\mathcal{V}_1 \cap \mathcal{B} =< 0 >$, but $\mathcal{V}_1 \leq_{sem} \mathcal{W}$ then $\mathcal{B} =< 0 >$, therefore, $(\mathcal{V}_1 \cap \mathcal{V}_2) \leq_{sem} \mathcal{W}$.□

Lemma 2.18: Let $\mathcal{W}, \mathcal{W}'$ be a $\mathcal{T}$-semimodule, $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ be a $\mathcal{T}$-homomorphism. If $\mathcal{B}'$ is a prime subsemimodule of $\mathcal{W}'$ then $\Psi^{-1}(\mathcal{B}')$ is a prime subsemimodule of $\mathcal{W}$.
Proof: Let $t \in \mathcal{T}$, $w \in \mathcal{W}$ such that $tw \in \Psi^{-1}(\mathcal{B}')$ then $\Psi(tw) \in \mathcal{B}'$, since Ψ is a $\mathcal{T}$-homomorphism then $t\Psi(w) \in \mathcal{B}'$. But $\mathcal{B}'$ is a prime subsemimodule in $\mathcal{W}'$ then either $\Psi(w) \in \mathcal{B}'$ or $t\mathcal{W}' \leq \mathcal{B}'$. Therefore, $w \in \Psi^{-1}(\mathcal{B}')$ or $\Psi^{-1}(t\mathcal{W}') = t\mathcal{W} \subseteq \Psi^{-1}(\mathcal{B}')$, consequently, $\Psi^{-1}(\mathcal{B}')$ is a prime subsemimodule of $\mathcal{W}$.□

Corollary 2.19: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule, $\mathcal{V} \leq \mathcal{B} \leq \mathcal{W}$, where $\mathcal{V}$ is subtractive, and $\phi: \mathcal{W} \longrightarrow \frac{\mathcal{W}}{\mathcal{V}}$ be the natural map, then $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$ if and only if $\frac{\mathcal{B}}{\mathcal{V}}$ is a prime subsemimodule in $\frac{\mathcal{W}}{\mathcal{V}}$.

Proof: Let $t \in \mathcal{T}$, $w + \mathcal{V} \in \frac{\mathcal{W}}{\mathcal{V}}$ such that $t(w + \mathcal{V}) \in \frac{\mathcal{B}}{\mathcal{V}}$, implies $tw + \mathcal{V} \in \frac{\mathcal{B}}{\mathcal{V}}$, therefore, $tw \in \mathcal{B}$, since $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, then either $w \in \mathcal{B}$ or $t\mathcal{W} \subseteq \mathcal{B}$, then either $w + \mathcal{V} \in \frac{\mathcal{B}}{\mathcal{V}}$ or $t\left(\frac{\mathcal{W}}{\mathcal{V}}\right) \subseteq \frac{\mathcal{B}}{\mathcal{V}}$ .Therefore, $\frac{\mathcal{B}}{\mathcal{V}}$ is a prime subsemimodule in $\frac{\mathcal{W}}{\mathcal{V}}$.
Conversely, by Lemma 2.18 we have then $\phi^{-1}\left(\frac{\mathcal{B}}{\mathcal{V}}\right) = \mathcal{B}$ (where ϕ is the natural map of $\mathcal{W}$ onto $\frac{\mathcal{W}}{\mathcal{V}}$) is a prime subsemimodule of $\mathcal{W}$.□

Proposition 2.20: [3] Let $\mathcal{W}, \mathcal{W}'$ be a $\mathcal{T}$-semimodules, $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ be an isomorphism. If $\mathcal{V} \leq_{sem} \mathcal{W}$, then $\Psi(\mathcal{V}) \leq_{sem} \mathcal{W}'$.

Lemma 2.21: Let $\mathcal{W}$ be a semisubtractive, $\mathcal{W}'$ cancellative $\mathcal{T}$-semimodules and $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ be a homomorphism of semimodules. If $\mathcal{B}$ is a subtractive subsemimodule of $\mathcal{W}$ such that $ker\Psi \subseteq \mathcal{B}$, then $\Psi^{-1}(\Psi(\mathcal{B})) = \mathcal{B}$.
Proof: $\mathcal{B} \subseteq \Psi^{-1}(\Psi(\mathcal{B}))$ in general, true.
Let $w \in \Psi^{-1}(\Psi(\mathcal{B}))$, then $\Psi(w) \in \Psi(\mathcal{B})$, that is $\Psi(w) = \Psi(\mathcal{B})$ for some $b \in \mathcal{B}$, since $\mathcal{W}$ is semisubtractive, then there exists $h \in \mathcal{W}$ such that $w + h = b$ or $w = b + h$, in any case $h \in ker\Psi$, hence $h \in \mathcal{B}$ by our hypothesis, so $w = b + h \longrightarrow w \in \mathcal{B}, w + h = b \longrightarrow w \in \mathcal{B}$ as $\mathcal{B}$ is a subtractive. Therefore, $\Psi^{-1}(\Psi(\mathcal{B})) = \mathcal{B}$.□
Note, if $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ such that Ψ is onto then $[\mathcal{B}: \mathcal{W}] \subseteq [\Psi(\mathcal{B}): \mathcal{W}']$.

