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Generalized Jacobian Weighted Modulus of Smoothness

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Abstract
The topic of modulus of smoothness still gets the interest of many researchers due to its applicable usage in different fields, especially for function approximation. In this paper, we define a new modulus of smoothness of weighted type. The properties of our modulus are studied. These properties can be easily used in different fields, in particular, the functions in the Besov spaces $\mathbb{B}_p^r$ when $0 < p < 1$.

Keywords: Lebesgue Space , Jacobi Weight , Besov Space , Modulus of Smoothness

مقياس نعومة جاكوبي الموزون المعمم

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الخلاصة
لا يزال موضوع معامل النعومة يحظى باهتمام العديد من الباحثين على عدة أصعدة ، لاستخدامه القابل للتطبيق في مجالات مختلفة ، وخاصة لتقريب الدوال. في هذا البحث ، عرفنا معاملاً جديدًا للنعومة من النوع الموزون لأغراض تقريب الدوال. تمت دراسة خصائص المعامل الخاص بنا هنا لاستخدامها بسهولة في مجالات عدة ولا سيما فيما يتعلق بالدوال من فضاء بيسوف $\mathbb{B}_p^r$ ، خاصة عندما $0 < p < 1$.

1. Introduction and preliminaries

Moduli of smoothness varies among researchers in their structures, weighted modulus of smoothness has several uses in function approximation, especially for estimating the degree of approximation. The most important application of all moduli is their rule in approximating functions, especially by polynomials or/and neural networks, for more information, see [1]-[10]. It is not clear when the first weighted moduli of smoothness were introduced, but it is probably defined first by Ditizian and Totik in their book [1], they were working on linking the weighted moduli of smoothness to the weighted approximation [2] in weighted spaces such as [3]. The idea began with defining the weighted norm in [4] and it was followed by the investigating other weights for spaces and moduli.

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Many versions of moduli of smoothness were defined later, for more details see [5],[6],[7] , [8], [9], [10] , [11] , and [12] .
In 2007, Jianjun [13] defined the weighted modulus in terms of the classical Jacobi weights

$$\omega_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta} \quad , \qquad (1.1)$$

where $\in [-1,1]$, α and $\beta \in J_p$, J_p is given by

$$J_p = \begin{cases} \left(-\frac{1}{p}, \infty\right) , & \text{If } 0 < p < \infty \\ [0, \infty) , & \text{If } p = \infty \end{cases}$$

Since 2014, Koputon et al. have made many generalizations to the modulus that is defined with weights (1.1) in their works [5], [10] and [14]. Their last weighted DT modulus of smoothness is given by

$$\omega_{k,r}^{\varphi}\left(f^{(r)}, t\right)_{\alpha,\beta,p} = \sup_{0\le h\le t} \left\| \mathcal{W}_{kh}^{\frac{r}{2}+\alpha,\frac{r}{2}+\beta}(\cdot)\Delta_{h\varphi(\cdot)}^{k}(f^{(r)},\cdot) \right\|_p \quad , \qquad (1.2)$$

where $f \in \mathbb{B}_p^r, \varphi(x) = \sqrt{1-x^2}$ and $\alpha, \beta \in J_p, k \in N$.

On the other hand, the generalized Jacobi weight is a good choice for usage in our work. The generalized Jacobi weight, for ≥ 2 , is given by [15] as follows:

$$\mathcal{W}(x) = \prod_{j=1}^{M} \left|x - z_j\right|^{\lambda j} \quad , \qquad (1.3)$$

where $-1 = z_1 \leq \cdots \leq z_M = 1, \lambda j \in J_p$.

Definition 1.1: [16]

The L_p space for $0 < p \leq \infty$ is defined as follows:

$$L_p = \{f : [0,1] \to \mathbb{R} \; such \; that \; f \; is \; measurable \; and \; \|f\|_p < \infty\},$$

where

$$\|f\|_p = \left(\int_a^b |f|^p\right)^{1/p} .$$

Definition 1.2: [14]

Let $0 < p \leq \infty$ and for $r \in \mathbb{N}_0$, we have

$$\mathbb{B}_p^r = \left\{f : \left\|f^{(r)}\varphi^r\right\|_p < \infty\right\}.$$

is the Besove space.

Definition 1.3: [14]

For $k \in \mathbb{N}_0$, $h \geq 0$, $x \in [-1,1]$, an interval J and $f : J \mapsto \mathbb{R}$, let

$$\Delta_h^k(f, x, J) = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right) & if \; x \pm \frac{kh}{2} \in J \\ 0 & otherwise , \end{cases}$$

be the kth symmetric difference, and let $\Delta_h^k(f, x) = \Delta_h^k(f, x, [-1,1])$.
By eq.(1.3), we define our new weights as follows:

Definition 1.4: For $k \in N, h > 0, M \geq 2, \varphi(x) = \sqrt{1-x^2}, \lambda j \in J_p, x \in (-1,1)$, we have:

$$\mathcal{W}_{\delta}^{J}(x) = \prod_{j=1}^{M} \left| x - z_j - \frac{\delta\varphi(x)}{M} \right|^{\lambda j} \quad . \tag{1.4}$$

So we define our new weighted modulus of smoothness is given as follows:

Definition 1.5: For $\in \mathbb{B}_p^r$, the generalized Jacobian weighted modulus of smoothness is defined as follows:

$$\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p = \sup_{0\le h\le\delta} \left\| \mathcal{W}_{kh}^{J}(x)\Delta_{h\varphi(x)}^{k}(f^{(r)},x) \right\|_p \quad , \tag{1.5}$$

where

$$\mathfrak{D}_{\delta} = \{x \mid 1 - \delta\varphi(x)/M \ge |x|\} \setminus \{\pm 1\}$$

Remark 1.1

- Note that, when $\delta = 0$, the improved generalized Jacobi weight eq. (1.4) returns to the generalized Jacobi weight eq.(1.3).
- Moreover, when $\delta = 0, M = 2,$ eq. (1.4) returns to eq. ((the classical Jacobi weight).
- If $J = \{\alpha, \beta\}$, with $M = 2$, eq. (1.4) returns to eq. (1.2), so we conclude a general case of eq. (1.4) .

