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Approximation Solution of Nonlinear Parabolic Partial Differential Equation via Mixed Galerkin Finite Elements Method with the Crank-Nicolson Scheme

Jamil Amir Al-Hawasy*, Marwa Ahmed Jawad
Department of Mathematics, College of Science, Mustansiriyah Univerisity, Baghdad, Iraq

Abstract
The approximate solution of a nonlinear parabolic boundary value problem with variable coefficients (NLPBVPVC) is found by using mixed Galekin finite element method (GFEM) in space variable with Crank Nicolson (C-N) scheme in time variable. The problem is reduced to solve a Galerkin nonlinear algebraic system (NLAS), which is solved by applying the predictor and the corrector method (PCM), which transforms the NLAS into a Galerkin linear algebraic system (LAS). This LAS is solved once using the Cholesky technique (CHT) as it appears in the MATLAB package and once again using the General Cholesky Reduction Order Technique (GCHROT), the GCHROT is employed here at first time to play an important role for saving a massive time. Illustrative examples are given to solve the NLPBVPVC with the GCHROT, the results are given by tables and figures which show from a side efficiency of this technique, and from another side show that the two methods GCHROT and CHM are given the same results, but the suggesting first technique is very fast than the second one.

Keywords: nonlinear parabolic (NLP) boundary value problem with variable coefficients, Galerkin finite element methods, Crank-Nicolson scheme.

حل تقريبي لمعادلة تفاضلية جزئية غير خطية ذات معاملات متغيرة من النمط المكافيء بواسطة مزج طريقة كاليركن للعناصر المنتهية مع مخطط كرانك نيكلسن

جميل أمير علي*، مروة احمد جواد
قسم الرياضيات، كلية العلوم ، الجامعة المستنصرية، بغداد، العراق

الخلاصة
في هذا البحث تم ايجاد الحل التقريبي لمسالة قيم حدودية غير خطية ذات معاملات متغيرة من النمط المكافيء باستخدام مزج طريقة كاليركن للعناصر المنتهية بالنسبة للمتغير الفضاء مع مخطط كرانك نيكلسن بالنسبة لمتغير الزمن والتي تحول المسالة لحل نظام"كاليركن" جبري غير خطي. يحل هذا النظام الغير خطي باستخدام طريقة التنبأ والتصحيح والتي تحوله الى نظام "كاليركن" جبري خطي . تم حل هذا النظام الخطي مرة باستخدام تقنية جولسكي كما هي موجودة في الماتلاب ومرة اخرى باستخدام تقنية جولسكي العامة لتخفيض الرتبة, هذه التقنية وظفت هنا لاول مرة لتلعب دور مهم في اختزال الزمن. تم اعطاء امثلة توضيحة لحل مسالة القيم الحدودية الغير خطية ذات المعاملات المتغيرة من النم المكافيء, النتائج اعطيت على شكل جدوال ورسمات وبينت من ناحية كفاءة تقنية جولسكي العامة ومن ناحية اخرى ان الطريقتين اعطت نفس النتائج الا ان تقنية جولسكي العامة والمقترحة هنا هي سريعة اكثر من الطريقة الاولى.

*Email: Jhawassy17@uomustansiriyah.edu.iq

1. Introduction

The solution of the boundary value problem (BVPS) in general and the solution of NLPBVPVC in particular are interested to study from many researchers in the last few decades. In fact there are many different methods for solving the NLPBVPVC, for example Eriksson and Johnson in 1995 used the Adaptive Finite Element Method (AFEM) for solving the NLP [1]. Amiya in 1998 studied an H^1-Galerkin mixed finite element method, proposed and analyzed for nonlinear parabolic (NLP) with non self adjoint elliptic parts [2]. Georgios and Charalambos in 2004 consider the GM for the time discretization of a class of nonlinear parabolic equations [3]. Buyang and Weiwei in 2017 present a general framework for numerical analysis of finite element methods (FEM) for NLP equations with nonsmooth coefficients [4], and many others.

The study of the solution for the parabolic BVP using the FEM back to the beginning of the 17th century, and are studied from many researchers so as Douglas and Dupont [5], in 1993 Reddy introduced in his book an introduction to the FEM applied to linear, one and two-dimensional problems of engineering and applied sciences [6]. In 1997-2006 Thomee [7] studied the GFEM with backward Euler method for NLPBVP, and many others studies. Actually these studies encourage us to study this work the approximate solution (APS) for NLPBVPVC using the GFEM method for the space variable and the C-N scheme for the time variable.

This paper starts with giving a description of proposed NLPBVPVC and its weak form. The APS of the problem is obtained by discretize the weak form by using the GFEM for the space variable and the C-N scheme for the time variable, the problem then reduces to solve a NLAS which is transformed upon using the PCM to a LAS. This LAS is solved once using the Cholesky Technique and once again using that we gave it the name General Cholesky Reduction Order Technique and it is employed here at first time to play an important role for saving a massive time. Finally illustrative examples are given to solve different problems using MATLAB R2013a software CPU@2.80GHz, the results show the efficiency of this method, and the General Cholesky Reduction Order Technique is very fast to solve the linear algebraic system than the Cholesky Technique.

In this work the inner product and norm in $L^2(\omega)$ will be denoted by $(\cdot,\cdot)$ and $\|\cdot\|_0$, the inner product and norm in Sobolev space $W = H_0^1(\omega)$ will be denoted by $(\cdot,\cdot)_1$ and $\|\cdot\|_1$, the duality bracket between W and its dual W^* will be denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_P$ be the norm in $L^2(P)$.