Lemma 2.22: Let $\mathcal{W}$ be an (s-P) semisubtractive, $\mathcal{W}'$ cancellative $\mathcal{T}$-semimodules and $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ is an epimorphism of semimodule. If $\mathcal{B}$ is a prime subsemimodule of $\mathcal{W}$ such that $ker\Psi \subseteq \mathcal{B}$, then $\Psi(\mathcal{B})$ is a prime subsemimodule of $\mathcal{W}'$.

Proof: Let $t \in \mathcal{T}, w' \in \mathcal{W}'$ and $tw' \in \Psi(\mathcal{B})$, since Ψ is epimorphism, then there exists $w \in \mathcal{W}$ such that $\Psi(w) = w'$, thus $t\Psi(w) \in \Psi(\mathcal{B})$ implies $\Psi(tw) \in \Psi(\mathcal{B}) \longrightarrow tw \in \Psi^{-1}(\Psi(\mathcal{B}))$, by using Lemma 2.21 we obtain $tw \in \mathcal{B}$. Since $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$, then either $w \in \mathcal{B}$ or $t \in [\mathcal{B}: \mathcal{W}]$, thus either $w' \in \Psi(\mathcal{B})$ or $t \in [\Psi(\mathcal{B}): \mathcal{W}']$. Therefore, $\Psi(\mathcal{B})$ is a prime subsemimodule of $\mathcal{W}'$.□

Proposition 2.23: Let $\mathcal{W}$ be an (s-P) semisubtractive, $\mathcal{W}'$ cancellative $\mathcal{T}$-semimodules and $\Psi: \mathcal{W} \longrightarrow \mathcal{W}'$ be a $\mathcal{T}$-epimorphism such that $ker(\Psi) \subseteq rad(\mathcal{W})$. If $\mathcal{H} \leq_{sem} \mathcal{W}'$, then $\Psi^{-1}(\mathcal{H}) \leq_{sem} \mathcal{W}$.

Proof: Let $\mathcal{B}$ be a non-zero prime subtractive subsemimodule of $\mathcal{W}$ such that $\Psi^{-1}(\mathcal{H}) \cap \mathcal{B} = < 0 >$. By Lemma 2.21 we have then $\Psi^{-1}(\mathcal{H}) \cap \Psi^{-1}(\Psi(\mathcal{B})) =< 0 >$ implies $\Psi^{-1}(\mathcal{H} \cap \Psi(\mathcal{B})) =< 0 >$, thus $\mathcal{H} \cap \Psi(\mathcal{B}) =< 0 >$, since $ker(\Psi) \subseteq rad(\mathcal{W}) \subseteq \mathcal{B}$, for every prime subsemimodule $\mathcal{B}$ of $\mathcal{W}$ and by Lemma 2.22 we get $\Psi(\mathcal{B})$ is a prime subsemimodule of $\mathcal{W}'$. But $\mathcal{H} \leq_{sem} \mathcal{W}'$ then $\Psi(\mathcal{B}) = 0$. Thus $\mathcal{B} \subseteq ker(\Psi) \subseteq \Psi^{-1}(\mathcal{H})$, hence $\mathcal{B} = \Psi^{-1}(\mathcal{H}) \cap \mathcal{B} = < 0 >$. Therefore, $\Psi^{-1}(\mathcal{H}) \leq_{sem} \mathcal{W}$.□

Remark 2.24: If $\mathcal{B}$ is a prime subsemimodule of $\mathcal{T}$-semimodule $\mathcal{W}$, and $\mathcal{B} < \mathcal{B}' \leq \mathcal{W}$, then $\mathcal{B}$ is a prime subsemimodule of $\mathcal{B}'$.

Proof: It is clear. □

Proposition 2.25: Let $\mathcal{W}$ be a finitely generated faithful and multiplication $\mathcal{T}$-semimodule, then $I \leq_{sem} J$ if and only if $I\mathcal{W} \leq_{sem} J\mathcal{W}$ for every two ideals I and J of $\mathcal{T}$.

