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Hollow Modules With Respect to an Arbitrary Submodule

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Abstract
In this paper ,we introduce hollow modules with respect to an arbitrary submodule .Let M be a non-zero module and T be a submodule of M .We say that M is aT-hollow module if every proper submodule K of M such that $T \nsubseteq K$ is a T-small submodule of M .We investigate the basic properties of a T-hollow module .

Keywords: T-small submodule, T-maximal submodule, T-radical submodule.

المقاسات المجوفة بالنسبة الى مقاس جزئي افتراضي

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الخلاصة
في هذا البحث قدمنا مفهوم المقاسات المجوفة بالنسبة الى مقاس جزئي افتراضي, ليكن M مقاسا غير صفري و T مقاس جزئي من M , يقال ان M مقاس مجوف بالنسبة للمقاس الجزئ T اذا كان لكل مقاس جزئي فعلي K من M بحيث ان المقاس الجزئي T غير محتوى في المقاس الجزئي K يكون مقاسا جزئيا صغيرا بالنسبة الى المقاس الجزئي T . افترضنا الخصائص الاساسية للمقاسات المجوفة بالنسبة الى مقاس جزئي افتراضي T .

1-Introduction

Throughout this paper , rings are associative with identity and modules are unital left R-modules. Recall that a submodule N of an R-module M is small, denoted by $N \ll M$,if for any submodule X of M , N+X = M implies that X = M .More detailes about small submodules can be found in [1-3]. The concept of small submodule has been generalized by some researchers , for this see [4,5]. In [6], the authors introduced the concept of small submodule with respect to an arbitrary submodul .Recall that a submodule N of M is called T-small in M , denoted by $N \ll_T M$, in case for any submodule $X \leq M$, $T \subseteq N+X$ implies that $T \subseteq X$ .In this paper ,we develop the properties of T-maximal submodules and introduce the concept of T-hollow module .Recall that a submodule K of M is called T-maximal in M if (T+K) / K is simple R-module ,see[6].Recall that the intersection of all T-maximal submodule in M is denoted by $Rad_T M$,see[6].

In section 2,we develop the properties of T- maximal submodule and the T-radical submodule of a module M ,We show that if T is a finitely generated submodule of a module M and N be submodule of M such that $T \nsubseteq N$,then there is a T-maximal submodule of M containing N ,see Theorem 2.2.Also we prove that Ra is not T-small submodule of a module M ,where $a \in M$ if and only if there is a T-maximal submodule N in M such that $a \notin N$ and $T \subseteq Ra + N$,see Theorem 2.9.

In section 3, we study the class of T-hollow module .We prove that if N is a non-zero submodule

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of T-hollow module M such that $T \subseteq N$,then N is T-hollow module ,see proposition 3.5.Also we prove that if T is finitely generated submodule of T-hollow module M ,then T is cyclic ,see proposition 3.10. We investigate the basic properties of T-hollow module .

Let R be a ring and M be a left R-module .If $X \subseteq M$,then $X \leq M$, $X \lneq M$, $X \ll M$, $X \ll_T M$ and $Rad_T M$ denote X is a submodule of M , X is a proper submodule of M , X is a small submodule of M , X is a T-small submodule of M and the T-radical of M , respectively .

2-The T-Radical of a module

In this section ,we develop the basic properties of the T-maximal submodules and the T-Radical of a module M .Following [6] ,let T, K be submodules of a module M .K is T-maximal submodule of M if (T+K)/ K is simple .The intersection of all T-maximal submodules of M is denoted by $Rad_T M$.

Proposition 2.1. Let M be a module and A ,T be submodules of M such that $A \not\subseteq T$ .Then A is T-maximal submodule of M if and only if $A+T = A+Rx$,for every $x \in A+T$ and $x \notin A$.

Proof .$\rightarrow$) Suppose that A is T-maximal submodule of M .Let $x \in A+T$ and $x \notin A$ .Then $A \subsetneq A+Rx \subseteq A+T$ .But A+T/A is simple , then $A+T = A+Rx$.