2. Basic Definitions and Theorems:

Definition 1 [8]: A point s^* in a subset $X \subset \mathbb{R}^2$ is said to be fixed point of a given function $f: X \to \mathbb{R}^2$, if $f(s^*) = s^*$.

Definition 2 [8]: A function $f: X \subset \mathbb{R}^2 \to \mathbb{R}^2$ is said to be contractive on X, if for each $s, t \in X$: $\|f(s) - f(t)\| \le \gamma\|s - t\|$, where $0 < \gamma < 1$ is a constant.

Theorem 1 [8]: A cf f on a complete normed space X has a unique fixed point s^* in X .

Theorem 2 [8]: Let $\|\cdot\|$ is a norm in $\mathbb{R}^2$ and $X \subset \mathbb{R}^2$. If $f: X \to \mathbb{R}^2$ is contractive on X, and one of the following is satisfied:

(i) For each s in X, the function $f(s)$ belongs X .

(ii) $X = \{s | \|s - t\| \le \beta\}$ and $\|q(t) - t\| \le (1 - \gamma)\beta$.

(iii) $X = \{s | \|s - s^*\| \le \beta\}$, where s^* is a fixed point of f Then $\{s^{(l)}\} \in X$, where $s^{(l)}$ is the $l-$th iterative value of s .

Theorem 3 [8]: Let $\|\cdot\|$ is a norm in $\mathbb{R}^2$ and X be a closed subset of $\mathbb{R}^2$. If $f: X \to \mathbb{R}^2$ is contractive function on , and $\{s^{(l)}\} \in X$, then

(i) The sequence $\{s^{(l)}\}$ converges to a fixed point $s^* \in X$

(ii) s^* is a unique in X.

3. Description of the (NLPBVPVC):

Let $\omega = \{\vec{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1, x_2 < 1\}$, with Lipischitz boundary $\partial\omega$, and let $I = [0, T]$, $0 < T < \infty$, and $p = \omega \times I$.

The NLP equation with variable coefficients is given by:

$$u_t - \sum_{s,q=1}^{2} \frac{\partial}{\partial x_q}\left[a_{sq}(\vec{x}, t)\frac{\partial u}{\partial x_s}\right] + \psi(\vec{x}, t)u = \phi(\vec{x}, t, u) \ , \text{ in } \ p \tag{1}$$

with the boundary condition (b.c)

$$u(\vec{x}, t) = 0 \ , \text{ on } \ \partial\omega \times I \tag{2}$$

and the initial condition (i.c)

$u(\vec{x},0) = u^0(\vec{x})$, in ω (3)
where $u = u(\vec{x},t)$, $a_{sq}(\vec{x},t), \psi(\vec{x},t), \phi(\vec{x},t,u) \in L^\infty(\omega)$ and $a_{sq}(\vec{x},t)$ are positive nonzero arbitral functions.

Now, the weak form of problem (1-3) is given by:

$\langle u_t, w \rangle + a(u,w,t) = (\phi(u), w)$, $\forall\ w \in W$ a.e on I , (4)

$(u(0), \xi) = (u^0, \xi)$, in W (5)

with u^0 belongs to $W \subset L^2(\omega)$ and $a(u,w,t) = \sum_{s,q=1}^{2} a_{sq}(\vec{x},t)\,(\frac{\partial u}{\partial x_q}, \frac{\partial w}{\partial x_s}) + \psi(\vec{x},t)(u,w)$, is the usual bilinear form .

Assumptions:

(1) for some positive constants δ_1, δ_2 and for each $w_1, w_2 \in W$, $t \in \bar{I}$, the following are hold

i) $|a(w_1,w_2,t)| \leq \delta_1 \|w_1\|_1 \|w_2\|_1$

ii) $a(w_1,w_1,t) \geq \delta_2 \| w_1 \|_1^2$

(2) the function $\phi$is of a Carathéodory type on p$\times \mathbb{R}$ and satisfies for $(\vec{x},t) \in p$:

i) $|\phi(\vec{x},t,u)| \leq \delta(\vec{x},t) + c_1|u|$, where $c_1 > 0$, $u \in \mathbb{R}$ and $\delta \in L^2(p,\mathbb{R})$

ii) $|\phi(\vec{x},t,u_1) - \phi(\vec{x},t,u_2)| \leq L|u_1 - u_2|$, where $u_1, u_2 \in \mathbb{R}$, L is a Lipischitz constant.

3.1 Discretization of the Continuous Equation:

The weak form of (4)-(5) is discretized by using the GFEM as follows, let the domain ω w is a polyhedron. For every integer n, let $\{\omega_i^n\}_{i=1}^{M(n)}$ be an admissible regular triangulation of $\bar{\omega}$ into closed disimplices [7], $\{I_k^n\}_{k=0}^{NT-1}$ be subdivision of the interval $\bar{I}$ into $NT(n)$ intervals , where $I_k^n = [t_k^n, t_{k+1}^n]$, of equal length $\Delta t = T/NT$, let $p_{ik} = \omega_i^n \times I_k^n$ *and* $W_n \subset W$ be the space of continuous piecewise affine in W. The Discrete equation (DEq) of the weak form (4-5) is obtained through applying the Crank-Nicolson scheme and it is

$(u_{k+1}^n - u_k^n, w) + \Delta t a\left(u_{\frac{1}{2}k}^n, w\right) = \Delta t\left(\phi\left(t_{\frac{1}{2}k}^n, u_{\frac{1}{2}k}^n\right), w\right)$, $k = 0,1,\dots,NT-1$ (6)

$(u_0^n, w) = (u^0, w)$, $u^0 \in W$ (7)

where $\xi \in W_n$, $u_{\frac{1}{2}k}^n = \frac{1}{2}(u_{k+1}^n + u_k^n)$, $t_{\frac{1}{2}k}^n = \frac{1}{2}(t_{k+1}^n + t_k^n)$, $u_k^n = u(t_k^n) \in W_n$, $k = 0,1,\dots,NT-1$.