Proof:
$\Longrightarrow$) Assume that $\leq_{sem} J$, to prove $I\mathcal{W} \leq_{sem} J\mathcal{W}$.
Let $\mathcal{B}$ be a prime subsemimodule in $J\mathcal{W}$ such that $I\mathcal{W} \cap \mathcal{B} =< 0 >$, since $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule then there exists a prime ideal E of $\mathcal{T}$ such that $\mathcal{B} = E\mathcal{W}$. Now, $< 0 >= I\mathcal{W} \cap E\mathcal{W} = (I \cap E)\mathcal{W}$ thus $(I \cap E)\mathcal{W} =< 0 >$, but $\mathcal{W}$ is a faithful $\mathcal{T}$-semimodule, then $(I \cap E) = 0$, since $\mathcal{B} = E\mathcal{W} \subsetneqq J\mathcal{W}$ and $\mathcal{W}$ is a finitely generated $\mathcal{T}$-semimodule then $E < J$ and since E is a prime in $\mathcal{T}$. By Remark 2.24 we have E is a prime in J. But $I \leq_{sem} J$ thus $E = 0$. Therefore, $I\mathcal{W} \leq_{sem} J\mathcal{W}$.
$\Longleftarrow$) Assume that $\mathcal{W} \leq_{sem} J\mathcal{W}$, to prove $I \leq_{sem} J$.
Let E be a prime ideal of J such that $(I \cap E) = 0$, then $(I \cap E)\mathcal{W} = 0.\mathcal{W}$ implies $I\mathcal{W} \cap E\mathcal{W} =< 0 >$, since E is a prime ideal of J and by Corollary 2.14 we obtain $E\mathcal{W}$ is a prime subsemimodule in $J\mathcal{W}$, but $I\mathcal{W} \leq_{sem} J\mathcal{W}$, then $E\mathcal{W} =< 0 >$ and since $\mathcal{W}$ is a faithful $\mathcal{T}$-semimodule, then $E = 0$. Therefore, $I \leq_{sem} J$.□

3- Fully prime semimodule, fully essential semimodule and semi-relative complement

The main objective of this section is to generalize the definition of each (semi-uniform, fully prime, fully essential and semi-complement) on semimodules after studying the above-mentioned definitions in modules. See [2], [5] and [8].

Definition 3.1: [1] A $\mathcal{T}$-semimodule $\mathcal{W}$ is called uniform if for each subsemimodule $\mathcal{V}$ of $\mathcal{W}$ is an essential subsemimodule in $\mathcal{W}$.

Definition 3.2: A $\mathcal{T}$-semimodule $\mathcal{W}$ is called semi-uniform if for any subsemimodule $\mathcal{V}$ of $\mathcal{W}$ is a semi-essential subsemimodule in $\mathcal{W}$.

Example 3.2.1: Consider the N-semimodule N_{36} (the semimodule of integers modulo 36). In this semimodule, there are nine subsemimodules which are $\mathcal{V}_1 =< \overline{0} >, \mathcal{V}_2 =< \overline{2} >, \mathcal{V}_3 =< \overline{3} >, \mathcal{V}_4 =< \overline{4} >, \mathcal{V}_5 =< \overline{6} >, \mathcal{V}_6 =< \overline{9} >, \mathcal{V}_7 =< \overline{12} >, \mathcal{V}_8 =< \overline{18} >$ and $\mathcal{V}_9 = N_{36}$. And there are two prime subsemimodules which are $\mathcal{V}_2$, and $\mathcal{V}_3$. Now, $\mathcal{V}_i \cap \mathcal{V}_2 \neq 0$ and $\mathcal{V}_i \cap \mathcal{V}_3 \neq 0$, for $i = 2,3, \ldots, 9$ then $\mathcal{V}_i \leq_{sem} N_{36}$. Therefore, N-semimodule N_{36} is semi-uniform.

Example 3.2.2: Consider the N-semimodule N_{24} (the semimodule of integers modulo 24). In this semimodule, there are eight subsemimodules which are $\mathcal{V}_1 =< \overline{0} >, \mathcal{V}_2 =< \overline{2} >, \mathcal{V}_3 =< \overline{3} >, \mathcal{V}_4 =< \overline{4} >, \mathcal{V}_5 =< \overline{6} >, \mathcal{V}_6 =< \overline{8} >, \mathcal{V}_7 =< \overline{12} >$ and $\mathcal{V}_8 = N_{24}$. And there are two prime subsemimodules which are $\mathcal{V}_2$, and $\mathcal{V}_3$. Now, $\mathcal{V}_8 \cap \mathcal{V}_3 = 0$ then $\mathcal{V}_8$ is not semi-essential in N_{24}. Therefore, N-semimodule N_{24} is not semi-uniform.

Definition 3.3: A $\mathcal{T}$-semimodule $\mathcal{W}$ is called fully prime if for each proper subsemimodule $\mathcal{V}$ of $\mathcal{W}$ is a prime subsemimodule in $\mathcal{W}$.

