Iraqi Journal of Science

ISSN: 0067-2904

En-prime Subacts over Monoids with Zero

Shireen O. Dakheel
Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad, Iraq.

Abstract
Throughout this paper S will be denote a monoids with zero. In this paper, we introduce the concept of En- prime subact, where a proper subact B of a right S- act A_s is called En- prime subact if for any endomorphism f of A_s and a $\in A_s$ with $f(a)S\subseteq B$implies that either $a \in B$ or $f(A_s) \subseteq B$. The right S-act A_s is called En-prime if the zero subact (θ) of A_s is En-prime subact. Some various properties of En-prime subact are considered, and also we study some relationships between En-prime subact and some other concepts such as prime subact and maximal subact.
Keywords: En-prime subact, Prime subact, Fully invariant subact, Maximal subacts.

الأثر الجزئي الأولي من النمط En بالنسبة لشبه زمرة أحادية مع الصفر

شيرين عودة دخيل
قسم الرياضيات، كلية العلوم للبنات، جامعة بغداد، بغداد، العراق.

الخلاصة
لتكن Sشبه زمرة أحادية مع الصفر. في هذا البحث قدمنا مفهوم الأثر الجزئي الأولي من النمط En حيث يقال للأثر الجزئي B من A_s بأنه أولي من النمط En أذا كان لأي تشاكل fعلىA_s و a مع$\in A_s$ $f(a)S\subseteq B$فإنه يؤدي الى اما $a \in B$ أو $f(A_s) \subseteq B$. كما يقال للأثرA_sبأنه أولي من النمط En أذا كان الأثر الجزئي الصفري(θ)هو أثر جزئي أولي من النمطEn. وقددرسنا بعض الخواص المختلفة للأثر الجزئي الأولي من النمط–En و كذلك درسنا بعض العلاقات بين الأثر الجزئي الأولي من النمطEn – مع بعضالأثار الجزئية الأخرى ذات الصلة مثل الأثر الجزئي الأولي و الأثر الجزئي الأعظم.

Introduction

Recall that a nonempty set A is called a right S-act where S is monoid that is semigroup with identity element, if there exists a mapping $\emptyset$: A$\times$ S $\rightarrow$ A define as (a, s) $\rightarrow$ as and satisfying a .1= a and a(st) = (as) t, for alla $\in$ A and s, t $\in$ S. We call A a right S-act or right act over S and write A_s.

Similarly, we define a left S-acts A and write sA [1]. If S is a commutative monoid, then every left S-act is right S-act. A non empty subset B of a right S-act A_s is called subact of A_s and written by B $\leq A_s$, if bs$\in$ B for all b $\in$ B and s $\in$ S. An element $\theta \in A_s$ is called a zero of A_s or fixed element if $\theta s = \theta$ for all s $\in$ S, i.e. $\{\theta\}$ is a one-element subact [1]. In this paper θ is a unique fixed element of all S-act A_s. A nonempty subset I is called an ideal or two sided ideal of S (left and right) if SI $\subseteq$ I and IS $\subseteq$ I. A mapping f: $A_s\rightarrow B_s$, where A_s and B_s are two right S-acts is called S- homomorphism if f(as) = f(a)s, for all a $\in A_s$ and s $\in$ S. The set of all S-homomorphism from A in to B denoted by Hom(A_s, B_s)

Email: shireenouda@yahoo.com

or $Hom_s(A,B)$. An S-homomorphism $f:A_s \rightarrow A_s$ is called an endomorphism of A_s. The composition $g_o f$ of homomorphism f: $A_s \rightarrow B_s$ and g: $B_s \rightarrow C_s$ of a right S-acts is a homomorphism of a right S-act, i.e.$g_o f \in Hom(A_s,B_s)$[1].

Throughout this paper S will denote monoid that is semigroup with identity element and all acts are right S-act. A subact B of an S-act A_s is a prime subact if for any a ∈ A_s and r ∈ S, ar∈ B implies that a ∈ B or Ar⊆ B [2]. An S-act A is itself called prime if the subact {θ} of A is prime subact [2]. In this paper we introduce the concept of En-prime subact which is define as follows: a proper subact B of an S- act A_s is called En- prime subact if for any endomorphism f of A_s and a ∈ $A_s$with f(a)S⊆ B implies that either a ∈ B or f(A_s) ⊆ B. An S-act A_s is called En-prime act if the zero subact (θ) of A_s is En-prime subact.Also, we study some properties of En-prime subact and the relationships between En-prime subact and other concepts like prime subact and maximal subact.

1.En-prime subacts:

In this section we introduce the concept of En-prime subact and study some properties of this concept.

Definition 1.1

A proper subact B of an S-act A_s is called En- prime subact if for any endomorphism f of A_s anda ∈$A_s$with f(a)S⊆ B implies that either a ∈ B or f(A_s) ⊆ B. The right S-act A_s is called En-prime if the zero subact (θ) of A_s is En-prime subact.