For the proofs of equivalence to K-functional, we can be helped by the following averaged moduli of smoothness,

Definition 1.6

Let $k \in \mathbb{N}, r \in \mathbb{N}_0, 0 < p < 1$ and $f \in \mathbb{B}_p^r$,

$$\omega_{k,r}^{*\varphi}\left(f^{(r)},\delta\right)_p = \left(\frac{1}{t} \int_0^t \int_{\mathfrak{D}_{kh}} \left| \mathcal{W}_{kh}^{J}(x)\Delta_{\tau\varphi(x)}^{k}(f^{(r)},x) \right|^p dx d\tau \right)^{1/p} ,$$

It is clear that by Definition 1.5 , Definition 1.6 and Lemma2.3.(ii)

$$\omega_{k,r}^{*\varphi}\left(f^{(r)},\delta\right)_{\alpha,\beta,p} \le \omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_{\alpha,\beta,p} \quad ,\delta > 0 \tag{1.6}$$

Definition 1.7 [14]

Let $k \in \mathbb{N}, r \in \mathbb{N}_0, 1 \le p < \infty$ and $f \in \mathbb{B}_p^r$,

$$\omega_{\varphi}^{*k}\left(f^{(r)},\delta\right)_{w,p} = \left(\frac{1}{t} \int_0^t \int_{-1+t^*}^{1-t^*} \left| w(x)\Delta_{\tau\varphi(x)}^{k}\left(f^{(r)},x\right) \right|^p dx d\tau \right)^{\frac{1}{p}}$$
$$+ \left(\frac{1}{t^*} \int_0^{t^*} \int_{-1}^{-1+At^*} \left| w(x)\overrightarrow{\Delta}_{u\varphi(x)}^{k}\left(f^{(r)},x\right) \right|^p dx du \right)^{\frac{1}{p}}$$
$$+ \left(\frac{1}{t^*} \int_0^{t^*} \int_{1-At^*}^{1} \left| w(x)\overleftarrow{\Delta}_{u\varphi(x)}^{k}(f^{(r)},x) \right|^p dx du \right)^{1/p} ,$$

where $t^* = 2k^2t^2$, A is constant and $w = \omega_{\alpha,\beta}\varphi^r$.

The functional modulus of smoothness is important to our theorems, it can be used to show the relation among moduli. It can be also used in proofing properties of the new modulus and for the function approximation.

Definition 1.8 : [14]
For $k \in N$, $r \in \mathbb{N}_0$ and $\in \mathbb{B}_p^r(\omega_{\alpha,\beta})$, $0 < p \leq \infty$, we define

$$K_{k,r}^{\varphi}\left(f^{(r)}, \delta^k\right)_{\alpha,\beta,p} = \inf_{g\in\mathbb{B}_p^{k+r}(\omega_{\alpha,\beta})} \left\{\left\|\omega_{\alpha,\beta}\varphi^r\left(f^{(r)} - g^{(r)}\right)\right\|_p + \delta^k\left\|\omega_{\alpha,\beta}\varphi^{k+r}g^{(k+r)}\right\|_p\right\}$$

To have a closer look to the modulus of smoothness, we give the following example for the modulus of smoothness with General Jacobian Weight of a function on $\mathbb{B}_p^r[-1,1]$

Example1.9.
Let $f(x) = x^2 + 1, M = 3$ be the general Jacobian weight $\mathcal{W}_\delta^J(x)$. It can be estimated with $J = \{1,1,1\}$, and taking the maximum value of h as δ so that the modulus of smoothness of order $k = 2$ of f is as follows:

$$\omega_2^{\varphi}(f,\delta)_p^p = \sup_{0\leq h\leq\delta}\left\|\mathcal{W}_\delta^J(x)\Delta_{h\varphi(x)}^2(f,x)\right\|_p^p$$

$$\leq \int_{-1}^{1}\left(x^3 - \delta\sqrt{(1-x^2)}x^2 + \frac{\delta\sqrt{1-x^2}}{3} + \frac{\delta^2x(1-x^2)}{3} - x - \frac{\delta^3\left(\sqrt{1-x^2}\right)^3}{27}\right)^p (2\delta^2(1 - x^2))\,dx$$

$$\leq \int_{-1}^{1}\left(x^3 - \delta\sqrt{(1-x^2)}x^2 + \frac{\delta\sqrt{1-x^2}}{3} + \frac{\delta^2x(1-x^2)}{3} - x - \frac{\delta^3\left(\sqrt{1-x^2}\right)^3}{27}\right)(2\delta^2(1-x^2))\,dx$$

$$= \int_{-1}^{1}\left(\left(2x^3\delta^2(1-x^2)\right) - 2\delta^3x^2(1-x^2)^{\frac{3}{2}} + \frac{2\delta^3\sqrt{(1-x^2)}}{3} - \frac{2\delta^3x^2\sqrt{(1-x^2)}}{3} + \frac{2\delta^4x(1-x^2)^2}{3} - 2x\delta^2 + 2x^3\delta^2 - \frac{1}{27}\left(2\delta^5(1-x^2)^2\sqrt{(1-x^2)}\right)\right)dx$$

$$= \frac{\pi}{8}\delta^3 - \frac{5\pi}{216}\delta^5$$

2. Auxiliary lemmas

Here, some general properties of Jacobi weight (1.1) are given, for more details see [10], these properties are useful to our work