Example 3.3.1: Consider the N-semimodule N_{15} (the semimodule of integers modulo 15). In this semimodule, there are two a nonzero proper subsemimodules of N_{15} which are $\mathcal{V}_1 =< \overline{3} >$ and $\mathcal{V}_2 =< \overline{5} >$. And they prime subsemimodules in N_{15}. Therefore, N-semimodule N_{15} is fully prime semimodule.

Example 3.3.2: Consider the N-semimodule N_{24} (the semimodule of integers modulo 24). See (*Example 3.2.2*), note, $\mathcal{V}_6 =< \overline{8} >$ is not prime subsemimodule in N_{24}. Therefore, N-semimodule N_{24} is not fully prime.

Definition 3.4: A non-zero $\mathcal{T}$-semimodule $\mathcal{W}$ is called fully essential, if every nonzero semi-essential subsemimodule of $\mathcal{W}$ is an essential subsemimodule in $\mathcal{W}$.

Example 3.4.1: Consider the N-semimodule N_8 (the semimodule of integers modulo 8). In this semimodule, there are two a nonzero proper subsemimodules of N_8 which are $\mathcal{V}_1 =< \overline{2} >$ and $\mathcal{V}_2 =< \overline{4} >$. Now, $\mathcal{V}_1 =< \overline{2} > \leq_{sem} N_8 \Rightarrow < \overline{2} > \leq_e N_8$ And $\mathcal{V}_2 =< \overline{4} > \leq_{sem} N_8 \Rightarrow < \overline{4} > \leq_e N_8$. Therefore, N-semimodule N_8 is fully essential semimodule.

Example 3.4.2: Consider the N-semimodule N_{24} (the semimodule of integers modulo 24). See (*Example 3.2.2*), note, $\mathcal{V}_5 =< \overline{6} >$ is semi-essential of N_{24}. But, $\mathcal{V}_5 =< \overline{6} >$ is not essential subsemimodule in N_{24}, (since, $< \overline{6} > \cap < \overline{8} >= 0$. Therefore N-semimodule N_{24} is not fully essential.

Definition 3.5: Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule and $\mathcal{V} \leq \mathcal{W}$. A prime subsemimodule $\mathcal{H}$ of $\mathcal{W}$ is called semi-relative intersection complement (shortly semi complement) of $\mathcal{V}$ in $\mathcal{W}$ if $\mathcal{V} \cap \mathcal{H} = 0$ and whenever $\mathcal{V} \cap \mathcal{B} = 0$ with $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W}$ such that $\mathcal{H} \subseteq \mathcal{B}$, then $\mathcal{H} = \mathcal{B}$.

Example 3.5.1: Consider the N-semimodule N_{24} (the semimodule of integers modulo 24). See (*Example 3.2.2*), note, $\mathcal{V}_3 =< \overline{3} >$ is semi complement of $\mathcal{V}_6 =< \overline{8} >$ in N_{24}, (since $< \overline{3} >$ is a prime subsemimodule in N_{24} and $< \overline{8} > \cap < \overline{3} >= 0$).

Remark 3.6: If $\mathcal{W}$ is a fully prime $\mathcal{T}$-semimodule and $< 0 > \neq \mathcal{V} \leq \mathcal{H} \leq \mathcal{W}$, then $\mathcal{V}$ is semi-essential subsemimodule of $\mathcal{H}$ if and only if it is essential subsemimodule of $\mathcal{H}$.

Proof: Clear. □

Remark 3.7: [3] Let $\mathcal{W}$ be a $\mathcal{T}$-semimodule. Then $\mathcal{W}$ is a uniform $\mathcal{T}$-semimodule if and only if $\mathcal{W}$ is a semi-uniform and fully essential $\mathcal{T}$-semimodule.

Proposition 3.8: If $\mathcal{W}$ is a faithful and multiplication $\mathcal{T}$-semimodule. Then $\mathcal{W}$ is a semi-uniform $\mathcal{T}$-semimodule if and only if $\mathcal{T}$ is a semi-uniform semiring.

Proof:

$\Rightarrow$) $\mathcal{W}$ is a semi-uniform $\mathcal{T}$-semimodule and $I \leq \mathcal{T}$, then $\mathcal{V} = I\mathcal{W} \leq_{sem} \mathcal{W}$, and by Proposition 2.25, $I \leq_{sem} \mathcal{T}$ therefore $\mathcal{T}$ is a semi-uniform.