$\leftarrow$) Suppose that $A+T = A+Rx$,for every $x \in A+T$ and $x \notin A$ .To show that A+T / A is simple .Let N be submodule of A+T such that $A \subsetneq N$ .Then there is $x \in N$ and $x \notin A$ .Therefore $A \subsetneq A+Rx \subseteq N \subseteq A+T$ .Thus $N = A+T$.

Theorem 2.2. Let N and T be submodules of a module M such that T is finitely generated and $T \not\subseteq N$ Then there is a T-maximal submodule of M containing N .

Proof. Let N and T be a submodules of M such that T is finitely generated and $T \not\subseteq N$ .Consider the set **S** $=\{ K \mid K$ is a submodule of M such that $T \not\subseteq K$ and $N \subseteq K \}$. Since $T \not\subseteq N$,then $N \in$ **S** .Thus **S**$\neq \varnothing$. Let $\{C_\alpha\}_{\alpha\in\Lambda}$ be a chain in **S** .To show that $\bigcup_{\alpha\in\Lambda} C_\alpha \in$ **S** .Clearly $\bigcup_{\alpha\in\Lambda} C_\alpha$ is submodule of M and $N \subseteq \bigcup_{\alpha\in\Lambda} C_\alpha$ .To show that $T \not\subseteq \bigcup_{\alpha\in\Lambda} C_\alpha$ .Assume that $T \subseteq \bigcup_{\alpha\in\Lambda} C_\alpha$ .Let $T = Rm_1+Rm_2+\ldots+Rm_n \subseteq \bigcup_{\alpha\in\Lambda} C_\alpha$, $m_1,m_2,\ldots,m_n \in T$ .Hence there exists $\alpha_1,\alpha_2,\ldots,\alpha_n \in \Lambda$ such that $m_1 \in C_{\alpha 1}$,$m_2 \in C_{\alpha 2}$,…,$m_n \in C_{\alpha n}$ .Let $C_{\alpha j} = \max\{C_{\alpha 1,} C_{\alpha 2},\ldots,C_{\alpha n}\}$.Thus $T \subseteq C_{\alpha j}$ which is a contradiction .Thus$\bigcup_{\alpha\in\Lambda} C_\alpha \in$**S** .By Zorn's Lemma **S** has maximal element , say H .

Claim that H is a T-maximal submodule of M .To show that .Since $H \in$**S** ,then $T \not\subseteq H$ and hence $H+T\ /H \neq 0$ .Now let $W \leq H+T$ such that $H \lneq W \leq H+T$ .Then by maximality of H ,$W \notin S$ and hence $T \subseteq W$. So $W = W + T$ and hence $W = H + T$ .Thus H is a T-maximal submodue of M containing N .

Proposition 2.3. Let T , K be submodules of a module M .If K is a T-maximal submodule of M , then K is A-maximal submodule of M , for all submodule A of T such that $A \not\subseteq K$.

Proof . Let K be a T-maximal submodule of M and let A be a submodule of T such that $A \not\subseteq K$ .Since $A \subseteq T$,then $(K+A / K) \subseteq (K+T/K)$ .But K+T / K is simple .Therefore either $K + A / K = 0$ and hence $A \subseteq K$ which is a contradiction. or $(K+A / K) = (K+T / K)$ .Since K is T-maximal ,then K+A / K is simple .Thus K is A-maximal submodule of M .

Proposition 2.4. Let A and B be submodules of a module M such that $M = A + B$ .Then A is a B-maximal submodule of M if and only if A is a maximal submodule of M .

Proof . $\rightarrow$) Let A be a B-maximal submodule of M .Then A+B/A is simple and hence M/A is simple. Thus A is a maximal submodule of M .

$\leftarrow$) Let A be maximal submodule of M , then$(M/A) = (A+B/A)$ is simple .Thus A is a B-maximal submodule of M .

Proposition 2.5. Let A and B be submodules of a module M such that $M = A \oplus B$ and B is simple .Then A is a B-maximal submodule of M .

Proof . Let $M = A \oplus B$ where B is simple .By the second isomorphism theorem ,$(A+B /A) \cong (B /A \cap B) \cong B$ .But B is simple ,therefore A+B /A is simple .Thus A is a B-maximal submodule of M .

Proposition 2.6. Let T, K be submodules of a module M .Then $T \cap K$ is a T-maximal submodule of M if and only if K is a T-maximal submodule of M .