Lemma 2.1:
Kopotun [10] studied the properties of φ in his set $\{x \mid 1 - \delta\varphi(x)/2 \geq |x|\} \setminus \{\pm1\}$, by using some similar steps, we get the following:

i. $(1-x) \leq 2(1-u)$ and $(1+x) \leq 2(1+u)$, $if\ u \in [min\{0,x\}, max\{0,x\}]$,
ii. $\varphi(x) \leq \varphi(u)$, $if\ |u| \leq |x| \leq 1$, $u \in [min\{0,x\}, max\{0,x\}]$ and $x \in \mathfrak{D}_\delta$,
iii. $\delta|\acute{\varphi}(x)| \leq 1$ for $x \in \mathfrak{D}_\delta$,
iv. $If\ y(x) = x + \delta_1\varphi(x)/2$ and $|\delta_1| \leq \delta$ then $\frac{1}{2} \leq y'(x) \leq \frac{3}{2}$ for all $x \in \mathfrak{D}_\delta$,
v. $If\ \delta_1 > \delta_2$ then $\mathfrak{D}_{\delta_1} \subset \mathfrak{D}_{\delta_2}$.

The following lemma relates our improved generalized Jacobi weight $\mathcal{W}$ that is given in (1.3) to the classical Jacobi weight (1.1).

Lemma 2.2:

i. $\mathcal{W}_\delta^J(x) \leq \varphi(u)$ for $x \in \mathfrak{D}_\delta$ and $u \in \left[-|x| - \frac{\delta\varphi(x)}{M}, |x| + \frac{\delta\varphi(x)}{M}\right]$,

ii. $\mathcal{W}_\delta^J(x) \leq \varphi(x)$ for $x \in \mathfrak{D}_\delta$,

iii. $\varphi(x)^M \leq M^M \mathcal{W}_\delta^J(x)$ $for\ x \in \mathfrak{D}_\delta$.

Proof :

i) From Lemma 2.1 (ii), we have

$$\varphi^2(u) - \mathcal{W}_\delta^2(x) \geq \varphi^2\left(|x| + \frac{\delta\varphi(x)}{M}\right) - \prod_{j=1}^{M}\left|x - z_j - \frac{\delta\varphi(x)}{M}\right|^{2\lambda j}$$

$$= 1 - \left(|x| + \frac{\delta\varphi(x)}{M}\right)^2 - \prod_{j=1}^{M}\left|x - z_j - \frac{\delta\varphi(x)}{M}\right|^{2\lambda j}$$

$$\geq 0 \qquad ,\forall\, x \in \mathfrak{D}_\delta .$$

The proof of (ii) is a special case of (i) when u $= x$.

iii) Let $z_j \in I_j = [z_{j-1}, z_j]$, then

Since $1 + |x| \geq |x - z_j|$, but for $M \geq 2$, we have

$1 + |x| \leq M|x - z_j|$ and $1 - |x| \leq M|x - z_j|$

So that

$$\varphi^2(x) \leq M^2 \left|x - z_j - \frac{\delta\varphi(x)}{M}\right|^2$$

$$\prod_{j=1}^{M} \varphi^2(x) \leq \prod_{j=1}^{M} M^2 \left|x - z_j - \frac{\delta\varphi(x)}{M}\right|^{2\lambda_j}$$

$$\varphi^{2M}(x) \leq \left(M^M \prod_{j=1}^{M} |x - z_j - \frac{\delta\varphi(x)}{M}|^{\lambda_j}\right)^2$$

$$\varphi^M(x) \leq M^M \mathcal{W}_\delta^J(x) .$$

More properties are proved in the next lemma, we get an equivalence between $\mathcal{W}_\delta^J$ and $\omega_{\alpha,\beta}$

Lemma 2.3: For $x \in \mathfrak{D}_\delta$, $\alpha, \beta \in J_p$ and ≥ 2 :

i. $\mathcal{W}_\delta^J(x) \leq M^{|\alpha|+|\beta|}\omega_{\alpha,\beta}(x)$ and

ii. $\omega_{\alpha,\beta}(x) \leq M^{|\alpha|+|\beta|}\mathcal{W}_\delta^J(x)$.