$\Leftarrow$) $\mathcal{T}$ is a semi-uniform semiring and $\mathcal{V} \leq \mathcal{W} \longrightarrow \mathcal{V} = I\mathcal{W}$ then $I \leq \mathcal{T} \xrightarrow{assumption} I \leq_{sem} \mathcal{T}$, by Proposition 2.25) then $\mathcal{V} \leq_{sem} \mathcal{W}$. Therefore, $\mathcal{W}$ is a semi-uniform $\mathcal{T}$-semimodule. □

Corollary 3.9: [3] If $\mathcal{W}$ is a fully prime $\mathcal{T}$-semimodule. Then $\mathcal{W}$ is a uniform $\mathcal{T}$-semimodule if and only if $\mathcal{W}$ is a semi-uniform $\mathcal{T}$-semimodule.

Proposition 3.10: Let $\mathcal{W}$ be a nonzero faithful and multiplication $\mathcal{T}$-semimodule. Then $\mathcal{W}$ is fully essential $\mathcal{T}$-semimodule if and only if $\mathcal{T}$ is fully essential semiring.

Proof:
$\Longrightarrow$) Assume that $\mathcal{W}$ is fully essential $\mathcal{T}$-semimodule. To prove $\mathcal{T}$ is fully essential semiring.

Let $I \leq_{sem} \mathcal{T}$, since $\mathcal{W}$ is a multiplication $\mathcal{T}$-semimodule, then there exists $\mathcal{V} \leq \mathcal{W}$ such that $\mathcal{V} = I\mathcal{W}$ and $I \subseteq [\mathcal{V}: y]$ for each $y \in \mathcal{W}$ thus, $[\mathcal{V}: y] \leq_{sem} \mathcal{T}\ \ \forall\, y \in \mathcal{W}$. By Theorem 2.15 we obtain:
$\mathcal{V} \leq_{sem} \mathcal{W} \xrightarrow{assumption} \mathcal{V} \leq_e \mathcal{W}$, and by Lemma 2.9 we get $[\mathcal{V}: y] \leq_e \mathcal{T}\ \ \forall\, y \in \mathcal{W}$. If $J \cap I = 0$, where $J \leq \mathcal{T}$, either $J \cap [\mathcal{V}: y] = 0$ for some $y \in \mathcal{W}$ then $J = 0$.
Or we have $J \cap [\mathcal{V}: y] \neq 0$ for any $y \in \mathcal{W}$, that is $\forall\, y \in \mathcal{W}, \exists\, t \in J$ such that $0 \neq ty \in \mathcal{V} = I\mathcal{W}$, hence $J \cap I \neq 0$, which is not possible. Therefore, $\mathcal{T}$ is fully essential semiring.
$\Longleftarrow$) Assume that $\mathcal{T}$ is fully essential semiring, to prove $\mathcal{W}$ is fully essential $\mathcal{T}$-semimodule.
Let $\mathcal{V} \leq_{sem} \mathcal{W}$, then:
$[\mathcal{V}: y] \leq_{sem} \mathcal{T}\ \ \forall\, y \in \mathcal{W} \xrightarrow{assumption} [\mathcal{V}: y] \leq_e \mathcal{T},\ \forall\, y \in \mathcal{W}$, by Lemma 2.9 we get $\mathcal{V} \leq_e \mathcal{W}$. Therefore, $\mathcal{W}$ is fully essential $\mathcal{T}$-semimodule. □

Proposition 3.11: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a $\mathcal{T}$-semimodule, where $\mathcal{W}_i \leq \mathcal{W}$, $(i = 1,2)$, $< 0 > \neq \mathcal{V}_i \leq \mathcal{W}_i$, $(i = 1, 2)$. If $\mathcal{V}_i \leq_{sem} \mathcal{W}_i$, $(i = 1, 2)$ then $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leq_{sem} \mathcal{W}$. But the converse is not true.