3.2 The Approximation Solution of the Nonlinear Parabolic Equation:

To find the APS $u^n = (u_0^n, u_1^n, \dots, u_{NT}^n)$ of (6-7) using the GFEM, the following procedure is used:

(1) Let $N = M_1 \times M_1$ with $M_1 = M - 1$, for any fixed k with $0 \leq k \leq NT-1$, let $\{w_i, i = 1,2,\dots,N$, with $w_i(\vec{x}) = 0$, on $\partial w\}$ be a continuous piecewise affine finite basis of W_n in W , then for any $u_k^n, u_{k+1}^n \in W_n$ (6-7) can be rewritten as:

$(u_{k+1}^n - u_k^n, w_i) + \Delta t a\left(u_{\frac{1}{2}k}^n, w_i\right) = \Delta t\left(\phi\left(t_{\frac{1}{2}k}^n, \frac{1}{2}u_{k+1}^n + \frac{1}{2}u_k^n\right), w_i\right)$, $i = 1,2,\dots,N$ (8)

$(u_o^n, w_i) = (u^0, w_i)$, $w_i \in W_n$,$i = 1,2,\dots,N$ (9)

(2) Apply the Galerkin method [7], to approximate the discrete functions u_0^n, u_k^n and u_{k+1}^n by their approximation forms using the basis $(w_1, w_2, \dots, w_N)$ of W_n , i.e.

$u_0^n = \sum_{j=1}^{N} \alpha_j^0 w_j$, $u_k^n = \sum_{j=1}^{N} \alpha_j^k w_j$ and $u_{k+1}^n = \sum_{k=1}^{N} \alpha_j^{k+1} w_j$

where ,$\alpha_j^k = \alpha_j(t_k^n)$, for each $k = 0,1,\dots,NT$ are unknown constants to be determine.

(3) Substitute u_k^n *and* u_{k+1}^n in (8) to get the following NLAS

$\left(C + \frac{1}{2}\Delta t D\right)\vec{\alpha}^{k+1} = \left(C - \frac{1}{2}\Delta t D\right)\vec{\alpha}^k + \vec{\beta}\left(t_{\frac{1}{2}k}^n\right)$, $k = 0,1,\dots,NT-1$ (10)

and substituting u_0^n in (9) to get the following LAS $\bar{\alpha}$

$C\vec{\alpha}^0 = \vec{\beta}^0$ (11)

where $C = (c_{ij})_{N\times N}$, $c_{ij} = (w_j, w_i)$,$D = (d_{ij})_{N\times N}$, $d_{ij} = a(w_j, w_i)$, $\vec{\alpha}_{N\times 1}^k = (\alpha_1^k, \alpha_2^k, \dots, \alpha_N^k)^T$, $\vec{\beta} = (\beta_i)_{N\times 1}$, $\beta_i = \Delta t\left(\phi\left(\frac{1}{2}\vec{b}^T\vec{\alpha}^{k+1} + \frac{1}{2}\vec{b}^T\vec{\alpha}^k\right), w_i\right)$, $\vec{b}_{N\times 1} = (w_1, w_2, \dots, w_N)^T$ *and* $\vec{\beta}^0 = (\beta_i^0)_{N\times 1}$, $\beta_i^0 = (u^0, w_i)$, $\forall\ i,j = 1,2,\dots,N$.

It is clear that the matrices C and $C + \frac{1}{2}\Delta t D$ in the system (10)-(11) are symmetric and positive definite (SAPD) hence the system has a unique solution [9]. To solve it, the LAS (11) is solved at first to get $\vec{\alpha}^0$, then to solve the NLAS (10) the PCM is used here [7], as follows : For each k $(0 \leq k \leq NT-1)$ the value (predictor solution PS) of the vector $\vec{\alpha}^{k+1}$ is predicate at first by using the explicit form (just the value of $\vec{\alpha}^{k}$) in the vector $\vec{\beta}$ in the right hand side (RHS) of (10) , then by setting $\vec{\tilde{\alpha}}^{k+1} = \vec{\alpha}^{k+1}$, in the vector $\vec{\beta}$ in the RHS of (10), again it becomes a LAS w.r.t. $\vec{\alpha}^{k+1}$, which is solved to get the corrector solution(CS) $\vec{\alpha}^{k+1}$. (In this point it is important to mention here that this procedure can be repeated "more than one time" if we need to get more accuracy results by substitute the CS $\vec{\tilde{\alpha}}^{k+1} = \vec{\alpha}^{k+1}$ in the RHS of the LAS (10) and solve it again to get a new CS $\vec{\alpha}^{k+1}$). Hence the corrector equation described as follows:

$$\left(u_{k+1}^{(l+1)} - u_k, w\right) + \Delta t a\left(\tfrac{1}{2}u_{k+1}^{(l+1)} + \tfrac{1}{2}u_k, w\right) = \Delta t\left(\phi\left(\tfrac{1}{2}u_{k+1}^{(l)} + \tfrac{1}{2}u_k\right), w\right) \tag{12}$$

where $u_{k+1}^{(l)} := u_{k+1}^{n}$ is the PS at the iteration $l+1$, $u_{k+1}^{(l+1)} := u_{k+1}^{n}$ is its corresponding CS at the iteration l and $u_k = u_k^n$ is the known CS for the previous step k, i.e. (12) can be written as :

$$u^{(l+1)} = f(u^{(l)}) \tag{13}$$

Theorem 4 (Existence and Uniqueness of Solution): The discrete equation (6-7) with fixed point and for Δt sufficiently small has a unique solution $u^n = (u_0^n, u_1^n, \dots, u_N^n)$, and the sequence of corrector solutions is convergence in $\mathbb{R}$.