Proof :

i) Let $\lambda j = \{\alpha, \beta, 0,0,0,0,0, \dots\}$, then

$$\mathcal{W}_\delta^J(x) = \prod_{j=1}^{M}\left|x - z_j - \frac{\delta\varphi(x)}{M}\right|^{\lambda j}$$

$$= \left|x - z_j - \frac{\delta\varphi(x)}{M}\right|^\alpha \left|x - z_j - \frac{\delta\varphi(x)}{M}\right|^\beta$$

$$\leq \left|x - 1 - \frac{\delta\varphi(x)}{M}\right|^\alpha \left|x + 1 - \frac{\delta\varphi(x)}{M}\right|^\beta$$

$$= \left|1 - \left(x - \frac{\delta\varphi(x)}{M}\right)\right|^\alpha \left|1 + \left(x - \frac{\delta\varphi(x)}{M}\right)\right|^\beta$$

$$= \omega_{\alpha,0}\left(x - \frac{\delta\varphi(x)}{M}\right)\omega_{0,\beta}\left(x - \frac{\delta\varphi(x)}{M}\right)$$

$$\leq M^{|\alpha|}\omega_{\alpha,0}(x) M^{|\beta|}\omega_{0,\beta}(x)$$

$$= M^{|\alpha|+|\beta|}\omega_{\alpha,\beta}(x) .$$

ii) $\omega_{\alpha,\beta}(x) = \omega_{\alpha,0}(x)\omega_{0,\beta}(x)$

$\leq M^{|\alpha|}\omega_{\alpha,0}\left(x-\frac{\delta\varphi(x)}{M}\right)M^{|\beta|}\omega_{0,\beta}\left(x-\frac{\delta\varphi(x)}{M}\right)$

$= M^{|\alpha|+|\beta|}\mathcal{W}_{\delta}^{J}(x)$.

Now, we begin studying the properties of our improved generalized weighted modulus of smoothness (1.5) in the following lemmas:

Lemma 2.4:

Let $k \in N, r \in N_0$, $0 < p < 1, f, g \in \mathbb{B}_p^r$ then for all $\delta > 0$, we have the following:

i. $\omega_{k,r}^{\varphi}\left(f^{(r)}+g^{(r)},\delta\right)_p \leq c\left(\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p+\omega_{k,r}^{\varphi}\left(g^{(r)},\delta\right)_p\right)$,

ii. $\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p \leq c(p,k)\left\|f^{(r)}\right\|_p$,

iii. $\omega_{k,r}\left(f^{(r)},\delta\right) \leq \omega_{k,r}\left(f^{(r)},\acute{\delta}\right)$, for $\delta \leq \acute{\delta}$

iv. $\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right) \leq c\left\|\omega_{\alpha,\beta}f^{(r)}\right\|_p$,

v. $\omega_{k,r}^{\varphi}\left(f^{(r)},\gamma\delta\right) \leq (1+\gamma)^k\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)$.

Proof :

i) $\omega_{k,r}^{\varphi}\left(f^{(r)}+g^{(r)},\delta\right)_p = \sup_{0\leq h\leq\delta}\left\|\mathcal{W}_{kh}^{J}(x)\Delta_{h\varphi(x)}^{k}(f^{(r)}+g^{(r)},x)\right\|_p$

$$= \sup_{0\leq h\leq\delta}\left\|\mathcal{W}_{kh}^{J}(x)\left(\Delta_{h\varphi(x)}^{k}f^{(r)}+\Delta_{h\varphi(x)}^{k}g^{(r)},x)\right)\right\|_p$$

By quasi-triangle inequality of $\|\cdot\|_p$ $, when\ 0 < p < 1$ we have :

$$\omega_{k,r}^{\varphi}\left(f^{(r)}+g^{(r)},\delta\right)_p$$

$$\leq c\left(\sup_{0\leq h\leq\delta}\left\|\mathcal{W}_{kh}^{J}(x)\Delta_{h\varphi(x)}^{k}(f^{(r)},x)\right\|_p+\sup_{0\leq h\leq\delta}\left\|\mathcal{W}_{kh}^{J}(x)\Delta_{h\varphi(x)}^{k}(g^{(r)},x)\right\|_p\right)$$

$$= c\left(\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p+\omega_{k,r}^{\varphi}\left(g^{(r)},\delta\right)_p\right)$$

ii) $\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p = \sup_{0\leq h\leq\delta}\left\|\mathcal{W}_{kh}^{J}(x)\Delta_{h\varphi(x)}^{k}(f^{(r)},x)\right\|_{L_p(\mathfrak{D}_{kh})}$

$$= \sup_{0\leq h\leq\delta}\left\{\int_{\mathfrak{D}_{kh}}\left|\mathcal{W}_{kh}^{J}(x)\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}f^{(r)}\left(x+\left(i-\frac{k}{M}\right)h\varphi(x)\right)\right|^p dx\right\}^{1/p}$$

$$\leq \left(\sum_{i=0}^{k}\left|\binom{k}{i}\right|^p\int_{\mathfrak{D}_{kh}}\sup_{0\leq h\leq\delta}\left|\mathcal{W}_{kh}^{J}(x)f^{(r)}\left(x+\left(i-\frac{k}{M}\right)h\varphi(x)\right)\right|^p dx\right)^{1/p}$$

$$\leq c(p,k)\left\|f^{(r)}\right\|_{L_p(\mathfrak{D}_{kh})}$$

iii) Since $\mathcal{W}_{\delta}^{J}(x)$ is monotone non-decreasing with respect to δ, then

$$\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p = \sup_{0\leq h\leq\delta}\left\|\mathcal{W}_{kh}^{J}(x)\Delta_{h\varphi(x)}^{k}(f^{(r)},x)\right\|_p$$

$$\leq \sup_{0\leq h\leq\acute{\delta}}\left\|\mathcal{W}_{kh}^{J}(x)\Delta_{h\varphi(x)}^{k}(f^{(r)},x)\right\|_p$$