Proof: Let $0 \neq \mathcal{B} = \lambda_1(\mathcal{B}) \oplus \lambda_2(\mathcal{B})$ be a prime subsemimodule in $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$, where $\lambda_i(\mathcal{B})$ is natural projection function of $\mathcal{W}$ on $\mathcal{W}_i$, $(i = 1, 2)$. If $\mathcal{B} \cap \mathcal{W}_1 \neq 0$, by Lemma 2.5 we have $\mathcal{B} \cap \mathcal{W}_1$ is a prime subsemimodule in $\mathcal{W}_1$, since $\mathcal{V}_1 \leq_{sem} \mathcal{W}_1$ then $\mathcal{V}_1 \cap (\mathcal{B} \cap \mathcal{W}_1) \neq 0$, consequently, there exists $0 \neq b \in \mathcal{V}_1 \cap \mathcal{B} \cap \mathcal{W}_1$, then $0 \neq b \in \mathcal{V}_1 \cap \mathcal{B}$, thus $0 \neq b \in (\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B}$, therefore, $(\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B} \neq 0$. Similarly, if $\mathcal{B} \cap \mathcal{W}_2 \neq 0$, then $(\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B} \neq 0$.
Assume that $\mathcal{B} \cap \mathcal{W}_1 = \mathcal{B} \cap \mathcal{W}_2 = 0$, thus 0 is a prime subsemimodule in $\mathcal{W}_i$, $(i = 1, 2)$. Then any subsemimodule of $\mathcal{W}_i$ is prime subsemimodule by Remark 2.24 $(i = 1, 2)$. Then $\lambda_i(\mathcal{B})$ is a prime subsemimodule in $\mathcal{W}_i$, $i = 1,2$, since $\mathcal{V}_i \leq_{sem} \mathcal{W}_i$ then $\mathcal{V}_i \cap \lambda_i(\mathcal{B}) \neq 0$ for $(i = 1, 2)$.
Let $0 \neq b = \lambda_1(b) + \lambda_2(b) \in \mathcal{B}$, if $\lambda_2(b) = 0$ then $0 \neq b = \lambda_1(b) \in \lambda_1(\mathcal{B}) \cap \mathcal{W}_1 \subseteq \mathcal{B} \cap \mathcal{W}_1 = 0$ contradiction. Then $\lambda_2(b) \neq 0$, also $\lambda_1(b) \neq 0$, Hence $0 \neq b = \lambda_1(b) + \lambda_2(b) \in (\lambda_1(\mathcal{B}) \cap \mathcal{V}_1) \oplus (\lambda_2(\mathcal{B}) \cap \mathcal{V}_2) \subseteq \mathcal{V}_1 \oplus \mathcal{V}_2$. That is, $0 \neq b \in (\mathcal{V}_1 \oplus \mathcal{V}_2) \cap \mathcal{B}$. Therefore, $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leq_{sem} \mathcal{W}$.□

Example 3.12: In this examples we will explain that the convers of Proposition 3.11 is not true.
We will consider the Z-semimodule $\mathcal{W} = Z_4 \oplus Z_6$. And $\mathcal{V}_1 =< \overline{2} > \leq Z_4, \mathcal{V}_2 =< \overline{2} > \leq Z_6$, then,

$(\mathcal{V}_1 \oplus \mathcal{V}_2) = (<\overline{2}>\oplus<\overline{2}>) \leq_{sem} (Z_4 \oplus Z_6) = (\mathcal{W}_1 \oplus \mathcal{W}_2)$ (since $\left((<\overline{2}>\oplus<\overline{2}>) \cap \mathcal{B} \neq 0\right)$) for any $\mathcal{B}$ is a prime subsemimodule in $\mathcal{W} = Z_4 \oplus Z_6$. But $<\overline{2}>$ is not semi-essential subsemimodule in Z_6 (since $<\overline{2}>\cap<\overline{3}>=<\overline{0}>$). □

Proposition 3.13: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a $\mathcal{T}$-semimodule, where $\mathcal{W}_i \leq \mathcal{W}, i = 1,2$, $< 0 >\neq \mathcal{V}_1 \leq \mathcal{W}_1$ and $< 0 >\neq \mathcal{V}_2 \leq \mathcal{W}_2$, if $(\mathcal{V}_1 \oplus \mathcal{V}_2) \leq_{sem} \mathcal{W}$ then $\mathcal{V}_1 \leq_{sem} \mathcal{W}_1$, provided that every a prime subsemimodule in $\mathcal{W}_1$ is a prime subsemimodule of $\mathcal{W}$. [3, see Prop. 2.6].

Proposition 3.14: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a fully essential $\mathcal{T}$-semimodule, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are subsemimodules of $\mathcal{W}$, $< 0 >\neq \mathcal{N}_i \leq \mathcal{W}_i, i = 1, 2$, then $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$ if and only if $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, i = 1, 2$.

Proof:
$\Longrightarrow$) Assume that $(\mathcal{N}_1\oplus\mathcal{N}_2) \leq_{sem} \mathcal{W}$, to prove $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, i = 1, 2$.

Since $(\mathcal{N}_1\oplus\mathcal{N}_2) \leq_{sem} \mathcal{W}$ and $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ is fully essential $\mathcal{T}$-semimodule then $(\mathcal{N}_1\oplus\mathcal{N}_2) \leq_e \mathcal{W}$, let $< 0 >\neq \mathcal{H}_1 \leq \mathcal{W}_1 \leq \mathcal{W}$, then $(\mathcal{N}_1\oplus\mathcal{N}_2) \cap \mathcal{H}_1 \neq< 0 >$, implies $(\mathcal{N}_1 \cap \mathcal{H}_1) + (\mathcal{N}_2 \cap \mathcal{H}_1) \neq< 0 >$. But $(\mathcal{N}_2 \cap \mathcal{H}_1) = < 0 >$, conesquntly $(\mathcal{N}_1 \cap \mathcal{H}_1) \neq< 0 >$, then $\mathcal{N}_1 \leq_e \mathcal{W}_1$, therefore $\mathcal{N}_1 \leq_{sem} \mathcal{W}_1$. Similarly, $\mathcal{N}_2 \leq_{sem} \mathcal{W}_2$.