Proof: Let $u^{(l+1)} = \left(u_0^{(l+1)}, \dots, u_k^{(l+1)}, \dots, u_N^{(l+1)}\right)$ and $\bar{u}^{(l+1)} = (\bar{u}_0^{(l+1)}, \dots, \bar{u}_k^{(l+1)}, \dots, \bar{u}_N^{(l+1)})$ are two solutions of (12) , i.e.

$$(u_{k+1}^{(l+1)} - u_k, w) + \Delta t a(\tfrac{1}{2}u_{k+1}^{(l+1)} + \tfrac{1}{2}u_k, w) = \Delta t(\phi\left(\tfrac{1}{2}u_{k+1}^{(l)} + \tfrac{1}{2}u_k\right), w) \tag{14}$$

$$(\bar{u}_{k+1}^{(l+1)} - u_k, w) + \Delta t a(\tfrac{1}{2}\bar{u}_{k+1}^{(l+1)} + \tfrac{1}{2}u_k, w) = \Delta t(\phi\left(\tfrac{1}{2}\bar{u}_{k+1}^{(l)} + \tfrac{1}{2}u_k\right), w) \tag{15}$$

By subtracting (15) from (14), setting $w = u_{j+1}^{(l+1)} - v_{j+1}^{(l+1)}$ in the obtained equation and using in assumption (2-ii), to get

$$\left\|u_{k+1}^{(l+1)} - \bar{u}_{k+1}^{(l+1)}\right\|_0^2 + \tfrac{1}{2}\Delta t a\left(u_{k+1}^{(l+1)} - \bar{u}_{k+1}^{(l+1)}, u_{k+1}^{(l+1)} - \bar{u}_{k+1}^{(l+1)}\right)$$
$$\leq \tfrac{1}{2}\Delta t L\left(\left|u_{k+1}^{(l)} - \bar{u}_{k+1}^{(l)}\right|, \left|u_{k+1}^{(l+1)} - \bar{u}_{k+1}^{(l+1)}\right|\right)$$

From assumption (1-ii) the 2^{nd} term in the left hand side (LHS) is nonnegative and then applying the Cauchy Schwarz (CS) inequality on the RHS of above inequality, it becomes

$$\left\|u_{k+1}^{(l+1)} - \bar{u}_{k+1}^{(l+1)}\right\|_0 \leq \gamma\left\|u_{k+1}^{(l)} - \bar{u}_{k+1}^{(l)}\right\|_0 \quad \text{, where } \gamma = \tfrac{1}{2}\Delta t L \text{ ,}$$

Upon using (13), the above inequality gives

$$\left\|f\left(u_{k+1}^{(l)}\right) - f(\bar{u}_{k+1}^{(l)})\right\|_0 \leq \gamma\left\|u_{k+1}^{(l)} - \bar{u}_{k+1}^{(l)}\right\|_0$$

It means that f is contractive (since Δt is sufficiently small and $\gamma < 1$), hence we get $u^{(l+1)} = \bar{u}^{(l+1)}$ (by theorem (1)), which means the DEq has a unique solution, on the other hand since for each l, that $\{u^{(l)}\} \in \mathbb{R}$, then so $f(u^{(l)}) = u^{(l+1)} \in \mathbb{R}$ which implies to $f(u) \in \mathbb{R}$, for each $u \in \mathbb{R}$, finally we get that $\{u^{(l)}\}$ converges to a point in $\mathbb{R}$ (by Theorem (3) with $X = \mathbb{R}$) .

4. General Cholesky Reduction Order Technique:

This technique is based in fact on an idea which is introduced at first in [10] which it reduces the diagonal elements in the Galerkin matrix in the LHS of the LAS) into columns, for this reasons we gave it the name Cholesky reduction order technique (GCHROT) and we formulate it by the following steps:

First, Let A be a SAPD $N \times N$ matrix, then A is reduced to a new matrix B of order $N \times M1$ by transforming the lower diagonals $(M1)$ of the matrix A to columns, **second** the new $N \times M1$ matrix B which is computed by using the following formula :

for $= 1,2, \dots, N$, $j = i+1, \dots, \min(i+M, N)$

- if $i = 1$, then $B_{iM1} = \sqrt{A_{iM1}}$ and $B_{jl} = \frac{A_{jl}}{R_{iM1}}$, $l = i - j + M1$
- if $i > 1$, then $B_{iM1} = \sqrt{A_{iM1} - \sum_{r=K-i+M1} B_{ir}^2}$, $K = \max(i-M, 1) : i-1$

  $B_{jl} = \frac{1}{R_{iM1}}\left(A_{jl} - \sum_{r=K-i+M1} B_{ir}B_{js}\right)$, $s = r + i - j$, with $j - K \leq M$.