Proof.$\rightarrow$)Suppose that $T \cap K$ is a T-maximal submodule of M .Then $(T + (T \cap K))/(T \cap K) = T/T \cap K$ is simple .By the second isomorphism theorem $(T/T \cap K) \cong (T+K/K)$ .Hence T+K/K is simple .Thus K is a T-maximal submodule of M .

$\leftarrow$) Suppose that K is T-maximal submodule of M.Then T+K/K is simple.By the second isomorphism theorem ,$(T+K / K) \cong (T / T \cap K) = (T+(T \cap K) / T \cap K)$ .Hence $T+(T \cap K) /(T \cap K)$ is simple .Thus $T \cap K$ is a T-maximal submodule of M .

Proposition 2.7. Let T be a finitely generated submodule of a module M and $a \in M$ .Then $T \subseteq Ra$ if and only if a belong to no T-maximal submodule of M .

Proof .→) Assume that there exists a T-maximal submodule K of M such that $a \in K$. Then $Ra \subseteq K$ .But $T \subseteq Ra$,therefore $T \subseteq Ra \subseteq K$ .Thus $T+K/K = 0$ which is a contradiction .

←) Suppose that $a \in M$ and a belong to no T-maximal submodule of M .To show that $T \subseteq Ra$ .Assume not .By Theorem 2.2, there exists a T-maximal submodule K of M such that $Ra \subseteq K$ .Thus $a \in K$ which is a contradiction .

Proposition 2.8. Let $0 \neq T$ be a finitely generated ideal of a ring R and let a be an idempotent element of R .Then either a or 1-a belong to a T-maximal ideal of R .

Proof . Assume not .Then for all T-maximal ideal M of R ,$a \notin M$ and $1-a \notin M$ .By proposition 2.7,$T \subseteq Ra$ and $T \subseteq R(1-a)$ .Hence $T \subseteq Ra \cap R(1-a)$ .But a is an idempotent element ,therefore $Ra \cap R(1-a) = 0$. So $T = 0$ which is a contradiction .

Theorem 2.9. Let T be a finitely generated submodule of a module M and $a \in M$ .Then Ra is not T-small submodule of M if and only if there is a T-maximal submodule N of M such that $a \notin N$ and $T \subseteq Ra + N$.

Proof**.**→) Suppose that Ra is not T-small submodule of M ,then there exists $K \leq M$ such that $T \subseteq Ra + K$ and $T \nsubseteq K$ .Now Let $\mathbf{F} = \{ N \mid N \leq M$ and $T \subseteq Ra+N$, $T \nsubseteq N \}$.Clearly that $K \in \mathbf{F}$ and hence $\mathbf{F} \neq \emptyset$. Let $\{C_\alpha\}_{\alpha \in \Lambda}$ be a chain in **F** .To show that $\bigcup_{\alpha \in \Lambda} C_\alpha \in \mathbf{F}$ .Clearly $\bigcup_{\alpha \in \Lambda} C_\alpha$ is submodule of M and $T \subseteq Ra + \bigcup_{\alpha \in \Lambda} C_\alpha$ .To show that $T \nsubseteq \bigcup_{\alpha \in \Lambda} C_\alpha$. Assume that $T \subseteq \bigcup_{\alpha \in \Lambda} C_\alpha$. Since T is finitely generated ,then $T = Rm_1 + Rm_2 + \ldots + Rm_n$, $m_1, m_2, \ldots, m_n \in T$ and hence there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $m_1 \in C_{\alpha 1}$,$m_2 \in C_{\alpha 2}$,…,$m_n \in C_{\alpha n}$ .Let $C_{\alpha j} = \max\{C_{\alpha 1}, C_{\alpha 2}, \ldots, C_{\alpha n}\}$.Thus $T \subseteq C_{\alpha j}$ which is a contradiction . By Zorn's lemma **F** has a maximal element, say H . Claim that H is a T-maximal submodule of M .Since $H \in \mathbf{F}$, then $T \nsubseteq H$ and hence $H+T/H \neq 0$.Now, let $W \leq M$ such that $H \subsetneq W \leq H+T$ .Since $T \subseteq Ra+ H \subseteq Ra+ W$ and H is the maximal element of **F** ,then $W \notin \mathbf{F}$ and hence $T \subseteq W$ implies $W \subseteq H+T \subseteq W+T = W$ .So $W=H+T$ .Thus H is a T-maximal submodule of M .Since $T \subseteq Ra+ H$ and $T \nsubseteq H$,then $Ra+ H \neq H$ and hence $a \notin H$.
←) Suppose that there exists a T-maximal submodule N of M with $a \notin N$ and $T \subseteq Ra+ N$ .Since $T+N/N$ is simple ,then $T \nsubseteq N$ .Thus Ra is not T-small submodule of M .

