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The Fuzzy Length of Fuzzy Bounded Operator

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Abstract
In this paper we recall the definition of fuzzy length space on a fuzzy set after that we recall basic definitions and properties of fuzzy length. We define fuzzy bounded operator as an introduction to defined fuzzy length of an operator then we proved that the fuzzy length space FB$(\tilde{A},\tilde{E})$ consisting of all fuzzy bounded linear operators from a fuzzy length space $\tilde{A}$ into a fuzzy length space $\tilde{E}$ is fuzzy complete if $\tilde{E}$ is fuzzy complete. Also we proved that every finite dimensional fuzzy length space is fuzzy complete.

Keywords: Fuzzy length space on fuzzy set, Fuzzy bounded operator, Fuzzy continuous operator, FB $(\tilde{A},\mathbb{R})$.

الطول الضبابي لمؤثر مقيد ضبابيا

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الخلاصة
في هذا البحث استعنا بتعريف فضاء الطول الضبابي على مجموعة ضبابية بعد ذلك استعرضنا التعاريف والخواص الاساسية للطول الضبابي. عرفنا التقيد الضبابي للمؤثرات كمقدمة لتقديم تعريف الطول الضبابي لمؤثربعد ذلك برهنا ان فضاء الطول الضبابي FB$(\tilde{A},\tilde{E})$ الذي يتالف من مجموعة كل المؤثرات الخطية والمقيدة ضبابيا من فضاء الطول الضبابي $\tilde{A}$ الى فضاء الطول الضبابي $\tilde{E}$ يكون كامل اذا كانت $\tilde{E}$ كامل. وكذلك برهنا فضاء الطول الضبابي منتهي البعد يكون كامل.

1. Introduction

The theory of fuzzy set was introduced by Zadeh in 1965[1]. In 1984[2], Katsaras is the first one who introduced the notion of fuzzy norm on a linear space during his studying the notion fuzzy topological vector spaces. In 1984 Kaleva and Seikkala [3] introduced a fuzzy metric space. In 1992 Felbin [4] introduced the notion of fuzzy norm on a linear space so that the corresponding fuzzy metric is of Kaleva and Seikkala type Kramosil and Michalek introduced another idea of fuzzy metric space [5]. In 1994 Cheng and Mordeson [6] introduced the notion fuzzy norm on a linear space so that the corresponding fuzzy metric is of Kramosil and Michalek type. Bag and Samanta [7] in 2003 studied finite dimensional fuzzy normed linear spaces. In 2005 Saadati and Vaezpour [8] studied some results on fuzzy complete fuzzy normed spaces. In 2005 Bag and Samanta [9] studied fuzzy bounded linear proved the fixed point theorems on fuzzy normed linear spaces. In 2009 Sadeqi and Kia [10] studied fuzzy normed linear space and its topological structure. In 2010 Si, Cao and Yang [11] studied the continuity in an intuitionistic fuzzy normed space. In 2015 Nadaban [12] studied properties of

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fuzzy continuous mapping on a fuzzy normed linear spaces. The concept of fuzzy norm has been used in developing the fuzzy functional analysis and its applications and a large number of papers by different authors have been published, for reference, please see [13 - 22].

In this paper we recall that the definition of fuzzy length on fuzzy set as a modification of the notion of fuzzy norm on a linear space duo to Bag and Samanta [7].Then we define fuzzy bounded operator as an introduction to define the fuzzy length of a fuzzy bounded operator. The structure of the present paper is as follows: In section one we recall basic properties of fuzzy length space on a fuzzy set that's will be needed later. In section two we introduce the definition of fuzzy length of fuzzy bounded linear operator and functional to prove that the set of all fuzzy bounded linear operators from a fuzzy length space to another fuzzy length space is fuzzy complete.

2. Basic Concept about fuzzy set

Definition 2.1: [1]

Let U be a classical set of object, the fuzzy set $\widetilde{A}$ is completely characterized by the set of pairs: $\widetilde{A} = \{ (x, \mu_{\widetilde{A}}(x)): x \in U, 0\leq \mu_{\widetilde{A}}(x) \leq 1\}$.Where $\mu_{\widetilde{A}}(x))$ is membership of $\widetilde{A}$.

Definition 2.2: [11]

Suppose that $\widetilde{D}$ and $\widetilde{B}$ be two fuzzy sets in $V\neq \emptyset$ and $W\neq \emptyset$ respectively, then $\widetilde{D} \times \widetilde{B}$ is a fuzzy set whose membership is defined by:

$\mu_{\widetilde{D}\times \widetilde{B}}(d, b) = \mu_{\widetilde{D}}(d) \wedge \mu_{\widetilde{B}}(b)\ \forall\ (d, b) \in V\times W$.

Definition 2.3: [18]

A fuzzy point p in U is a fuzzy set with single element and is denoted by x_α or (x,α). Two fuzzy points x_α and y_β are said to be different if and only if $x\neq y$.

Definition 2.4: [7]

Suppose that d_β is a fuzzy point and $\widetilde{D}$ is a fuzzy set in U. then d_β is said to belongs to $\widetilde{D}$ which is written by $d_\beta \in \widetilde{D} \Leftrightarrow \mu_{\widetilde{D}}(x) > \beta$.