Example (1): Consider the following NLBPVPVC:

$$u_t - \frac{\partial}{\partial x_1}\left[(x_1^2 + 1)\frac{\partial u}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[(x_2^2 + 1)\frac{\partial u}{\partial x_2}\right] + u = \phi(x_1, x_2, t, u)$$

$u(x_1, x_2, t) = 0$, on $\partial\omega \times \text{I}$
$u(x_1, x_2, 0) = 0.1x_1x_2(1 - x_1)(1 - x_2)$, on ω
where $\omega = [0,1] \times [0,1]$, $\text{I} = [0,1]$

$$\phi(x_1, x_2, t, u) = e^t\{0.2(x_1^2x_2 - 2x_1^2x_2^2 + x_1x_2^2 - x_1^2 + x_1) + 0.2(x_2^2x_1 - 2x_1^2x_2^2 + x_2x_1^2 - x_2^2 + x_2) \\ -0.1(x_1x_2 - x_1x_2^2 - x_2x_1^2 + x_1^2x_2^2)(1 + \sin(0.1(x_1x_2 - x_1x_2^2 - x_2x_1^2 + x_1^2x_2^2)e^t) \\ +u\sin u$$

The exact solution of this problem is: $u(x_1, x_2, t) = 0.1x_1x_2(1 - x_1)(1 - x_2)e^t$

The above BVP of NLBPVPVC is solved using the GFEM with the PCM, the LAS which are obtained from the PCM is solved either by the CHM or by the CHROT, with M=9 and NT=20, a computer program is written in MATLAB Software to code the solution of this problem, it takes ***5-hours and 4-minutes*** when we use the CHM, while it takes ***56- minutes*** when we use the CHROT. The approximate solution $\bar{u}(x_1, x_2, t)$, the exact solution $u(x_1, x_2, t)$ and the absolute error at (x_1, x_2) are given at the time $\hat{t} = 0.5$ in Table-1 and are shown in Figure-1.

Table 1-Comparison between exact and approximation solutions

x_1	x_2	$u(x_1,x_2,t)$	$\bar{u}(x_1,x_2,t)$	absolute error	x_1	x_2	$u(x_1,x_2,t)$	$\bar{u}(x_1,x_2,t)$	absolute error
0.1	0.1	0.0014	0.0014	0.0000	0.2	0.1	0.0024	0.0025	0.0001
0.3	0.1	0.0032	0.0032	0.0000	0.4	0.1	0.0037	0.0037	0.0000
0.5	0.1	0.0038	0.0039	0.0001	0.6	0.1	0.0037	0.0037	0.0000
0.7	0.1	0.0032	0.0032	0.0000	0.8	0.1	0.0024	0.0025	0.0001
0.9	0.1	0.0014	0.0014	0.0000	0.1	0.2	0.0024	0.0025	0.0001
0.2	0.2	0.0043	0.0044	0.0001	0.3	0.2	0.0057	0.0058	0.0001
0.4	0.2	0.0065	0.0066	0.0001	0.5	0.2	0.0068	0.0069	0.0001
0.6	0.2	0.0065	0.0066	0.0001	0.7	0.2	0.0057	0.0058	0.0001
0.8	0.2	0.0043	0.0044	0.0001	0.9	0.2	0.0024	0.0025	0.0001
0.1	0.3	0.0032	0.0032	0.0000	0.2	0.3	0.0057	0.0058	0.0001
0.3	0.3	0.0075	0.0076	0.0001	0.4	0.3	0.0085	0.0087	0.0002
0.5	0.3	0.0089	0.0090	0.0001	0.6	0.3	0.0085	0.0087	0.0002
0.7	0.3	0.0075	0.0076	0.0001	0.8	0.3	0.0057	0.0058	0.0001
0.9	0.3	0.0032	0.0033	0.0001	0.1	0.4	0.0037	0.0037	0.0000
0.2	0.4	0.0065	0.0066	0.0001	0.3	0.4	0.0085	0.0087	0.0002
0.4	0.4	0.0097	0.0099	0.0002	0.5	0.4	0.0101	0.0103	0.0002
0.6	0.4	0.0097	0.0099	0.0002	0.7	0.4	0.0085	0.0087	0.0002
0.8	0.4	0.0065	0.0066	0.0001	0.9	0.4	0.0037	0.0037	0.0000
0.1	0.5	0.0038	0.0039	0.0001	0.2	0.5	0.0068	0.0069	0.0001
0.3	0.5	0.0089	0.0090	0.0001	0.4	0.5	0.0101	0.0103	0.0002
0.5	0.5	0.0106	0.0108	0.0002	0.6	0.5	0.0101	0.0103	0.0002
0.7	0.5	0.0089	0.0091	0.0002	0.8	0.5	0.0068	0.0069	0.0001
0.9	0.5	0.0038	0.0039	0.0001	0.1	0.6	0.0037	0.0037	0.0000
0.2	0.6	0.0065	0.0066	0.0001	0.3	0.6	0.0085	0.0087	0.0002
0.4	0.6	0.0097	0.0099	0.0002	0.5	0.6	0.0101	0.0103	0.0002
0.6	0.6	0.0097	0.0099	0.0002	0.7	0.6	0.0085	0.0087	0.0002
0.8	0.6	0.0065	0.0066	0.0001	0.9	0.6	0.0037	0.0037	0.0000
0.1	0.7	0.0032	0.0032	0.0000	0.2	0.7	0.0057	0.0058	0.0001
0.3	0.7	0.0075	0.0076	0.0001	0.4	0.7	0.0085	0.0087	0.0002
0.5	0.7	0.0089	0.0091	0.0002	0.6	0.7	0.0085	0.0087	0.0002
0.7	0.7	0.0075	0.0076	0.0001	0.8	0.7	0.0057	0.0058	0.0001
0.9	0.7	0.0032	0.0032	0.0000	0.1	0.8	0.0024	0.0025	0.0001
0.2	0.8	0.0043	0.0044	0.0001	0.3	0.8	0.0057	0.0058	0.0001
0.4	0.8	0.0065	0.0066	0.0001	0.5	0.8	0.0068	0.0069	0.0001
0.6	0.8	0.0065	0.0066	0.0001	0.7	0.8	0.0057	0.0058	0.0001
0.8	0.8	0.0043	0.0044	0.0001	0.9	0.8	0.0024	0.0025	0.0001