$$= \omega_{k,r}^{\varphi}\left(f^{(r)},\acute{\delta}\right)_p .$$

iv) $\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p = \sup_{0\leq h\leq\delta}\left\|\mathcal{W}_{kh}^{J}(x)\Delta_{h\varphi(x)}^{k}(f^{(r)},x)\right\|_{L_p(\mathfrak{D}_{kh})}$

from (ii) and Lemma 2.3(i) we get :

$$\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right) \leq c\left\|\omega_{\alpha,\beta}f^{(r)}\right\|_p$$

v) Noting that $\gamma\delta \leq \delta$, v comes immediately from (iii)

$$\omega_{k,r}^{\varphi}\left(f^{(r)},\gamma\delta\right)_p \leq \omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p \leq (1+\gamma)^k \omega_{k,r}^{\varphi}\left(f^{(r)},\gamma\delta\right)_p$$

The most important property of any modulus is its convergence to zero, so we prove that in the next lemma,

Lemma 2.5:
If $r \in N_0, 0 < p < 1$ and $f \in \mathbb{B}_p^r$, then $\lim_{\delta\to 0^+} \omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p = 0.$

Proof:
If $\epsilon > 0$ then $\delta > 0$ such that

$$\int_{[-1,1]\setminus\mathfrak{D}_\delta} \left|\varphi^r(x) f^{(r)}(x)\right|^p dx < \left(\frac{\epsilon}{2^k+2}\right)^p$$

Set

$$g^{(r)}(x) = \begin{cases} f^{(r)}(x) & , \quad if\ x \in \mathfrak{D}_\delta \\ 0 & , \ otherwise \end{cases}$$

Since $g^{(r)} \in L_P[-1,1\,]$, $\exists\ \delta_0 > 0$ such that

$$\omega_k^{\varphi}\left(g^{(r)},\delta\right)_p < \frac{\epsilon}{2}\ , \quad 0 < \delta \leq \delta_0$$

For each $h > 0$, we have

$$\left\|\mathcal{W}_{kh}^{j}(x)\Delta_{h\varphi(x)}^{k}\left(f^{(r)},x\right)\right\|_p$$
$$\leq c\left(\left\|\mathcal{W}_{kh}^{j}(x)\Delta_{h\varphi(x)}^{k}\left(g^{(r)},x\right)\right\|_p + \left\|\mathcal{W}_{kh}^{j}(x)\Delta_{h\varphi(x)}^{k}\left(f^{(r)}-g^{(r)},x\right)\right\|_p\right)$$
$$\leq c\left(\left\|\Delta_{h\varphi(x)}^{k}\left(g^{(r)},x\right)\right\|_p + \left\|\mathcal{W}_{kh}^{j}(x)\Delta_{h\varphi(x)}^{k}\left(f^{(r)}-g^{(r)},x\right)\right\|_p\right)$$
$$\leq c\left(\frac{\epsilon}{2} + \sum_{i=0}^{k}\binom{k}{i}\left(\int_{\mathfrak{D}_{kh}}(\mathcal{W}_{kh}^{j}(x)\left|f^{(r)}(x+\left(i-\frac{k}{M}\right)h\varphi(x) - g^{(r)}(x+\left(i-\frac{k}{M}\right)h\varphi(x)\right|^p\ dx\right)^{\frac{1}{p}}\right)$$

From Lemma 2.2 (i) and letting $u(x) = x + \left(i-\frac{k}{M}\right)h\varphi(x)$, we get from $M \geq 2$ and monotonicity of φ that:

$$\left\|\mathcal{W}_{kh}^{j}(x)\Delta_{h\varphi(x)}^{k}\left(f^{(r)},x\right)\right\|_p$$
$$\leq c\left(\frac{\epsilon}{2} + \sum_{i=0}^{k}\binom{k}{i}\left(\int_{\mathfrak{D}_{kh}}\left(\varphi^r(x+\left(i-\frac{k}{M}\right)h\varphi(x)\right)\left|f^{(r)}\left(x+\left(i-\frac{k}{M}\right)h\varphi\right)\right.\right.\right.$$
$$\left.\left.\left. - g^{(r)}\left(x+\left(i-\frac{k}{M}\right)h\varphi\right|^p dx\right)^{\frac{1}{p}}\right)$$
$$\leq c\left(\frac{\epsilon}{2} + 2\sum_{i=0}^{k}\binom{k}{i}\left(\int_{-1}^{1}\left(\varphi^r(u)\left|f^{(r)}(u) - g^{(r)}(u)\right|\right)^p du\right)^{\frac{1}{p}}\right)$$
$$\leq c\left(\frac{\epsilon}{2} + 2\sum_{i=0}^{k}\binom{k}{i}\left(\int_{[-1,1]\setminus\mathfrak{D}_\delta}\left|\varphi^r(u)f^{(r)}(u)\right|^p\right)^{\frac{1}{p}}\right) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon$$

Then the proof is done.
Kopotun [10] proved the following property for $1 \le p \le \infty$, later Sharba et al. proved the same property for $0 < p < 1$ in [9].

Lemma 2.6: [10] [9]
Let $0 < p \le \infty$ and $\in N_0$, if $g \in \mathbb{B}_p^{r+1}$ then

$$\left\|\varphi^{\gamma} g^{(r)}\right\|_p < \infty$$

for any $\gamma \ge 0$ such that $\gamma > r - 1$.