Proposition 2.5: [22]

Suppose that h: V→ W is a function. Then the image of the fuzzy point d_β in V, is the fuzzy point $h(d_\beta)$ in W with $h(d_\beta)=(h(d), \beta)$.

Definition 2.6: [23]

A binary operation $*: [0,1]^2 \to [0,1]$ is said to be t-norm (or continuous triangular norm) if for all p, q, t, r ∈ [0, 1] the conditions are satisfied:

(i) $p*q = q*p$

(ii) $p*1 = p$

(iii) $(p*q) * t = p*(q*t)$

(iv) If $p \leq q$ and $t \leq r$ then $p*t \leq q*r$.

Examples 2.7: [23]

When $p*q = p.q$ and $p*q=p\wedge q$ for all p, q ∈[0, 1] then $*$ is a continuous t-norm, where p.q is the ordinary multiplication and $\wedge$=min.

Remark 2.8: [23]

For all p>q, there is t such that $p*t \geq q$ and for every r, there is e such that $r* r \geq e$, where p, q, t, r and e belongs to [0, 1].

First we recall the main definition in this paper

Definition 2.9: [24]

Let U be a linear space over field $\mathbb{F}$ and let $\widetilde{A}$ be a fuzzy set in X. let $*$ be a t-norm and $\widetilde{F}$ be a fuzzy set from $\widetilde{A}$ to [0,1] such that:

(FL_1) $\widetilde{F}(x_\alpha) > 0$ for all $x_\alpha \in \widetilde{A}$.

(FL_2) $\widetilde{F}(x_\alpha) = 1$ if and only if $x_\alpha = 0$.

(FL_3) $\widetilde{F}(cx, \alpha)= \widetilde{F}(x, \frac{\alpha}{|c|})$, where $0\neq c \in \mathbb{F}$.

(FL_4) $\widetilde{F}(x_\alpha+y_\beta) \geq \widetilde{F}(x_\alpha) * \widetilde{F}(y_\beta)$.

(FL_5) $\widetilde{F}$ is a continuous fuzzy set for all $x_\alpha, y_\beta \in \widetilde{A}$ and α, β ∈[0,1].

Then the triple $(\widetilde{A}, \widetilde{F}, *)$ is called a fuzzy length space on the fuzzy *set* $\widetilde{A}$.

Definition 2.10: [24]

Suppose that $(\widetilde{D},\widetilde{F},*)$ is a fuzzy length space on the fuzzy set $\widetilde{D}$ then $\widetilde{F}$ is continuous fuzzy set if whenever $\{(x_n,\propto_n)\}$ converges to x_α in $\widetilde{D}$ then $\widetilde{F}\{(x_n,\propto_n)\}$ converges to $\widetilde{F}(x_\alpha)$ that is $\lim_{n\to\infty}\widetilde{F}[(x_n,\propto_n)] = \widetilde{F}(x_\alpha)$.

Proposition 2.11: [24]

Let $(U,\|.\|)$ be a normed space, suppose that $\widetilde{D}$ is a fuzzy set in U. Put $\|x_\alpha\| = \|x\|$. Then $(\widetilde{D},\|.\|)$, is a normed space.

Proposition: [24]

Suppose that $(U,\|.\|)$ is a normed space and assume that $\widetilde{D}$ is a fuzzy set in U. Put $p * q = p.q$ for all p, q$\in$ [0,1]. Define $\widetilde{F}_{\|.\|}(x_\alpha) = \frac{\alpha}{\alpha+\|x\|}$: Then $(\widetilde{D},\widetilde{F}_{\|.\|},*)$ is a fuzzy length space on the fuzzy set $\widetilde{D}$, is called the fuzzy length induced by $\|.\|$.

Definition 2.13: [24]

Let $\widetilde{A}$ be a fuzzy set in U, and assume that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space on the fuzzy set $\widetilde{A}$ .let $\widetilde{B}(x_\alpha,r)=\{y_\beta\in\widetilde{A} : \widetilde{F}(y_\beta - x_\propto)>(1-p)\}$.So $\widetilde{B}(x_\alpha,p)$ is said to be a fuzzy open fuzzy ball of center $x_\propto \in \widetilde{A}$ and radius r.

Definition 2.14: [24]

The sequence $\{(x_n,\propto_n)\}$ in a fuzzy length space $(\widetilde{A},\widetilde{F},*)$ on the fuzzy $set\ \widetilde{A}$ is fuzzy converges to a fuzzy point $x_\alpha\in \widetilde{A}$ if for a given ε, $0 < \varepsilon < 1$, then there exists a positive number K such that $\widetilde{F}[(x_n,\propto_n) - x_\alpha] > (1-\varepsilon)$ for all $n \geq K$.

Definition 2.15: [24]

The sequence $\{(x_n,\propto_n)\}$ in a fuzzy length space $(\widetilde{A},\widetilde{F},*)$ on the fuzzy $set\ \widetilde{A}$ is fuzzy converges to a fuzzy point $x_\alpha\in\widetilde{A}$ if $\lim_{n\to\infty}\widetilde{F}[(x_n,\propto_n) - x_\alpha] = 1$.

Theorem 2.16: [24]

The two Definitions 2.14 and 2.15 are equivalent.