Before we give our next result ,we need the following theorem .

Theorem 2.10.[6] Let M be a right R-module and $0 \neq T$ be a proper finitely generated submodule of M .Then $\sum_{L \in A} L = \bigcap_{K \in B} K$,where $A = \{L \leq M \mid L \ll_T M$ and $L+K \subseteq T+K$,for all T-maximal submodule K of M $\}$ and $B = \{K \leq M \mid K$ is an T-maximal submodule of M $\}$.

Theorem 2.11. Let T be a submodule of a module M .Then $\sum_{L \in A} L = \sum_{L \in A_1} L$,where $A=\{L \leq M \mid L \ll_T M$ and $L+K \subseteq T+K$,for all T-maximal submodule K of M$\}$ and $A_1 = \{L \leq M \mid L \ll_T M$ and either $L \subseteq K$ or $L+K = T+K$,for all T-maximal submodule K of M$\}$.

Proof. Let $L \in A$ .Then $L \ll_T M$ and $L+K \subseteq T+K$ for all T-maximal submodule K of M .Then $L+K/K \subseteq T+K/K$ .Since K is a T-maximal submodule of M ,then $T+K/K$ is simple and hence either $L+K/K = 0$ implies that $L \subseteq K$ or $L+K/K = T+K/K$ implies that $L+K = T+K$ .Therefore $L \in A_1$.

Now , let $L \in A_1$.Then $L \ll_T M$ .Let K be a T-maximal submodule of M .Then either $L \subseteq K$ and hence $L+K = K \subseteq T+K$ or $L+K = T+K$ .Therefore $L \in A$.

Proposition 2.12. Let T be a finitely generated submodule of a module M and $m \in M$ such that $Rm+K \subseteq T+K$,for all T-maximal submodule K of M .Then $Rm \ll_T M$ if and only if $m \in Rad_T M$.

Proof**.→)** Suppose that $Rm \ll_T M$ and $Rm+K \subseteq T+K$,for all K is a T-maximal submodule of M . By Theorem 2.9.[6] , $Rm \subseteq A$ and hence $Rm \subseteq Rad_T M$.

←) Let $m \in Rad_T M$ .To show that $Rm \ll_T M$ .Assume that Rm is not T-small submodule of M . By Theorem 2.9., then there exists a T-maximal submodule K of M with $m \notin K$ which is a contradiction .Thus Rm is a T-small submodue of M .

Proposition 2.13. Let M be a module .Then $Rad_T M \ll_T M$ if and only if The sum of any family of submodules of M $\{C_\alpha$; Where $C_\alpha \ll_T M$ and $C_\alpha+K \subseteq T+K$,for all T-maximal submodule K of M $\}$ is T-small in M .

Proof**.**→) Assume that $Rad_T M \ll_T M$ .Let $\{C_\alpha\}_{\alpha \in \Lambda}$ be a family of T-small submodules of M with $C_\alpha + K \subseteq T + K$,for all T-maximal submodule K of M .Since $\sum_{\alpha \in \Lambda} C_\alpha \subseteq Rad_T M$ and $Rad_T M \ll_T M$,then $\sum_{\alpha \in \Lambda} C_\alpha \ll_T M$.

←) Clear by Theorem 2.9.[6] .

3- The T-hollow module .

In this section , we develop the basic properties of the T-hollow module .

Definition 3.1.

Let M be a non-zero module and T be a submodule of M .We say that M is a T-hollow module if every submodule K of M such that $T \not\subseteq K$ is a T-small submodule of M .