Lemma 2.7:
Let $k \in \mathbb{N}, r \in \mathbb{N}_0, 0 < p \le \infty$,if $f \in \mathbb{B}_p^r$ then for $\delta \ge 2/k$

$$\omega_{k,r}^{\varphi}\left(f^{(r)}, 2/k\right)_p = \omega_{k,r}^{\varphi}\left(f^{(r)}, \delta\right)_p$$

Lemma 2.8
If $k \in \mathbb{N}, r \in \mathbb{N}_0, \alpha, \beta \in J_p,\ 0 < p < \infty$ and $f \in \mathbb{B}_p^r$ then

$$\omega_{\varphi}^{*k}\left(f^{(r)}, \delta\right)_{\omega_{\alpha,\beta}\varphi^r,p} \le c(k,r,\alpha,\beta)\omega_{k,r}^{*\varphi}\left(f^{(r)}, c(k)\delta\right)_{\alpha,\beta,p} \qquad ,0 < \delta < c(k)$$

Proof:
For $0 < p < 1$, by Definition 1.6 and 1.7 we have

$$\omega_{\varphi}^{*k}\left(f^{(r)}, \delta\right)_{\omega_{\alpha,\beta}(x)\varphi^r(x),p} = \left(\frac{1}{t}\int_0^t \int_{-1+t^*}^{1-t^*} \left|\omega_{\alpha,\beta}(x)\varphi^r(x)\Delta_{\tau\varphi(x)}^k\left(f^{(r)}, x\right)\right|^p dx d\tau\right)^{\frac{1}{p}}$$

$$+\left(\frac{1}{t^*}\int_0^{t^*} \int_{-1}^{-1+At^*} \left|\omega_{\alpha,\beta}(x)\varphi^r(x)\vec{\Delta}_{u\varphi(x)}^k\left(f^{(r)}, x\right)\right|^p dx du\right)^{\frac{1}{p}}$$

$$+\left(\frac{1}{t^*}\int_0^{t^*} \int_{1-At^*}^{1} \left|\omega_{\alpha,\beta}(x)\varphi^r(x)\overleftarrow{\Delta}_{u\varphi(x)}^k\left(f^{(r)}, x\right)\right|^p dx du\right)^{1/p}$$

Now, by Lemma 2.3 (ii) and Lemma 2.2(iii) we get:

$$\omega_{\varphi}^{*k}\left(f^{(r)}, \delta\right)_{\omega_{\alpha,\beta}(x)\varphi^r(x),p} \le c(k,r,\alpha,\beta)\left(\frac{1}{t}\int_0^t \int_{-1+t^*}^{1-t^*} \left|\mathcal{W}_{kh}^J(x)\Delta_{\tau\varphi(x)}^k\left(f^{(r)}, x\right)\right|^p dx d\tau\right)^{\frac{1}{p}}$$

$$+\left(\frac{1}{t^*}\int_0^{t^*} \int_{-1}^{-1+At^*} \left|\mathcal{W}_{kh}^J(x)\vec{\Delta}_{u\varphi(x)}^k\left(f^{(r)}, x\right)\right|^p dx du\right)^{\frac{1}{p}}$$

$$+\left(\frac{1}{t^*}\int_0^{t^*} \int_{1-At^*}^{1} \left|\mathcal{W}_{kh}^J(x)\overleftarrow{\Delta}_{u\varphi(x)}^k\left(f^{(r)}, x\right)\right|^p dx du\right)^{1/p}$$

So, we obtain that:

$$\omega_{\varphi}^{*k}\left(f^{(r)}, \delta\right)_{\omega_{\alpha,\beta}\varphi^r,p} \le c(k,r,\alpha,\beta)\omega_{k,r}^{*\varphi}\left(f^{(r)}, c(k)\delta\right)_{\alpha,\beta,p}.$$

Lemma 2.9 [14]
If $k \in \mathbb{N}, r \in \mathbb{N}_0, \alpha, \beta \in J_p, 1 \le p < \infty$ and $f \in \mathbb{B}_p^r$, *then*

I. $K_{k,\varphi}\left(f, \delta^k\right)_{w,p} \le c\omega_{\varphi}^{*k}(f,\delta)_{w,p} \quad , \quad 0 < \delta < \delta_0$

II. $K_{k,r}^{\varphi}\left(f^{(r)}, \delta^k\right)_{\alpha,\beta,p} \le \left\|\omega_{\alpha,\beta}\varphi^r f^{(r)}\right\|_p < \infty.$

We note that because of lemma 2.3, the last lemma is also valid for $0 < p < 1$.

3. Main results

In this section, we discuss the relationship between our General Jacobian Weighted Modulus of Smoothness and the K-functional of Koputon [14]. This relation is very important and useful to improve the degree of the best approximation in terms of our above modulus.