Lemma 2.17: [24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space on the fuzzy set $\widetilde{A}$. Then $\widetilde{F}(x_\alpha - y_\beta) = \widetilde{F}(y_\beta - x_\alpha)$, for any $x_\alpha, y_\beta \in \widetilde{A}$.

Definition 2.18: [24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space and $\widetilde{D} \subseteq \widetilde{A}$ then $\widetilde{D}$ is called fuzzy open if for every $y_\beta \in \widetilde{D}$ there is $\widetilde{B}(y_\beta,q) \subseteq \widetilde{D}$. A subset $\widetilde{E} \subseteq \widetilde{A}$ is called fuzzy closed if $\widetilde{E}^c = \widetilde{A} - \widetilde{E}$ is fuzzy open.

Theorem 2.19: [24]

Any $\widetilde{B}(y_\beta,q)$ in a fuzzy length space $(\widetilde{A},\widetilde{F},*)$ is a fuzzy open.

Definition 2.20: [24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space, and assume that $\widetilde{D} \subseteq \widetilde{A}$ .Then the fuzzy closure of $\widetilde{D}$ is denoted by $\overline{\widetilde{D}}$ or FC($\widetilde{D}$) and is defined by $\overline{\widetilde{D}}$ is the smallest fuzzy closed fuzzy set that contains $\widetilde{D}$.

Definition 2.21: [24]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space, and assume that $\widetilde{D} \subseteq \widetilde{A}$ .Then $\widetilde{D}$ is said to be fuzzy dense in $\widetilde{A}$ if $\overline{\widetilde{D}} = \widetilde{A}$ or $FC(\widetilde{D}) = \widetilde{A}$.

Lemma 2.22: [25]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space, and assume that $\widetilde{D} \subseteq \widetilde{A}$, Then $d_\alpha \in \overline{\widetilde{D}}$ if and only if we can find $\{(d_n,\propto_n)\}$ in $\widetilde{D}$ such that $(d_n,\propto_n) \to d_\alpha$.

Theorem 2.23: [25]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space and let that $\widetilde{D} \subseteq \widetilde{A}$, then $\widetilde{D}$ is fuzzy dense in $\widetilde{A}$ if and only if for any $a_\alpha \in \widetilde{A}$ we can find $d_\beta \in \widetilde{D}$ with $\widetilde{F}[a_\alpha - d_\beta] > (1-\varepsilon)$ for some $0 < \varepsilon < 1$.

Definition 2.24: [25]

Suppose that $(\widetilde{A},\widetilde{F},*)$ is a fuzzy length space. A sequence of fuzzy points $\{(x_n,\propto_n)\}$ is said to be a fuzzy Cauchy if for any given ε, $0 < \varepsilon < 1$, there is a positive number K such that $\widetilde{F}[(x_n,\propto_n) - (x_m,\propto_m)] > (1-\varepsilon)$ for all $n, m \geq K$.

Definition 2.25: [25]

Suppose that $(\widetilde{A}, \widetilde{F}, *)$ is a fuzzy length space and $\widetilde{D} \subseteq \widetilde{A}$. Then $\widetilde{D}$ is said to be fuzzy bounded if we can find q ,0 < q < 1 such that, $\widetilde{F}(x_\alpha) > (1-q)$, $\forall\, x_\alpha \in \widetilde{A}$.

Definition 2.26: [25]

Let $(\widetilde{A}, \widetilde{F}_{\widetilde{A}}, *)$ and $(\widetilde{D}, \widetilde{F}_{\widetilde{D}}, *)$ be two fuzzy length space on fuzzy set $\widetilde{A}$ and $\widetilde{D}$ respectively, let $\widetilde{E} \subseteq \widetilde{A}$ then The operator $T: \widetilde{E} \to \widetilde{D}$ is said to be fuzzy continuous at $a_\alpha \in \widetilde{E}$, if for every $0 < \varepsilon < 1$, there exist $0 < \delta < 1$, such that $\widetilde{F}_{\widetilde{D}}\, [T(x_\beta) - T(a_\alpha)] > (1-\varepsilon)$ whenever $x_\beta \in \widetilde{E}$ satisfying $\widetilde{F}_{\widetilde{A}}(x_\beta - a_\alpha) > (1-\delta)$. If T is fuzzy continuous at every fuzzy point of $\widetilde{E}$, then T it is said to be fuzzy continuous on $\widetilde{E}$.

Theorem 2.27: [25]

Let $(\widetilde{A}, \widetilde{F}_{\widetilde{A}}, *)$ and $(\widetilde{D}, \widetilde{F}_{\widetilde{D}}, *)$ be two fuzzy length space, let $\widetilde{E} \subseteq \widetilde{A}$ The operator $T: \widetilde{E} \to \widetilde{D}$ is fuzzy continuous at $a_\alpha \in \widetilde{E}$ if and only if whenever a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ in $\widetilde{E}$ fuzzy converge to a_α, then the sequence of fuzzy points $\{(T(x_n), \alpha_n)\}$ fuzzy converges to $T(a_\alpha)$.