$\Longleftarrow$) Assume that $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, (i = 1, 2)$, to prove $(\mathcal{N}_1\oplus\mathcal{N}_2) \leq_{sem} \mathcal{W}$.

By Proposition. 3.11 we obtain $(\mathcal{N}_1\oplus\mathcal{N}_2) \leq_{sem} \mathcal{W}$.□

Proposition 3.15: Let $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ be a distributive and fully prime $\mathcal{T}$-semimodule, where $\mathcal{W}_1$ and $\mathcal{W}_2$ are subsemimodules of $\mathcal{W}$, $< 0 >\neq \mathcal{N}_i \leq \mathcal{W}_i$ $(i = 1, 2)$, then $(\mathcal{N}_1\oplus\mathcal{N}_2) \leq_{sem} \mathcal{W}$ if and only if $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, (i = 1, 2)$.

Proof:
$\Longrightarrow$)Assume that $(\mathcal{N}_1\oplus\mathcal{N}_2) \leq_{sem} \mathcal{W}$, to prove $\mathcal{N}_i \leq_{sem} \mathcal{W}_i$ $(i = 1, 2)$.

Let $\mathcal{P}_1$ be a prime subsemimodule of $\mathcal{W}_1$ such that $\mathcal{N}_1 \cap \mathcal{P}_1 =< 0 >$, since $\mathcal{W}$ is fully prime and $\mathcal{P}_1 \leq \mathcal{W}_1 \leq \mathcal{W}$ then $\mathcal{P}_1$ is a prime subsemimodule in $\mathcal{W}$. Since $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ thus $\mathcal{W}_1 \cap \mathcal{W}_2 =< 0 >$, implies $\mathcal{P}_1 \cap \mathcal{N}_2 =< 0 >$ for each $\mathcal{N}_2 \leq \mathcal{W}_2$. Now, $(\mathcal{N}_1 \cap \mathcal{P}_1)\oplus(\mathcal{N}_2\cap\mathcal{P}_1) =< 0 >$, since $\mathcal{W}$ is distributive $\mathcal{T}$-semimodule thus $(\mathcal{N}_1 \oplus \mathcal{N}_2) \cap \mathcal{P}_1 =< 0 >$, but $(\mathcal{N}_1 \oplus \mathcal{N}_2) \leq_{sem} \mathcal{W}$, therefore, $\mathcal{P}_1 =< 0 >$, consequently $\mathcal{N}_1 \leq_{sem} \mathcal{W}_1$. Similarly, $\mathcal{N}_2 \leq_{sem} \mathcal{W}_2$.

$\Longleftarrow$) Assume that $\mathcal{N}_i \leq_{sem} \mathcal{W}_i, (i = 1, 2)$, to prove $(\mathcal{N}_1\oplus\mathcal{N}_2) \leq_{sem} \mathcal{W}$.

Then by using Proposition. 3.11 we get $(\mathcal{N}_1\oplus\mathcal{N}_2) \leq_{sem} \mathcal{W}$.□

Proposition 3.16: Let $\mathcal{W}$ be an (s-P) $\mathcal{T}$-semimodule, $0 \neq \mathcal{V} \leq \mathcal{W}$ and $\mathcal{H}$ is a non-zero prime subsemimodule in $\mathcal{W}$. Then $\mathcal{H}$ is semi- complement of $\mathcal{V}$ in $\mathcal{W}$ if and only if $\frac{(\mathcal{V}\oplus\mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$.

Proof: Let $h: \mathcal{W} \longrightarrow \frac{\mathcal{W}}{\mathcal{H}}$ be the natural map.

$\Longrightarrow$) Assume that $\mathcal{H}$ is a semi-relative intersection complement of $\mathcal{V}$ in $\mathcal{W}$, to prove $\frac{(\mathcal{V}\oplus\mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$.