Remarks 3.2. (a)

Let M be a non-zero module .Then M is M-hollow module if and only if M is hollow module
Proof. Clear .

It is known that Z as Z-module is not hollow module .Then Z is not Z-hollow module .

(b) A T-hollow module need not to be hollow module as the following example shows :

Consider the module Z_6 as Z-module .If T = $\{\bar{0},\bar{3}\}$,then one can easily show Z_6 is T-hollow module . But Z_6 is not hollow module .

Proposition 3.3. Let M be a module with submodules $K \leq T \leq L \leq M$ .If $K \ll_T M$, then $K \ll_T L$.

Proof. Suppose that $K \ll_T M$ .To show that $K \ll_T L$ .Let $T \subseteq K+X$ for some $X \leq L$ .Since $K \ll_T M$, then $T \subseteq X$ .Thus $K \ll_T L$.

Proposition 3.4. Let M be a T-hollow module and let N be a non-zero submodule of M such that $T \subseteq N$ .Then N is a T-hollow module .

Proof.Let M be a T-hollow module .To show that N is T-hollow module ,let L be a proper Submodule of N such that $T \not\subseteq L$ .Since M be a T-hollow module ,then $L \ll_T M$ .By proposition 3.4., then $L \ll_T N$ Thus N is T-hollow module .

Proposition 3.5 Let M be a T-hollow module and let $f:M \rightarrow M`$ be an epimorphism ,where M` is a non -zero module .Then M` is f(T)-hollow module .

Proof . Suppose that M is a T-hollow module and let $f:M \rightarrow M`$ be an epimorphism .To show that M` is f(T)-hollow .Let $N` \subsetneq M`$ such that $f(T) \not\subseteq N`$.To show that $N` \ll_{f(T)} M`$ .Let $f(T) \subseteq N`+X$,for some $X \leq M`$.Then $f^{-1}(f(T)) \subseteq f^{-1}(N`+X)$ .Therefore $T+ Ker f \subseteq f^{-1}(N`) + f^{-1}(X)$ .Thus $T \subseteq f^{-1}(N`) + f^{-1}(X)$ .To show that $T \not\subseteq f^{-1}(N`)$.Assume $T \subseteq f^{-1}(N`)$ .Then $f(T) \subseteq N`$ which is a contradiction .Thus $T \not\subseteq f^{-1}(N`)$ .Since M is T-hollow module ,then $f^{-1}(N`) \ll_T M$ and hence $T \subseteq f^{-1}(X)$.Therefore $f(T) \subseteq X$ .Thus M` is f(T)-hollow module .

Proposition 3.6. Let T and K be submodules of a module M such that $K \subseteq T$. If K is T-small submodule of M and M/K is T/K-hollow module ,then M is T-hollow .

Proof . Assume that $K \ll_T M$ and M/K is T/K-hollow module .We want to show that M is T-hollow. Let $N \leq M$ such that $T \not\subseteq N$ and let $T \subseteq N+X$ for some $X \leq M$ .Then $T/K \subseteq (N+X)/K$ and hence $T/K \subseteq (N+K)/K + (X+K)/K$ .To show that $T/K \not\subseteq N+K/K$ .Assume that $T/K = N+K/K$ .Then $T = N+K$ and hence $T \subseteq N+K$ .Since $K \ll_T M$,then $T \subseteq N$ which is a contradiction .Thus $T/K \not\subseteq N+K/K$ .Since M/K is a T/K-hollow module, then $N+K/K \ll_{T/K} M/K$ .Therefore $T/K \subseteq X+K/K$ .Thus $T \subseteq X+K$ .Since $K \ll_T M$,then $T \subseteq X$ .Thus M is T-hollow module .

Before we give our next result ,we need the following proposition .

Proposition 3.7 [6]. Let M be an R-module with submodules $N \leq K \leq M$ and $T \leq K$ .If $N \ll_T K$, then $N \ll_T M$.

Proposition 3.8. Let L and T be a submodules of a module M with $T \subseteq L$ .If L is T-hollow module , then either L is a T-small submodule of M or whenever $L/K \ll_{T/K} M/K$, for some submodule $K \leq L$ with $T \neq K$,then L=K .But not both .