Remarks 1.2

1. The one element subact {θ } is the only En-prime subactof simple act.

Proof:

Let A_s be a simple act and let f(a)S = θ , where f an endomorphism of A_s and a ∈ A_s. Suppose that a ≠θ, henceθ≠aS is a subact of A_s. But A_s is asimple act thus aS = A_s. Now, θ = f(a)S = f(A_s). Therefore f(A_s) = θ.

2. Every simple act is En-prime act.

Proof:

Follows directly from (1).

3. If B and C are En-prime subacts of an S-act A_s, then B ⋃C is also En-prime subact of A_s.

Proof:

Let B and C be En-prime subacts of an S-act A_s and let f(a)S⊆ B ⋃ C , where f an endomorphism of $A_s$and a ∈ A_s and suppose thata ∉ B⋃C and then we must prove that f(A_s)⊆B ⋃ C. Now, we have f(a)S ⊆ B or f(a)S ⊆ C and a ∉B⋃C . If f(a)S ⊆ B and a ∉B by assumption then we get f(A_s) ⊆B since B is En-prime subact. Also, iff(a)S ⊆C and a ∉C, then f(A_s)⊆C. Hence f(A_s)⊆ B ⋃ C.

Recall that: A subact B of S-act A_s is a prime subact if for any a ∈ A_s and r ∈ S, ar∈ B implies that a ∈ B or Ar⊆ B. An S-act A is itself called prime if the subact {θ} of A is prime subact [2].

Proposition 1.3

Every En-prime subact is prime subact.

Proof:

Let B be En- prime subact of an S-act $A_s$and let as ∈ B where a ∈ A_s and s ∈ S and suppose that a ∉ B. We have to prove that As ⊆ B. Define f: $A_s \rightarrow A_s$ by f(x) = xs for all x ∈ A_s. Now, f(a) = as ∈ B and thus f(a)S⊆ B. But B is En-prime subact of A_s and a ∉ B by assumption, so f(A_s) ⊆ B. Hence As ⊆ B.

Remark 1.4

The converse of Prop. (1.3) is not true in general, for example: Consider Z⊕Z as (Z,.) –act with multiplication by integers as operation. Now, it is clear that 2Z ⊕ Z is a prime subact of Z⊕Z but not En-prime subact since if we define an endomorphism f : Z⊕Z →Z⊕Z as f(x,y) = (y,x) for all (x,y) ∈ Z⊕Z and it is clear that f S-homomorphism, then we have f(3,2) = (2,3)∈ 2Z ⊕ Z . Hence f(3,2) Z ⊆ 2Z ⊕ Z , but (3,2)∉2Z ⊕ Zand f(Z⊕Z) = Z⊕Z ⊄ 2Z ⊕ Z . Therefore 2Z ⊕ Z is not En-prime subactof Z⊕Z.

Recall that: A subact B of an S-act A_s is called fully invariant subact iff(B) ⊆ B for every endomorphism f of A_s and A_s is called duo act if every subact of A_s is fully invariant [3].

In [2] it was proved that every maximal subact is prime subact, but for En-prime subact we have:

Proposition 1.5

Let B be a fully invariant subact of an S-act A_s. If B is a maximal subact of A_s then B is En-prime subact of A_s.

Proof:

Let $f(a)S \subseteq B$, where f an endomorphism of A_s and $a \in A_s$. Assume that $a \notin B$ and B is a maximal subact of A_s, hence $\langle B \cup aS\rangle = A_s$. Now, we have $B \cup aS = A_s$ thus $f(B \cup aS) = f(A_s)$ and so $f(B) \cup f(a)S = f(A_s)$. Let x be any element of A_s, then $f(x) \in f(A_s)$ it follows that $f(x) \in f(B)$ or $f(x) \in f(a)S$. If $f(x) \in f(B)$ then $f(x) = f(b) \in B$ since B is fully invariant. Also, if $f(x) \in f(a)S$ then $f(x) = f(a)s$ for some $s \in S$. Thus $f(x) \in B$ in any case. Therefore $f(A_s) \subseteq B$.

Remark 1.6

The condition of fully invariant subact in the previous proposition is an essential as the following example shows: $2Z \oplus Z$ is a maximal subact of an act $Z \oplus Z_{(Z,.)}$ but not fully invariant and not En-prime subact, to prove that : define endomorphism $f : Z\oplus Z \to Z\oplus Z$ as $f(x,y) = (y,x)$ for all $(x,y) \in Z\oplus Z$ and it is clear that f an S-homomorphism. Now, $f(2Z \oplus Z) = Z \oplus 2Z \not\subseteq 2Z \oplus Z$, hence $2Z \oplus Z$ is not fully invariant subact of $Z\oplus Z$ and not En-prime subact as we define in Remark (1.4).

Corollary 1.7

Every maximal subact of duo act is En-prime subact.

Example 1.8

2Z is a maximal subact of duo act $Z_{(Z,.)}$ of all integers, and thus by corl. (1.7) 2Z is En-prime subact of Z.