0.1	0.9	0.0014	0.0014	0.0000	0.2	0.9	0.0024	0.0025	0.0001
0.3	0.9	0.0032	0.0033	0.0001	0.4	0.9	0.0037	0.0037	0.0000
0.5	0.9	0.0038	0.0039	0.0001	0.6	0.9	0.0037	0.0037	0.0000
0.7	0.9	0.0032	0.0032	0.0000	0.8	0.9	0.0024	0.0025	0.0001
0.9	0.9	0.0014	0.0014	0.0000					

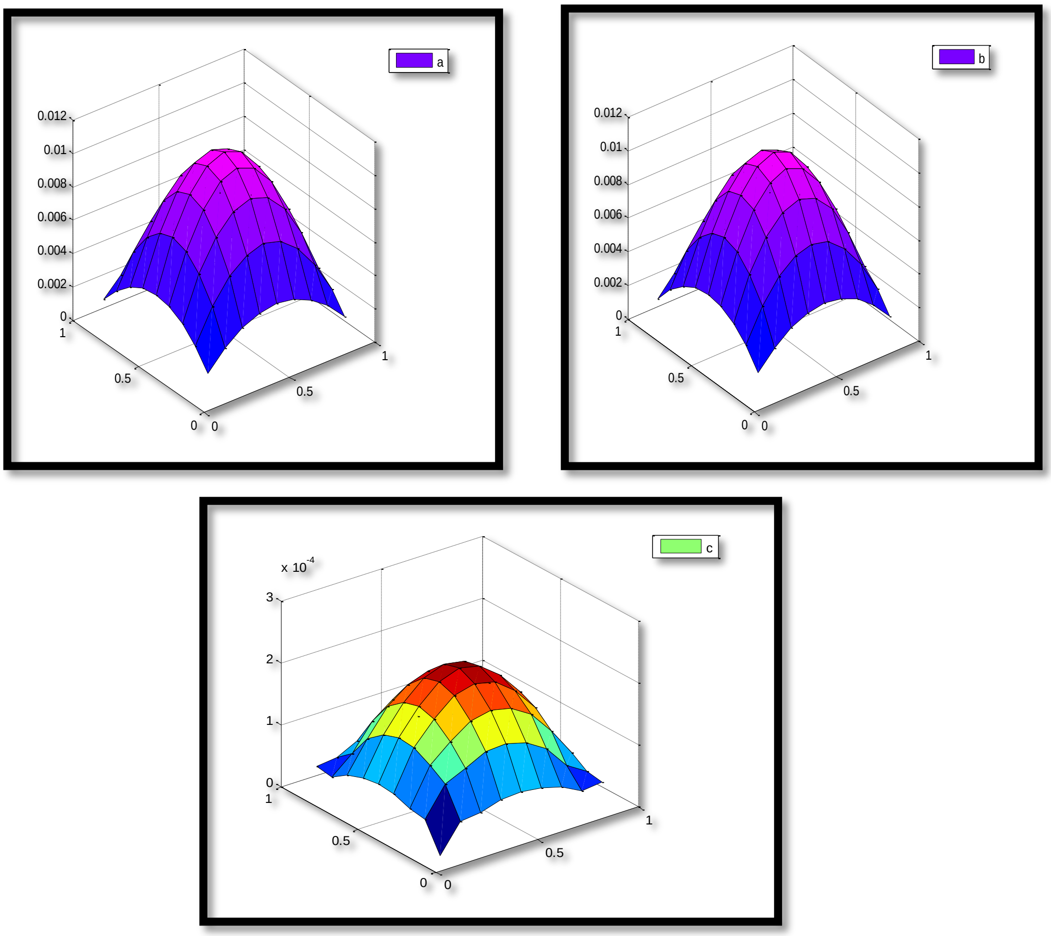

Figure 1-(a) shows the approximation solution, (b) shows the exact solution and (c) the absolute error

Example (2): Consider the following nonlinear parabolic equation

$$u_t - \frac{\partial}{\partial x_1}\left[(x_1+1)\frac{\partial u}{\partial x_1}\right] - \frac{\partial}{\partial x_2}\left[(x_2+1)\frac{\partial u}{\partial x_2}\right] + u = \phi(x_1,x_2,t,u)$$