Theorem 2.28: [25]

An operator $T: \widetilde{A} \to \widetilde{D}$ is fuzzy continuous if and only if $T^{-1}(\widetilde{G})$ is fuzzy open in $\tilde{A}$ for all fuzzy open $\widetilde{G}$ of $\widetilde{D}$ where $(\widetilde{A}, \widetilde{F}_{\widetilde{A}}, *)$ and $(\widetilde{D}, \widetilde{F}_{\widetilde{D}}, *)$ are fuzzy length space

3. The Fuzzy Length of the Space of Operators

Definition 3.1:

Let $(\tilde{A}, \tilde{F}_{\tilde{A}}, *)$ and $(\tilde{E}, \tilde{F}_{\tilde{E}}, *)$ be two fuzzy length spaces and $T: \widetilde{D}(T) \to \tilde{E}$ is a linear operator where $\widetilde{D}(T) \subset \tilde{A}$. The operator T is said to be fuzzy bounded if there is a real number c .$0 < c < 1$ such that for all $x_\alpha \in \widetilde{D}(T)$: $\tilde{F}_{\tilde{E}}(T(x_\alpha)) > (1-c)$ ... (2.2)

Define $\tilde{F}(T, \lambda) = inf_{x_\alpha \in \widetilde{D}(T)} \tilde{F}_{\tilde{E}}(T(x_\alpha))$ ... (3.1)

where $\lambda = \min\{\alpha : x_\alpha \in \widetilde{D}(T)\}$. $\tilde{F}(T, \lambda)$ is called the fuzzy length of the operator T . If $T = 0$, we define $\tilde{F}(T, \lambda) = 1$. In this case (3.1) can be written:

$\tilde{F}_{\tilde{E}}(T(x_\alpha)) \geq \tilde{F}(T, \lambda)$ ... (3.2) ■

Example 3.2:

Let X be the vector space of all polynomials on [0,1] with ordinary norm $||x|| = \max|x(t)|$, $t \in [0,1]$, let $\tilde{A}$ be a fuzzy set in X with $\tilde{F}_{||\cdot||}$ on $\tilde{A}$. Let $a * b = a.b$ for all $a, b \in [0,1]$. Suppose that $T: X \to X$ be defined by $T(x(t)) = x'(t)$. T is linear but T is not fuzzy bounded. Take $x_n(t) = t^n$ where $n \in \mathbb{N}$. $T[x_n] = x_n'(t) = nt^{n-1}$ so $||Tx_n|| = n$, $\tilde{F}_{||\cdot||}[T(x_n, \alpha_n)] = \frac{\alpha_n}{\alpha_n + n}$

Since $n \in \mathbb{N}$ is arbitrary, this shows that there is no $0 < r < 1$ such that $\tilde{F}_{||\cdot||}[T(x_n, \alpha_n)] \geq (1-r)$. Hence T is not fuzzy bounded.

Theorem 3.3:

Let $(\tilde{A}, \tilde{F}_{\tilde{A}}, *)$ and $(\tilde{E}, \tilde{F}_{\tilde{E}}, *)$ be two fuzzy length spaces and $T: \widetilde{D}(T) \to \tilde{E}$ is a linear fuzzy bounded operator where $\widetilde{D}(T) \subset \tilde{A}$. Then $\tilde{F}(T, \lambda) = inf_{x_\alpha \in \widetilde{D}(T)} \tilde{F}_{\tilde{E}}(T(x_\alpha))$ is a fuzzy length.

Proof:

(FL_1) Since $\tilde{F}_{\tilde{E}}(T(x_\alpha)) > 0$ so $\tilde{F}(T, \lambda) > 0$. where $\lambda = \min\{\alpha : x_\alpha \in \widetilde{D}(T)\}$

(FL_2) $\tilde{F}(T, \lambda) = 1$ if and only if $inf_{x_\alpha \in \widetilde{D}(T)} \tilde{F}_{\tilde{E}}(T(x_\alpha)) = 1$ if and only if $\tilde{F}_{\tilde{E}}(T(x_\alpha)) = 1$ if and only if $T(x_\alpha) = 0$ if and only if T=0.

(FL_3) For $0 \neq c \in \mathbb{F}$ we have

$$\tilde{F}(c\,T, \lambda) = inf_{x_\alpha \in \widetilde{D}(T)} \tilde{F}_{\tilde{E}}(c\,T(x_\alpha)) = inf_{x_\alpha \in \widetilde{D}(T)} \tilde{F}_{\tilde{E}}\left(T, \frac{\alpha}{|c|}\right) = \tilde{F}\left(T, \frac{\lambda}{|c|}\right)$$

$$\begin{aligned}(FL_4)\ \tilde{F}[(T_1, \lambda) + (T_2, \lambda)] &= inf_{x_\alpha \in \widetilde{D}(T_1) \cap \widetilde{D}(T_2)} \tilde{F}_{\tilde{E}}(T_1(x_\alpha) + T_2(x_\alpha)) \\ &\geq inf_{x_\alpha \in \widetilde{D}(T_1)} \tilde{F}_{\tilde{E}}(T_1(x_\alpha)) * inf_{x_\alpha \in \widetilde{D}(T_2)} \tilde{F}_{\tilde{E}}(T_2(x_\alpha)) \\ &\geq \tilde{F}(T_1, \lambda) * \tilde{F}(T_2, \lambda)\end{aligned}$$

(FL_5) *Since* $\tilde{F}_{\tilde{E}}$ is a continuous so is $\tilde{F}$.