Let $\frac{\mathcal{B}}{\mathcal{H}}$ be a prime subsemimodule of $\frac{\mathcal{W}}{\mathcal{H}}$ such that $\frac{(\mathcal{V}\oplus\mathcal{H})}{\mathcal{H}} \cap \frac{\mathcal{B}}{\mathcal{H}} =< 0 >$, by Corollary 2.19 we get $h^{-1}(\frac{\mathcal{B}}{\mathcal{H}})$ is a prime subsemimodule in $\mathcal{W}$. Then $\mathcal{B} = h^{-1}(\frac{\mathcal{B}}{\mathcal{H}})$ is prime in $\mathcal{W}$, thus $\frac{(\mathcal{V}\oplus\mathcal{H})}{\mathcal{H}} \cap \frac{\mathcal{B}}{\mathcal{H}} =< 0 >$, then $\frac{(\mathcal{V}\oplus\mathcal{H})\cap\mathcal{B}}{\mathcal{H}} =< 0 >$,thus $(\mathcal{V} \oplus \mathcal{H}) \cap \mathcal{B} = \mathcal{H}$, therefore, $(\mathcal{V} \cap \mathcal{B}) \oplus \mathcal{H} = \mathcal{H}$, then $(\mathcal{V} \cap \mathcal{B}) \leq \mathcal{H}$, and since $(\mathcal{V} \cap \mathcal{B}) \leq \mathcal{V}$ then $(\mathcal{V} \cap \mathcal{B}) \leq (\mathcal{V} \cap \mathcal{H})$. Since $\mathcal{H}$ is a semi-relative intersection complement of $\mathcal{V}$, then $\mathcal{V} \cap \mathcal{B} =< 0 >$, but $\mathcal{V} \cap \mathcal{H} =< 0 >$, and $\mathcal{H} \subseteq \mathcal{B}$ then $\mathcal{H} = \mathcal{B}$, consequently, $\frac{\mathcal{B}}{\mathcal{H}} =< 0 >$. Therefore, $\frac{(\mathcal{V} \oplus \mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$.

$\Longleftarrow$) Assume that $\frac{(\mathcal{V} \oplus \mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$, to prove $\mathcal{H}$ is semi-relative intersection complement of $\mathcal{V}$ in $\mathcal{W}$.

Let $\mathcal{B}$ be a prime subsemimodule of $\mathcal{W}$ such that $\mathcal{H} \subseteq \mathcal{B}$ and $\mathcal{V} \cap \mathcal{B} =< 0 >$. Suppose that $b \in (\mathcal{V} \oplus \mathcal{H}) \cap \mathcal{B}$, thus $b = v + h$, where $v \in \mathcal{V}, h \in \mathcal{H}\ and\ b \in \mathcal{B}$. Since $\mathcal{H} \subseteq \mathcal{B}$ then $h \in \mathcal{B}$, but $\mathcal{B}$ is subtractive subsemimodule of $\mathcal{W}$ and $b = (v + h) \in \mathcal{B}$ then $v \in \mathcal{B}$, thus $v \in \mathcal{V} \cap \mathcal{B}$, therefore, $v = 0$, implies $b = h$, consequently $(\mathcal{V} \oplus \mathcal{H}) \cap \mathcal{B} = \mathcal{H}$. It follows that $\left(\frac{(\mathcal{V} \oplus \mathcal{H})}{\mathcal{H}}\right) \cap \frac{\mathcal{B}}{\mathcal{H}} =< 0 >$. But by Corollary 2.19 we have $\frac{\mathcal{B}}{\mathcal{H}}$ is a prime subsemimodule of $\frac{\mathcal{W}}{\mathcal{H}}$ and $\frac{(\mathcal{V} \oplus \mathcal{H})}{\mathcal{H}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{H}}$, then $\frac{\mathcal{B}}{\mathcal{H}} =< 0 >$, implies $\mathcal{B} = \mathcal{H}$, therefore $\mathcal{H}$ is semi-relative intersection complement of $\mathcal{V}$ in $\mathcal{W}$.□

Proposition 3.17: Let $\mathcal{W}$ be an (s-P) $\mathcal{T}$-semimodule, $< 0 > \neq \mathcal{V} \leq \mathcal{W}$. If $\mathcal{U}$ is a semi-complement of $\mathcal{V}$ in $\mathcal{W}$, then $(\mathcal{V} \oplus \mathcal{U}) \leq_{sem} \mathcal{W}$.

Proof: Let $h: \mathcal{W} \longrightarrow \frac{\mathcal{W}}{\mathcal{U}}$ be the natural map. Since $\mathcal{U}$ is a semi-complement of $\mathcal{V}$ in $\mathcal{W}$, then by Proposition 3.16 we have $\frac{(\mathcal{V} \oplus \mathcal{U})}{\mathcal{U}} \leq_{sem} \frac{\mathcal{W}}{\mathcal{U}}$, and by Proposition 2.23 we get $h^{-1}\left(\frac{\mathcal{V} \oplus \mathcal{U}}{\mathcal{U}}\right) \leq_{sem} \mathcal{W}$, implies $(\mathcal{V} \oplus \mathcal{U}) \leq_{sem} \mathcal{W}$.□

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