Associated with the i.c and b.c

$u(x_1,x_2,t) = 0$, on $\partial\omega \times \mathrm{I}$

$u(x_1,x_2,0) = 0$, in ω

where

$$\phi(x_1,x_2,t,u) = \sin(t)\,\{x_2(1-x_2)[x_1^2 - 5x_1 - 1] - x_1(1-x_1)$$
$$[x_2(4 + (1-x_2)\sin([x_1x_2 - x_1x_2^2 - x_2x_1^2 + x_1^2x_2^2]\sin(t))) + 1]\}$$
$$-(x_1x_2 - x_1x_2^2 - x_2x_1^2 + x_1^2x_2^2)\cos(t) + u\sin u$$

The exact solution of this problem is: $u(x_1,x_2,t) = x_1x_2(1-x_1)(1-x_2)\sin(-t)$

The above BVP of NLBPVPVC is solved using the GFEM with the PCM, the LAS which is obtained from the PCM solved either by the CHM or by the CHROT, with M=9 and NT=20, a computer program is written in MATLAB Software to code the solution of this problem, it takes ***5-hours and 2-minutes*** when we use the CHM to solve the LAS, while it takes ***55-minute***s when we use the CHROT. The approximate solution$\bar{u}(x_1,x_2,t)$, the exact solution $u(x_1,x_2,t)$ and the absolute error at (x_1, x_2) are given at the time $\hat{t} = 0.5$ in Table-2 and are shown in Figure-2.

Table 2-Comparison between exact and approximation solutions

x_1	x_2	$u(x_1,x_2,t)$	$\bar{u}(x_1,x_2,t)$	absolute error	x_1	x_2	$u(x_1,x_2,t)$	$\bar{u}(x_1,x_2,t)$	absolute error
0.1	0.1	-0.0041	-0.0042	0.0001	0.2	0.1	-0.0072	-0.0075	0.0003
0.3	0.1	-0.0095	-0.0098	0.0003	0.4	0.1	-0.0108	-0.0112	0.0004
0.5	0.1	-0.0113	-0.0117	0.0004	0.6	0.1	-0.0108	-0.0112	0.0004
0.7	0.1	-0.0095	-0.0098	0.0003	0.8	0.1	-0.0072	-0.0075	0.0003
0.9	0.1	-0.0041	-0.0042	0.0001	0.1	0.2	-0.0072	-0.0075	0.0003
0.2	0.2	-0.0128	-0.0133	0.0005	0.3	0.2	-0.0168	-0.0174	0.0006
0.4	0.2	-0.0192	-0.0199	0.0007	0.5	0.2	-0.0200	-0.0208	0.0008
0.6	0.2	-0.0192	-0.0199	0.0007	0.7	0.2	-0.0168	-0.0175	0.0007
0.8	0.2	-0.0128	-0.0133	0.0005	0.9	0.2	-0.0072	-0.0075	0.0003
0.1	0.3	-0.0095	-0.0098	0.0003	0.2	0.3	-0.0168	-0.0174	0.0006
0.3	0.3	-0.0221	-0.0229	0.0008	0.4	0.3	-0.0253	-0.0262	0.0009
0.5	0.3	-0.0263	-0.0273	0.0010	0.6	0.3	-0.0253	-0.0262	0.0009
0.7	0.3	-0.0221	-0.0229	0.0008	0.8	0.3	-0.0168	-0.0175	0.0007
0.9	0.3	-0.0095	-0.0098	0.0003	0.1	0.4	-0.0108	-0.0112	0.0004
0.2	0.4	-0.0192	-0.0199	0.0007	0.3	0.4	-0.0253	-0.0262	0.0009
0.4	0.4	-0.0289	-0.0299	0.0010	0.5	0.4	-0.0301	-0.0312	0.0011
0.6	0.4	-0.0289	-0.0300	0.0011	0.7	0.4	-0.0253	-0.0262	0.0009
0.8	0.4	-0.0192	-0.0200	0.0008	0.9	0.4	-0.0108	-0.0112	0.0004
0.1	0.5	-0.0113	-0.0117	0.0004	0.2	0.5	-0.0200	-0.0208	0.0008
0.3	0.5	-0.0263	-0.0273	0.0010	0.4	0.5	-0.0301	-0.0312	0.0011
0.5	0.5	-0.0313	-0.0325	0.0012	0.6	0.5	-0.0301	-0.0312	0.0011
0.7	0.5	-0.0263	-0.0273	0.0010	0.8	0.5	-0.0200	-0.0208	0.0008
0.9	0.5	-0.0113	-0.0117	0.0004	0.1	0.6	-0.0108	-0.0112	0.0004
0.2	0.6	-0.0192	-0.0199	0.0007	0.3	0.6	-0.0253	-0.0262	0.0009
0.4	0.6	-0.0289	-0.0300	0.0011	0.5	0.6	-0.0301	-0.0312	0.0011
0.6	0.6	-0.0289	-0.0300	0.0011	0.7	0.6	-0.0253	-0.0262	0.0009
0.8	0.6	-0.0192	-0.0200	0.0008	0.9	0.6	-0.0108	-0.0112	0.0004
0.1	0.7	-0.0095	-0.0098	0.0003	0.2	0.7	-0.0168	-0.0175	0.0007
0.3	0.7	-0.0221	-0.0229	0.0008	0.4	0.7	-0.0253	-0.0262	0.0009
0.5	0.7	-0.0263	-0.0273	0.0010	0.6	0.7	-0.0253	-0.0262	0.0009
0.7	0.7	-0.0221	-0.0229	0.0008	0.8	0.7	-0.0168	-0.0175	0.0007
0.9	0.7	-0.0095	-0.0098	0.0003	0.1	0.8	-0.0072	-0.0075	0.0003
0.2	0.8	-0.0128	-0.0133	0.0005	0.3	0.8	-0.0168	-0.0175	0.0007
0.4	0.8	-0.0192	-0.0200	0.0008	0.5	0.8	-0.0200	-0.0208	0.0008
0.6	0.8	-0.0192	-0.0200	0.0008	0.7	0.8	-0.0168	-0.0175	0.0007
0.8	0.8	-0.0128	-0.0133	0.0005	0.9	0.8	-0.0072	-0.0075	0.0003
0.1	0.9	-0.0041	-0.0042	0.0001	0.2	0.9	-0.0072	-0.0075	0.0003
0.3	0.9	-0.0095	-0.0098	0.0003	0.4	0.9	-0.0108	-0.0112	0.0004
0.5	0.9	-0.0113	-0.0117	0.0004	0.6	0.9	-0.0108	-0.0112	0.0004
0.7	0.9	-0.0095	-0.0098	0.0003	0.8	0.9	-0.0072	-0.0075	0.0003
0.9	0.9	-0.0041	-0.0042	0.0001					