Theorem 3.1 :

Let $0 < p < 1$ *and if* $k \in \mathbb{N}, r \in \mathbb{N}_0, f \in \mathbb{B}_p^r$ *then for* $0 < \delta \leq 2/k$

$$c(k,r,p)K_{k,r}^{\varphi}\left(f^{(r)},\delta^k\right)_p \leq \omega_{k,r}^{*\varphi}\left(f^{(r)},\delta\right)_p \leq \omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p \leq c(k,r,p)K_{k,r}^{\varphi}\left(f^{(r)},\delta^k\right)_p .$$

The upper estimate in Theorem 3.1 :

If $k \in \mathbb{N}, r \in \mathbb{N}_0, 0 < p < 1$ *and* $f \in \mathbb{B}_p^r$, *then*

$$\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p \leq c(k,r,p)K_{k,r}^{\varphi}\left(f^{(r)},\delta^k\right)_p \qquad ,\forall\, \delta > 0$$

Proof:

By Lemma 2.8 and monotonicity of K-functional, we suppose that $\delta \geq 2/k$ and take $g \in \mathbb{B}_p^{r+1}$, so we get $g \in B_p^r$ from Lemma 2.7, wherever:

$$\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p \leq \omega_{k,r}^{\varphi}\left(f^{(r)}-g^{(r)},\delta\right)_p + \omega_{k,r}^{\varphi}\left(g^{(r)},\delta\right)_p$$

Let $0 < h \leq \delta$ and $y_i := x + \left(i - \frac{k}{M}\right)h\varphi(x)$ for $0 \leq i \leq k$ and $M \geq 2$. From Lemma 2.1(ii), we get $y_i'(x) \geq 1/2$ for $x \in \mathfrak{D}_{kh}$.

For $0 < p < 1$

$$\omega_{k,r}^{\varphi}\left(f^{(r)}-g^{(r)},\delta\right)_p = \sup_{0\leq h\leq\delta}\left\|\mathcal{W}_{kh}^{J}(y_i)\Delta_{h\varphi(x)}^{k}(f^{(r)}-g^{(r)},x)\right\|_p^p$$

$$= \sup_{0\leq h\leq\delta}\int_{\mathfrak{D}_{kh}}\left|\sum_{i=0}^{k}\binom{k}{i}\mathcal{W}_{kh}^{J}(y_i)\left(f^{(r)}(y_i)-g^{(r)}(y_i)\right)\right|^p dx$$

Since $\mathcal{W}_{kh}^{J}(y) \leq M^{|\alpha|+|\beta|}\omega_{\alpha,\beta}(y)$ for $y \in \left[x-\frac{\delta\varphi(x)}{M}, x+\frac{\delta\varphi(x)}{M}\right]$ and $0 < \delta \leq 2$

We get:

$$\omega_{k,r}^{\varphi}\left(f^{(r)}-g^{(r)},\delta\right)_p \leq$$

$$M^{|\alpha|+|\beta|}\sup_{0\leq h\leq\delta}\left|\sum_{i=0}^{k}\binom{k}{i}\right|^p\int_{\mathfrak{D}_{kh}}\varphi^{rp}(y_i)\varphi^{-rp}(y_i)\omega_{\alpha,\beta}^{p}(y_i)\left|f^{(r)}(y_i)-g^{(r)}(y_i)\right|^p dx$$

$$= M^{|\alpha|+|\beta|}\sup_{0\leq h\leq\delta}\left|\sum_{i=0}^{k}\binom{k}{i}\right|^p\sup_{0\leq i\leq k}\left|\varphi^{-r}(y_i)\right|\left\|\varphi^{r}(y_i)\omega_{\alpha,\beta}(y_i)\left|f^{(r)}(y_i)-g^{(r)}(y_i)\right|\right\|_p^p$$

$$= c(p,k)\left\|\varphi^{r}\omega_{\alpha,\beta}\left(f^{(r)}-g^{(r)}\right)\right\|_p^p .$$

To estimate the second term $\omega_{k,r}^{\varphi}\left(g^{(r)},\delta\right)_p$ by using the identity

$$\Delta_h^k(f,x) = \int_{-\frac{h}{M}}^{\frac{h}{M}}\dots\int_{-\frac{h}{M}}^{\frac{h}{M}} f^{(k)}(x+u_1+\dots+u_k)\,d_{u_1}\dots d_{u_k},$$

we have

$$\omega_{k,r}^{\varphi}\left(g^{(r)},\delta\right)_p = \sup_{0\leq h\leq\delta}\left\|\mathcal{W}_{kh}^{J}(\cdot)\Delta_{h\varphi(x)}^{k}(g^{(r)},\cdot)\right\|_{L_p(\mathfrak{D}_{kh})}$$

$$= \sup_{0\le h\le\delta}\left\|\mathcal{W}_{kh}^{J}\int_{-\frac{h\varphi}{M}}^{\frac{h\varphi}{M}}\dots\int_{-\frac{h\varphi}{M}}^{\frac{h\varphi}{M}} g^{(k+r)}(x+u_1+\dots+u_k)\,d_{u_1}\dots d_{u_k}\right\|_{L_p(\mathfrak{D}_{kh})}$$

By lemma 2.3(i), we get

$\omega_{k,r}^{\varphi}\left(g^{(r)},\delta\right)_p \le$

$$M^{|\alpha|+|\beta|}\sup_{0\le h\le\delta}\left\|\int_{-\frac{h\varphi}{M}}^{\frac{h\varphi}{M}}\dots\int_{-\frac{h\varphi}{M}}^{\frac{h\varphi}{M}}\left(\omega_{\alpha,\beta}g^{(k+r)}\right)(x+u_1+\dots+u_k)\,d_{u_1}\dots d_{u_k}\right\|_{L_p(\mathfrak{D}_{kh})}$$