Recall that, an S-act A_s is called multiplication S-act if for each subact B of A_s , there exists an ideal I of S with B = AI. An S-act A_s is a multiplication S-act if and only if for each $a \in A_s$ there exists an ideal I of S such that aS = AI. If B is a subact of multiplication S-act A_s, the ideal $\{s \in S : As \subseteq B\}$ will be denoted by $(B : A_s)$. If B is a subact of a multiplication S-act A_s, then $B = A_s (B : A_s)$ [4].

The following proposition show that every maximal subact in a multiplication act is En-prime subact.

Proposition 1.9

Every maximal subact of a multiplication S-act is En-prime subact.

Proof:

Let B be a maximal subact of a multiplication S-act A_s, then $B = A_s (B : A_s)$ and thus for every endomorphism f of A_s , we have $f(B) = f(A_s) (B : A_s) \subseteq B$, hence B is fully invariant subact of A_s and by Prop. (1.5) we get that B is En-prime subact of A_s.

Recall that: An S-act A_s is called injective act if for each S-monomorphismg : $A_s \to B_s$ and each S-homomorphism f: $A_s \to M_s$, then there exists a homomorphism h: $B_s \to M_s$ such that $h_o g = h$ [1].

Proposition 1.10

If B is En-prime subact of an S-act A_s and C is a subact of A_s which isan injective . Then either $C \subseteq B$ or $B\cap C$ is En-prime subact of C.

Proof:

Suppose that $C \not\subseteq B$, hence $B\cap C$ is a proper subact of C. Let $f(x)S \subseteq B\cap C$, where f an endomorphism of Cand $x \in C$. Suppose that $x \notin B\cap C$, thus $x \notin B$. We must prove that $f(C) \subseteq B\cap C$. Since C is injective, then there exists an S-homomorphism h: $A_s \to C$ such that $h_o i = f$, where i is an inclusion map.

Now, $f(x)S = h_o i(x)S = h(x)S \subseteq B\cap C \subseteq B$. But B is En-prime subact of S-act A_s and $x \notin B$ by assumption, hence $h(A_s) \subseteq B$. Also, $f(C)) = h_o i(C)) = h(C)) \subseteq C$, and $f(C) = h(C) \subseteq h(A_s) \subseteq B$. This implies that $f(C) \subseteq B\cap C$.

Proposition 1.11

Let B be a subact of an S-act A_s and let P be En-prime subact of A_s such that $BI \subseteq P$ for some ideal I of S. If $I \not\subseteq (P:A_s)$ then $B \subseteq P$.

Proof:

Let B be a subact of A_s andlet P be En-prime subact of A_s such that $BI \subseteq P$ for some ideal I of S with $I \not\subseteq (P: A_s)$. Suppose that $x \in B$ and since $I \not\subseteq (P:A_s)$, then there exists $t \in I$ and $t \notin (P: A_s)$. Define f_t: $A_s \to A_s$ by $f_t(a) = at$ for all $a \in A_s$. Now, $f_t(x) = x\,t \in BI \subseteq P$ thus $f_t(x)S \subseteq P$. But P is En-prime subact of A_s and $f_t(A_s) = A_s t \not\subseteq P$, therefore $x \in P$.

Proposition 1.12

Let A_s be an S-act and let f be any endomorphism of A_s. If B is a fully invariant and En-prime subact of A_s, such that $f(A_s) \not\subseteq B$, then $f^{-1}(B)$ is also is En-prime subact of A_s.

Proof:

First, we have to prove that $f^{-1}(B)$ is a proper subact of A_s. Suppose that $f^{-1}(B) = A_s$, thus $f(A_s) \subseteq B$, which is a contradiction to assumption. Hence$f^{-1}(B)$ is a proper subact of A_s.Now, Let $h(a)S \subseteq f^{-1}(B)$, where h endomorphism of A_s and $a \in A_s$. Suppose that $a \notin f^{-1}(B)$, then $f(a) \notin B$ and follows that $a \notin B$, since B is fully invariant subact of A_s. We have to prove that $h(A_s) \subseteq f^{-1}(B)$. Now, since $h(a)S \subseteq f^{-1}(B)$, then $f_o h(a)S \subseteq B$. But B is En-prime subact of $A_s$and $a \notin B$, hence $f_o h(A_s) \subseteq B$ which implies that $h(A) \subseteq f^{-1}(B)$.

Reference

1. Knauer, U., Kilp, M. and Mikhalev, A. V. **2000**. *Monoids, Acts and Categories*,Walter de Gruyter, Berlin, New York.
2. Ahsan, J., Zhongkui, L. **2001**. Prime and semiprime acts over monoids with zero, *Math. J. Ibaraki*, **33**: 9-15.
3. Anjaneyulu, A. **1981**. Structure and ideal theory of duo semigruop, *Semigroup Forum*, **22**: 257-276.
4. Estaji, A.A., shabani, M. **2007**. A note on multiplication S-act, 38th Annual Iranian Mathematics Conference, University of Zanjan, Iran, 19-21.