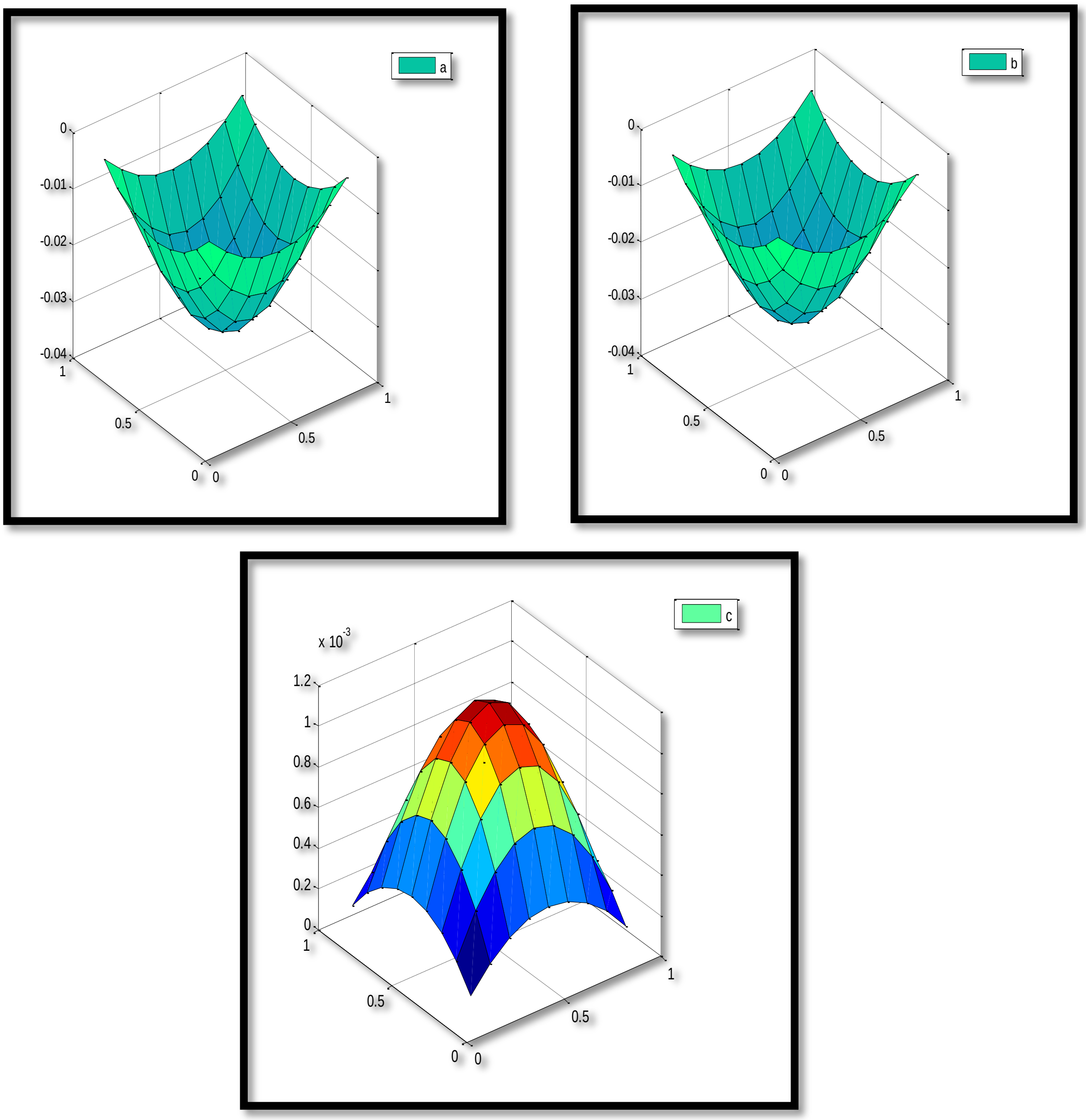

Figure 2-(a) shows the approximation solution, (b) shows the exact solution and (c) the absolute error

5. Conclusions

- The GFEM associated with the PCT is suitable, efficient and very fast to solve the nonlinear parabolic boundary value problems.
- The CHROT is very fast than the CHM with same results and this is important when we have problems gives very large algebraic systems which take a long time in the classical CHM.
- The value of $\hat{t}$ is chose arbitral in the interval I , same results with same accuracy will obtain if we can take any other value of $\hat{t}$ provided this value belongs to I .

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