For each u that satisfies $-1 < x+u-\frac{h\varphi(x)}{M} < x+u+\frac{h\varphi(x)}{M} < 1$, we have

$$\int_{-\frac{h\varphi}{M}}^{\frac{h\varphi}{M}}\left(\omega_{\alpha,\beta}g^{(k+r)}\right)(x+u+u_k)\,d_{u_k}$$

$$= \int_{x+u-\frac{h\varphi}{M}}^{x+u+\frac{h\varphi}{M}}\varphi^{-(k+r)}(a)\varphi^{k+r}(a)\left(\omega_{\alpha,\beta}g^{(k+r)}\right)(a)\,da$$

$$\le M\left(\left\|\varphi^{k+r}\omega_{\alpha,\beta}g^{(k+r)}\right\|_{L_p(\mathcal{A}(x,u))}\left\|\varphi^{-(k+r)}\right\|_{L_p(\mathcal{A}(x,u))}\right)$$

$$= M\left(\sup_{x\in\mathcal{A}(x,u)}\left|\varphi^{-(k+r)}\right|\left\|\varphi^{k+r}\omega_{\alpha,\beta}g^{(k+r)}\right\|_{L_p(\mathcal{A}(x,u))}\right),$$

where

$$\mathcal{A}(x,u) := \left[x+u-\frac{h\varphi(x)}{M}\,,x+u+\frac{h\varphi(x)}{M}\right]$$

To complete the proof, we have

$$M^{|\alpha|+|\beta|}\sup_{0\le h\le\delta}\int_{\mathfrak{D}_{kh}}\left(\int_{-\frac{h\varphi}{M}}^{\frac{h\varphi}{M}}\dots\int_{-\frac{h\varphi}{M}}^{\frac{h\varphi}{M}}\sup_{u_1u_2\dots u_k}\left|\varphi^{-(k+r)}\right|\left\|\varphi^{k+r}\omega_{\alpha,\beta}g^{(k+r)}\right\|_p d_{u_1}\dots d_{u_k}\right)^p dx$$

$$\le c(k\,,r\,,p)\left\|\varphi^{k+r}\omega_{\alpha,\beta}g^{(k+r)}\right\|_p^p\ .$$

So

$$\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p \le \omega_{k,r}^{\varphi}\left(f^{(r)}-g^{(r)},\delta\right)_p+\omega_{k,r}^{\varphi}\left(g^{(r)},\delta\right)_p$$
$$\le c\left(\left\|\omega_{\alpha,\beta}\varphi^{r}\left(f^{(r)}-g^{(r)}\right)\right\|_p+\delta^k\left\|\omega_{\alpha,\beta}\varphi^{k+r}g^{(k+r)}\right\|_p\right).$$

The lower estimate in Theorem 3.1

Let $0 < p < 1$ *and if* $k \in \mathbb{N}\,, r\in\mathbb{N}_0\,, f\in\mathbb{B}_p^r$ *then*

$$K_{k,r}^{\varphi}\left(f^{(r)},\delta^k\right)_{\alpha,\beta,p} \le c\omega_{k,r}^{*\varphi}\left(f^{(r)},\delta\right)_p \le c\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p \qquad (3.1)$$

Proof :

By Lemma 2.9(I) with weight $w = \omega_{\alpha,\beta}\varphi^r$ and Lemma 2.8, we obtain , for $0 < p < 1$,

$$K_{k,r}^{\varphi}\left(f^{(r)},\delta^k\right)_{\alpha,\beta,p} = K_{k,\varphi}\left(f^{(r)},\delta^k\right)_{\omega_{\alpha,\beta}\varphi^r,p} \le c\omega_{\varphi}^{*k}\left(f^{(r)},\delta\right)_{\omega_{\alpha,\beta}\varphi^r,p}$$

$$\le c\omega_{k,r}^{*\varphi}\left(f^{(r)},c(k)\delta\right)_p \quad , \quad 0<\delta<c(k).$$

Hence, we have

$$K_{k,r}^{\varphi}\left(f^{(r)},\delta^k\right)_{\alpha,\beta,p} \le c\omega_{k,r}^{*\varphi}\left(f^{(r)},c(k)\delta\right)_p \quad , \quad 0<\delta<c(k)\,, \qquad (3.2)$$

Now we assume that $0<\delta\le 2/\delta$, and let $M=\max\{1,c_1,2/kc(k)\}$ then from (3.2) we obtain

$$K_{k,r}^{\varphi}\left(f^{(r)},\delta^k\right)_{\alpha,\beta,p} \le M^k K_{k,r}^{\varphi}\left(f^{(r)},\left(\frac{\delta}{M}\right)^k\right)_{\alpha,\beta,p} \le c\omega_{k,r}^{*\varphi}\left(f^{(r)},\frac{c(k)\delta}{M}\right)_p \le c\omega_{k,r}^{*\varphi}\left(f^{(r)},\delta\right)_p,$$

then by (1.6) we get

$c\omega_{k,r}^{*\varphi}\left(f^{(r)},\delta\right)_p \le c\omega_{k,r}^{\varphi}\left(f^{(r)},\delta\right)_p$.

Conclusions and Future Work

The importance of moduli of smoothness pushes us to make improvements to the existing moduli, so that a new generalized moduli of smoothness is defined which is related to the very important tool, namely, the Jacobian weight. Further, a new defined Jacobian weight in a very general form is introduced. So in future work, this modulus can be used to approximate functions by applicable tools such as neural networks with a degree of approximation in terms of modulus of smoothness.

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