Hence $\tilde{F}(T, \lambda)$ is a fuzzy length. ■

Theorem 3.4:

Let U and V be any two vector spaces over same field $\mathbb{F}$. Let $\tilde{A}$ be a fuzzy set in U and $\tilde{E}$ is a fuzzy set in V. Suppose that $(\tilde{A}, \tilde{F}_{\tilde{A}}, *)$ and $(\tilde{E}, \tilde{F}_{\tilde{E}}, *)$ are two fuzzy length spaces, Assume that $T: \tilde{D}(T) \to \tilde{E}$ is a linear operator, where $\tilde{D}(T) \subset \tilde{A}$. Then:

(a)T is fuzzy continuous if and only if T is fuzzy bounded

(b)If T is fuzzy continuous at a single fuzzy point, then it is fuzzy continuous.

Proof:

(a)Assume that T is fuzzy bounded and consider any $x_\alpha \in \tilde{D}(T)$, let $0 < \varepsilon < 1$ be given. Then since T is linear for every $y_\beta \in \tilde{D}(T)$ such that $\tilde{F}_{\tilde{A}}(y_\beta - x_\alpha) = (1-\delta)$, take $(1-\delta) = (1-\varepsilon)$, we obtain $\tilde{F}_{\tilde{E}}[T(y_\beta) - T(x_\alpha)] = \tilde{F}_{\tilde{E}}[T(y_\beta - x_\alpha)] \geq \tilde{F}(T, \lambda) > (1-\varepsilon)$

Since $x_\alpha \in \tilde{D}(T)$ was arbitrary, this shows that T is fuzzy continuous

Conversely, assume that T is fuzzy continuous at $x_\alpha \in \tilde{D}(T)$. Then for given any $0 < \varepsilon < 1$ there is $0 < \delta < 1$ such that $\tilde{F}_{\tilde{E}}[T(y_\beta) - T(x_\alpha)] \geq (1-\varepsilon)$ for all $y_\beta \in \tilde{D}(T)$ satisfying $\tilde{F}_{\tilde{A}}[y_\beta - x_\alpha] > (1-\delta)$ ... (3.3)

We now take $z_\sigma \in \tilde{D}(T)$ and set $z_\sigma = y_\beta - x_\alpha$ where $\sigma = \min\{\alpha, \beta\}$

$\tilde{F}_{\tilde{E}}[T(y_\beta) - T(x_\alpha)] = \tilde{F}_{\tilde{E}}[T(y_\beta - x_\alpha)] = \tilde{F}_{\tilde{E}}[T(z_\sigma)] > (1-\varepsilon)$.

Hence T is fuzzy bounded. ■

(b)Fuzzy continuity of T at fuzzy point x_α implies fuzzy bounded of T by proof of (a) which implies fuzzy continuity of T.■

Corollary 3.5:

Let U and V be vector space over the same field $\mathbb{F}$. Let $\tilde{A}$ and $\tilde{E}$ be two fuzzy sets in U and V respectively. Assume that $(\tilde{A}, \tilde{F}_{\tilde{A}}, *)$ and $(\tilde{E}, \tilde{F}_{\tilde{E}}, *)$ are two fuzzy length spaces. Suppose that $T: \tilde{D}(T) \to \tilde{E}$ be a fuzzy bounded linear operator where $\tilde{D}(T) \subset \tilde{A}$. Then

a)$(x_n, \alpha_n) \to x_\alpha$ [Where $(x_n, \alpha_n), x_\alpha \in \tilde{D}(T)$] implies $(T(x_n), \alpha_n) \to T(x_\alpha)$

b)The null fuzzy set [for kernel of T] $\tilde{K}(T)$ is fuzzy closed where $\tilde{K}(T) = \{y_\beta \in \tilde{D}(T): T(y_\beta) = 0\}$

Proof:

a)Since T is fuzzy bounded, $\tilde{F}(T, \lambda) \geq (1-r)$ for some $0 < r < 1$ .since $\tilde{F}_{\tilde{E}}[(T(x_n), \alpha_n) - T(x_\alpha)] = \tilde{F}_{\tilde{E}}[(T(x_n - x), \lambda] \geq \tilde{F}(T, \lambda) \geq (1-r)$. Hence $(T(x_n), \alpha_n) \to T(x_\alpha)$ where $\lambda = \min\{\alpha_n, \alpha: n \in \mathbb{N}\}$.■

b)For $x_\alpha \in \tilde{K}(T)$ there is a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ in $\tilde{K}(T)$ such that $(x_n, \alpha_n) \to (x_\alpha)$ by lemma 2.22. Hence $(T(x_n), \alpha_n) \to T(x_\alpha)$ by part (a). Also $T(x_\alpha) = 0$ since $T(x_n, \alpha_n) = 0$ so that $x_\alpha \in \tilde{K}(T)$ since $x_\alpha \in \overline{\tilde{K}(T)}$, was arbitrary $\tilde{K}(T)$ is fuzzy closed.■

Definition 3.6:

Two operator $T_1 and\, T_2$ are defined to be equal written $T_1 = T_2$ if they have the same domain $\tilde{D}(T_1) = \tilde{D}(T_2)$ and if $T_1(x_\alpha) = T_2(x_\alpha)$ for all $x_\alpha \in \tilde{D}(T_1) = \tilde{D}(T_2)$

Definition 3.7:

Let $T: \tilde{D}(T) \to \tilde{E}$ be an operator where X and Y are vector space and $\tilde{D}(T) \subset \tilde{A}$ where $\tilde{A}$ and $\tilde{E}$ are two fuzzy sets in X and Y respectively. Let $\tilde{B} \subset \tilde{D}(T)$, then the restriction of the operator T to the subset $\tilde{B}$ is denoted by $T|_{\tilde{B}}$ and is defined by $T|_{\tilde{B}}: \tilde{B} \to \tilde{E}, T|_{\tilde{B}}(x_\alpha) = T(x_\alpha)$ for all $x_\alpha \in \tilde{B}$.

The extension of T to a fuzzy set $\tilde{M}$ where $\tilde{D}(T) \subset \tilde{M}$ is the operator $S: \tilde{M} \to \tilde{E}$ such that $S|_{\tilde{D}(T)} = T$ that is $S(x_\alpha) = T(x_\alpha)$ for all $x_\alpha \in \tilde{D}(T)$.

Theorem 3.8:

Let U and V be two vector spaces and let $\tilde{A}$ and $\tilde{E}$ be two fuzzy sets in U and V respectively. Let $T: \tilde{D}(T) \to \tilde{E}$ where $\tilde{D}(T) \subset \tilde{A}$. Let $(\tilde{A}, \tilde{F}_{\tilde{A}}, *)$ and $(\tilde{E}, \tilde{F}_{\tilde{E}}, *)$ be two fuzzy complete fuzzy length spaces. Let $T: \tilde{D}(T) \to \tilde{E}$ be fuzzy bounded linear operator. Then T has an extension $S: \overline{\tilde{D}(T)} \to \tilde{E}$ where S is fuzzy bounded linear operator and $\tilde{F}(T, \lambda) = \tilde{F}(S, \lambda)$.

Proof:

We consider any $x_\alpha \in \overline{\tilde{D}(T)}$. By lemma (2.22) there is a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ in $\tilde{D}(T)$ such that $(x_n, \alpha_n) \to (x_\alpha)$. Since T is linear and fuzzy bounded, we have $\tilde{F}(T, \lambda) \geq (1-r)$, where $0 < r < 1$. Hence $\tilde{F}_{\tilde{E}}[T(x_n, \alpha_n) - T(x_m, \alpha_m)]$=

$\tilde{F}_{\tilde{E}}[T(x_n - x_m), \lambda] \geq \tilde{F}(T, \lambda) \geq (1 - r)$, for all $n, m \geq K$. This shows that $\{T(x_n, \alpha_n)\}$ is fuzzy Cauchy because $\{(x_n, \alpha_n)\}$ fuzzy converges. By assumption $\tilde{E}$ is fuzzy complete, so that $\{T(x_n, \alpha_n)\}$ fuzzy converges, say, $T(x_n, \alpha_n) \to z_\sigma$, we defined S by $S(x_\alpha) = z_\sigma$.

It is shown that this definition is dependent of the particular choice of a sequence of fuzzy points in $\widetilde{D}(T)$ fuzzy converging to x_α. Suppose that $(x_n, \alpha_n) \to (x_\alpha)$ and $(y_n, \beta_n) \to (x_\alpha)$. Then $(v_m, \delta_m) \to x_\alpha$ where $\{(v_m, \delta_m)\}$ is the sequence of fuzzy points $\{(x_1, \alpha_1), (y_1, \beta_1), (x_2, \alpha_2), (y_2, \beta_2), \ldots\}$. Hence $\{(T(v_m, \delta_m))\}$ fuzzy converges by corollary 3.5(a) and the two subsequences of fuzzy points $\{(T(x_n, \alpha_n))\}$ and $\{(T(y_n, \beta_n))\}$ of $\{(\underline{T(v_m}, \delta_m))\}$ must have the same fuzzy limit. This proves that S is uniquely defined at every $x_\alpha \in \widetilde{D}(T)$. Clearly S is linear and $S(x_\alpha) = T(x_\alpha)$ for every $x_\alpha \in \widetilde{D}(T)$, so that S is an extension of T. We now use $\tilde{F}_{\tilde{E}}[(T(x_n, \alpha_n)] \geq \tilde{F}(T, \lambda)$ and let $n \to \infty$. Then $T(x_n, \alpha_n) \to z_\sigma = S(x_\alpha)$. Defines a fuzzy continuous operator, we thus obtain $\tilde{F}_{\tilde{E}}(S(x_\alpha)) \geq \tilde{F}(T, \lambda)$. Hence S is fuzzy bounded and $\tilde{F}(S, \lambda) \geq \tilde{F}(T, \lambda)$. Of course $\tilde{F}(S, \lambda) \leq \tilde{F}(T, \lambda)$ because the fuzzy length being defined by an infimum, cannot decrease is an extension. Together we have $\tilde{F}(T, \lambda) = \tilde{F}(S, \lambda)$.■

4. Linear Functional

A functional is an operator whose range lies in a fuzzy set in $\mathbb{R}$ or $\mathbb{C}$. We denote functional by lower case letters $, g, h, \ldots$, the domain of f by $\widetilde{D}(f)$, the range by $\tilde{R}(f)$, and the value of f at an $x_\alpha \in \widetilde{D}(f)$ by $f(x_\alpha)$.