Proof. Suppose that there exists $K \subseteq L$ with $T \neq K$ and $L/K \ll_{T/K} M/K$ .To show that $L \ll_T M$ .Let $T \subseteq L+X$,for some $X \leq M$ .Since L is T-hollow and $T \not\subseteq K$,then $K \ll_T L$ .By proposition 3.8.[6] ,then $K \ll_T M$ .By Theorem 2.16.[6] ,therefore $L \ll_T M$.

Now, let L be a T-hollow module and satisfies the condition 1 and 2.Assume $L \neq 0$.Since $L \ll_T M$,then $L/0 \ll_{T/0} M/0$ .By condition 2 ,L = 0 which is a contradiction .

Proposition 3.9. Let T be a finitely generated submodule of a module M .If M is a T-hollow module ,then T is cyclic.

Proof. Let $T = Rx_1+Rx_2+\ldots+Rx_n$, for $x_i \in M$,$\forall$ i=1,…,n .Then $T \subseteq Rx_1+Rx_2+\ldots+Rx_n$ .If $T \neq Rx_1$, then $T \not\subseteq Rx_1$ .Since M is a T-hollow module ,then $Rx_1 \ll_T M$ and hence $T \subseteq Rx_2+Rx_3+\ldots+Rx_n$ .Therefore $T = Rx_2+Rx_3+\ldots+Rx_n$ .So,we delete the component one by one until we have $T = Rx_i$, for some i .Thus T is cyclic .

Proposition 3.10. Let M be a module with unique T-maximal submodule H ,where T is a finitely generated submodule of M .Then M is T-hollow .

Proof.Suppose that M has aunique T-maximal submodule H,where T is afinitely generated submodule of M .To show that M is T-hollow module .Let L be a proper submodule of M such that $T \nsubseteq L$ .To show that $L \ll_T M$,let $T \subseteq L+K$, for some $K \leq M$ .If $T \nsubseteq K$,then there is a T-maximal containing K .By Theorem 2.2 .But H is the unique T-maximal in M , then $K \subseteq H$ .Since $T \nsubseteq L$,then by the same way $L \subseteq H$ and hence $L+K \subseteq H$ .Therefore $T \subseteq L+K \subseteq H$ and hence T+H/H is simple .This is a contradiction .So $T \subseteq K$ and $L \ll_T M$ .Thus M is T-hollow module .

Proposition 3.11. Let T be a non-zero submodule of a module M .If M is T-hollow module .Then T is indecomposable .

Proof. Suppose that there are proper submodules K and L of T such that $T = K \oplus L$ .Therefore $T \nsubseteq K$ .Since M is T-hollow module ,then $K \ll_T M$ .But $T \subseteq K \oplus L$,therefore $T \subseteq L$ and hence T = L .This is a contradiction .Thus T is indecomposable .

Proposition 3.12. Let N and T be submodules of a module M such that $N \lneq T$ .If M is T-hollow module andT/N is finitely generated ,then T is finitely generated .

Proof. Let $T/N = R(x_1+N)+R(x_2+N)+\ldots+R(x_n+N)$,$x_i \in T$,$\forall i=1,\ldots,n$ .Clearly $Rx_i \subseteq T$,$\forall i=1,\ldots,n$ and hence $Rx_1+Rx_2+\ldots+Rx_n \subseteq T$ .To show that $T \subseteq Rx_1+Rx_2+\ldots+Rx_n$ .Let $t \in T$ .Since T/N is finitely generated ,then $t + N = r_1(x_1+N) + r_2(x_2+N) +\ldots+ r_n(x_n+N) = (r_1x_1 + r_2x_2 +\ldots+ r_nx_n) + N$ and hence $t = (r_1x_1 + r_2x_2 +\ldots+ r_nx_n) + n$,for some $n \in N$ .Therefore $T = (Rx_1 + Rx_2 +\ldots+ Rx_n) + N$ .Since $N \lneq T$,then $T \nsubseteq N$ .But M is T-hollow module ,so $N \ll_T M$ .Therefore $T \subseteq Rx_1 + Rx_2 +\ldots+ Rx_n$ .Thus $T = Rx_1 + Rx_2 +\ldots+ Rx_n$.

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