Functional are operators, so that previous definitions apply. We shall need in particular the following two definitions because most of the functional to be considered will be linear and fuzzy bounded.

Definition 4.1:

A linear functional f is a linear operator with domain in a fuzzy set $\tilde{A}$ and range in the scalar field $\mathbb{F}$ of X. Thus f : $\widetilde{D}(f) \to \mathbb{F}$.

Definition 4.2:

A fuzzy bounded linear fuzzy functional f is a fuzzy bounded linear operator with range in the fuzzy set in the scalar field of the fuzzy of the fuzzy length space $\tilde{A}$ in which the domain $\widetilde{D}(f)$ lies. Thus there exists a fuzzy real number $(1 - c), 0 < c < 1$ such that for all $x_\alpha \in \widetilde{D}(f)$

$\tilde{F}_{\tilde{E}}(f(x_\alpha)) \geq (1 - c) \ldots (4.1)$

Furthermore, the fuzzy length of f is: $\tilde{F}_{\tilde{E}}(f, \lambda) = \inf_{x_\alpha \in \widetilde{D}(f)} \tilde{F}_{\tilde{E}}(f(x_\alpha) \ldots (2.7)$

Or $\tilde{F}_{\tilde{E}}(f, \lambda) = \inf_{x_\alpha \in \widetilde{D}(f)} \tilde{F}_{\tilde{E}}(f(x_\alpha) \ldots (4.2)$

Formula (2.6) implies: $\tilde{F}_{\tilde{E}}(f(x_\alpha)) \geq \tilde{F}_{\tilde{E}}(f, \lambda) \ldots (4.3)$.

The proof of the following theorem is similar to the proof of theorem 3.4 hence omitted.

Theorem 4.3:

A linear fuzzy functional f with domain $\widetilde{D}(f)$ in a fuzzy length space $(\widetilde{A}, \widetilde{F}, *)$ is fuzzy continuous if and only if f is fuzzy bounded.

We now give an example to show that the fuzzy length is not linear functional.

Example 4.4:

The fuzzy length $\tilde{F}(\cdot): \tilde{A} \to [0,1]$ on a fuzzy length space $\tilde{A}$ is a fuzzy functional on $\tilde{A}$ which is not linear.

5. Fuzzy length space of operators, fuzzy dual space

In the present section our goal is as follows: By taking any two fuzzy length spaces $\tilde{A}$ and $\tilde{E}$ (both fuzzy real or both fuzzy complex) and consider the set $FB(\tilde{A}, \tilde{E})$ consisting of all fuzzy bounded linear operators from $\tilde{A}$ into $\tilde{E}$, that is, each such operator is defined on all of $\tilde{A}$ and its range lies in $\tilde{E}$.We want to show that $FB(\tilde{A}, \tilde{E})$ can itself be made into a fuzzy length space. The whole matter is quite simple, first of all, $FB(\tilde{A}, \tilde{E})$ becomes a vector space

Lemma 5.1:

$FB(\tilde{A}, \tilde{E})$ is a vector space over the field $\mathbb{F}$.

Proof:

Since the sum $T_1 + T_2$ of two operators $T_1, T_2 \in FB(\tilde{A}, \tilde{E})$ is defined by: $[T_1 + T_2](x_\alpha) = T_1(x_\alpha) + T_2(x_\alpha)$. Then $T_1 + T_2$ is fuzzy bounded since if T_1 and T_2 are fuzzy bounded then there exists $0 < \varepsilon < 1, 0 < r < 1$ such that $\tilde{F}(T_1) > (1 - \varepsilon)$, $\tilde{F}(T_2) > (1 - r)$. Now $\tilde{F}(T_1 + T_2) >$

$\tilde{F}(T_1) * \tilde{F}(T_2) > (1-\varepsilon) * (1-r) = (1-\rho)$ for some $0 < \rho < 1$. Hence $T_1 + T_2$ is fuzzy bounded. Also the definition of the product cT of T$\in FB(\tilde{A}, \tilde{E})$, and scalar c by $(cT)(x_\alpha) = cT(x_\alpha)$.Then cT is fuzzy bounded, since if T is fuzzy bounded then there is $0 < \varepsilon < 1$ such that $\tilde{F}(T) > (1-\varepsilon)$. Then $\tilde{F}(cT) > (1-\varepsilon)$. Hence cT is fuzzy bounded.
By using Theorem 3.3 we get the following result

Theorem 5.2:

The vector space $FB(\tilde{A}, \tilde{E})$ of all fuzzy bounded linear operators from a fuzzy length space $\tilde{A}$ into a fuzzy length space $\tilde{E}$ itself a fuzzy length space with fuzzy length defined by: $\tilde{F}_{\tilde{E}}(T, \lambda) = \inf_{x_\alpha \in \tilde{D}(\tilde{T})} \tilde{F}_{\tilde{E}}(T(x_\alpha))$ ... (5.1), where $\lambda = \min\{\alpha, x_\alpha \in \tilde{D}(T)\}$.

There is a central question under what conditions $FB(\tilde{A}, \tilde{E})$ will be a fuzzy complete space? which is answered in the following theorem.

Theorem 5.3:

If $\tilde{E}$ is a fuzzy complete space, then $FB(\tilde{A}, \tilde{E})$ is a fuzzy complete space.

Proof:

Consider an arbitrary Cauchy sequence (T_n) in $FB(\tilde{A}, \tilde{E})$ and show that (T_n) converges to an operator T$\in FB(\tilde{A}, \tilde{E})$.since (T_n) is Cauchy, for every $0 < \varepsilon < 1$ there is an K such that $\tilde{F}_{\tilde{E}}(T_n - T_m) > (1-\varepsilon), \forall\, m, n > \mathbb{K}$. For all $x_\beta \in \tilde{A}$ and $n, m \geq K$, we obtain $\tilde{F}_{\tilde{E}}[T_n(x_\beta) - T_m(x_\beta)] = \tilde{F}_{\tilde{E}}[(T_n - T_m)(x_\beta)] \geq$
$\tilde{F}_{\tilde{E}}[T_n - T_m] > (1-\varepsilon)$ ... (5.2)
Now, for any fixed x_β and given $(1-\acute{\varepsilon})$. Then from (5.2) one can have $\tilde{F}_{\tilde{E}}[T_n(x_\beta) - T_m(x_\beta)] > (1-\acute{\varepsilon})$ and see that $\{(T_n(x_\beta))\}$ is fuzzy Cauchy in $\tilde{E}$. Since $\tilde{E}$ is fuzzy complete, $\{(T_n(x_\beta))\}$ fuzzy converges, say $T_n(x_\beta) \to z_\sigma$. Clearly the fuzzy limit $z_\sigma \in \tilde{E}$ depends on the choice of $x_\beta \in \tilde{A}$. This defines an operator $T: \tilde{A} \to \tilde{E}$, where $z_\sigma = T(x_\beta)$, the operator T is linear since $T(ax_\beta + by_\delta) = \lim_{n\to\infty} T_n(ax_\beta + by_\delta) = \lim_{n\to\infty}(a\, T_n\, x_\beta + b\, T_n\, y_\delta) = a \lim_{n\to\infty} T_n(x_\beta) + b \lim_{n\to\infty} T_n(y_\delta)$
$= a\, T(x_\beta) + b\, T(y_\delta)$
We prove that T is fuzzy bounded and $(T_n \to T)$, that is, $\tilde{F}_{\tilde{E}}\, (T_n - T) \to 1$. Since (5.2) holds for every $m \geq$K and $T_m(x_\beta) \to T(x_\beta)$, we may let $m \to \infty$. Using the fuzzy continuity of the fuzzy length. Then obtaining from (2.11) for every $n \geq$K and all$x_\beta \in \tilde{A}$. $\tilde{F}_{\tilde{E}}[T_n(x_\beta) - T(x_\beta)] = \tilde{F}_{\tilde{E}}\left[T_n(x_\beta) - \lim_{n\to\infty} T_n(x_\beta)\right] =$
$\lim_{n\to\infty} \tilde{F}_{\tilde{E}}[T_n(x_\beta) - T_n(x_\beta)] \geq (1-\varepsilon)$ ... (5.3)
This shows that $(T_n - T)$ with $n \geq$K is a fuzzy bounded linear operator. Since T_n is fuzzy bounded, $T = T_n - (T_n - T)$ is fuzzy bounded, that is $T \in FB(\tilde{A}, \tilde{E})$. Furthermore, if in (5.3) we take the infimum over all x_β, we obtain $\tilde{F}_{\tilde{E}}(T_n - T) \geq (1-\varepsilon)$ $n \geq$K. Hence $\tilde{F}_{\tilde{E}}(T_n - T) \to 1$.■
This theorem has an important consequence with respect to the fuzzy dual space $FB(\tilde{A}, \mathbb{R})$ of $\tilde{A}$, which is defined as follows:

Definition 5.4:

Let X be a vector space over the field $\mathbb{F}$ .let $\tilde{A}$ be a fuzzy set in X. Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy length space. Then the fuzzy set of all fuzzy bounded linear functional is on $\tilde{A}$ constitutes a fuzzy length space with fuzzy length defined by:

$$\tilde{F}(f, \lambda) = \inf_{x_\alpha \in \tilde{A}} \tilde{F}(f(x_\alpha))$$

Which is called the fuzzy dual space of $\tilde{A}$ and is denoted by $\tilde{A}'$ or $FB(\tilde{A}, \mathbb{R})$

Theorem 5.5:

The fuzzy dual space $FB(\tilde{A}, \mathbb{R})$ of fuzzy length space $\tilde{A}$ is a fuzzy complete space (whether or not $\tilde{A}$ is).

Conclusion

In the present paper our aim is to define the fuzzy length of a fuzzy bounded linear operator in order to prove that the set of all fuzzy bounded linear operators from a fuzzy length space to another fuzzy length space is fuzzy complete also to prove that fuzzy bounded linear operator is equivalent to fuzzy continuous.

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